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MULTIPLE KINK SOLUTIONS FOR THE (2+1)-DIMENSIONAL INTEGRABLE GARDNER EQUATION

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In this work we study the (2+1)-dimensional integrable Gardner equation. The simplified form of Hirota's direct method is used to derive multiple kink solutions and multiple singular kink solutions for this equation.

Key words: (2+1)-dimensional Gardner equation; Hirota bilinear method; multiple kink solutions.

1. INTRODUCTION

The (1+1)-dimensional Gardner equation reads

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x = 0, \tag{1}$$

that combines the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation. For $\alpha = 0$, Eq. (1) reduces to the KdV equation, whereas for $\beta = 0$ it reduces to the mKdV equation. The KdV equation was complemented with a higher-order cubic nonlinear term of the form $u^2 u_x$ to obtain the Gardner equation (1), which is completely integrable like the KdV equation and the mKdV equation. The function $u(x,t)$ is the amplitude of the relevant wave mode. Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [1–7]. It also describes a variety of wave phenomena in plasma and solid state physics [5, 6].

Kadomtsov and Petviashivilli [7] extended the KdV equation to obtain the Kadomtsov–Petviashivilli (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{2}$$

Kadomtsov and Petviashivilli [7] relaxed the restriction that the waves be strictly one dimensional, namely the *x*-direction of the KdV equation, to derive the completely integrable KP equation (2). The KP equation describes the evolution of quasi-one dimensional shallow-water waves when effects of the surface tension and the viscosity are negligible. However, Wazwaz [6] used the sense of Kadomtsov and Petviashivilli of the relaxation of the restriction that the waves be strictly one dimensional in the Gardner equation, and introduced the Gardner-KP equation given by

$$(u_t + 6uu_x + 6u^2 u_x + u_{xxx})_x + u_{yy} = 0. \tag{3}$$

The Gardner-KP equation was proved to be integrable. The Lie symmetries, conservation laws, reductions, and exact solutions via generalized double reduction theorem are computed for the Gardner-KP equation in [8]. However, in [9], the bifurcation concept was used to handle (3) and explicit parameter expressions of all types of bounded traveling wave solutions were derived.

In [2, 3], the (2+1)-dimensional Gardner equation was proposed in the integro-differential form

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma u_x \partial_x^{-1} u_y = 0, \tag{4}$$

where $\sigma^2 = \pm 1$, α and β are arbitrary constants, and ∂_x^{-1} is the inverse of ∂_x with $\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = 1$. The (2+1)-dimensional Gardner equation (4) was examined in [2,3] by using the inverse scattering theorem and proved to be a completely integrable equation. In [4], the authors presented the Casorati and Grammian determinant solutions to the (2+1)-dimensional Gardner equation (4). In [1–3], it was stated that the (2+1)-dimensional Gardner equation (4) has amazing properties that in appropriate limits it reduces to the KdV equation, mKdV equation, the Gardner equation, KP equation, and the modified KP (mKP) equation. For example for $\alpha = 0$, this equation reduces to the KP equation. For $\beta = 0$, it reduces to the mKP equation. For $\alpha = 0$ and $\sigma = 0$ it becomes the KdV equation [1–3]. For $\beta = 0$ and $\sigma = 0$ it becomes the mKdV equation. For $\sigma = 0$, the (2+1)-dimensional Gardner equation becomes the (1+1)-dimensional Gardner equation. We recall here that all the aforementioned equations are integrable.

The Lax pair [1]–[3] for the (2+1)-dimensional Gardner equation (4) is given by

$$\sigma\psi_y + \psi_{xx} + \alpha u \psi_x + \beta u \psi = 0, \tag{5}$$

$$\begin{aligned} &\psi_t + 4\psi_{xxx} + \alpha u \psi_{xx} + (3\alpha u_x + \frac{3}{2}\alpha^2 u^2 + 6\beta u - 3\alpha\sigma\partial_x^{-1} u_y)\psi_x + \\ &+ (3\beta u_x + \frac{3}{2}\alpha^2 \beta u^2 + 3\beta\sigma\partial_x^{-1} u_y)\psi = 0. \end{aligned} \tag{6}$$

The compatibility condition [1] between (5) and (6) gives the (2+1)-dimensional Gardner equation (4). Many kinds of solitons occur for this equation, including pulse-type solitons, positive and negative solitons, and kinks and table-top solitons [1].

Studies of various physical structures of nonlinear equations had attracted much attention in connection with the important problems that arise in scientific applications [10–23]. Many powerful methods, such the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis, and the generalized symmetry method are commonly used to integrate nonlinear evolution equations. The Hirota's bilinear method [10] and the simplified Hirota method [13], are rather heuristic and the most commonly used. These approaches possess powerful features that make them practical for the determination of multiple soliton solutions for a wide class of nonlinear evolution equations. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

It is the aim of this work to focus on deriving multiple kink solitons and multiple singular kink solutions for the (2+1)-dimensional Gardner equation (4). The constraint condition for the existence of multiple kink solutions will be established. The simplified Hirota's method, developed by Hereman-Nuseir [13] will be employed to achieve this goal.

2. MULTIPLE KINK SOLUTIONS

In this section, we will derive multiple kink solutions for the (2+1)-dimensional Gardner equation

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \partial_x^{-1} u_{yy} - 3\alpha\sigma\partial_x^{-1} u_y = 0, \tag{7}$$

where ∂_x^{-1} is the inverse of ∂_x with $\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = 1$, and

$$(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t, \tag{8}$$

under the decaying condition at infinity.

To get rid of the inverse operator, we use the potential transformation

$$u(x, y, t) = v_x(x, y, t), \tag{9}$$

that carries (7) to the potential form

$$v_{xt} + 6\beta v_x v_{xx} + v_{xxxx} - \frac{3}{2}\alpha^2 v_x^2 v_{xx} + 3\sigma^2 v_{yy} - 3\alpha\sigma v_{xx} v_y = 0. \tag{10}$$

To determine the dispersion relation c_i, we substitute

$$v(x,y,t) = \mathrm{e}^{k_i x + r_i y - c_i t}, \tag{11}$$

into the linear terms of (10); we obtain the dispersion relation

$$c_i = k_i^3 + \frac{3\sigma^2 r_i^2}{k_i}, \tag{12}$$

and hence we set the dispersion variable as

$$\theta_i = k_i x + r_i y - \left(k_i^3 + \frac{3\sigma^2 r_i^2}{k_i}\right)t. \tag{13}$$

The multi-kink solutions of (7), using the Cole-Hopf transformation, are assumed to be

$$u(x,y,t) = R\left(\ln f(x,y,t)\right)_x, \tag{14}$$

and therefore

$$v(x,y,t) = R\left(\ln f(x,y,t)\right). \tag{15}$$

The simplified Hirota's method admits the use of the auxiliary function $f(x,y,t)$ for the single kink solution by

$$f(x,y,t) = 1 + \mathrm{e}^{k_1 x + r_1 y - (k_1^3 + \frac{3\sigma^2 r_1^2}{k_1})t}. \tag{16}$$

Substituting (15) into (10) and solving for R we find

$$R = \frac{2}{\alpha}, \tag{17}$$

and the kink solutions exist if the coefficients r_i satisfy the constraint condition

$$r_i = \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha}, \quad i = 1, 2, \cdots, N. \tag{18}$$

Based on this result, the dispersion relation (12) becomes

$$c_i = \frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}, \quad i = 1, 2, \cdots, N. \tag{19}$$

Using the potential (9) gives the single kink solution

$$u(x,y,t) = \frac{2k_1 \mathrm{e}^{k_1 x + \frac{k_1(2\beta - \alpha k_1)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_1^2 - 3\alpha\beta k_1 + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 + \mathrm{e}^{k_1 x + \frac{k_1(2\beta - \alpha k_1)}{\sigma\alpha} y - \left(\frac{4k_1(\alpha^2 k_1^2 - 3\alpha\beta k_1 + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{20}$$

For the two kink solutions we set

$$f(x,y,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2}. \tag{21}$$

Substituting these results into (15) we find the two kink solutions given by

$$u(x,y,t) = \frac{2\sum_{i=1}^{2} k_i \mathrm{e}^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 + \sum_{i=1}^{2} \mathrm{e}^{k_i x + \frac{k_i(2\beta - \alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{22}$$

For the three kink solutions, we set

$$f(x,y,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}. \tag{23}$$

Proceeding as before, we find the three kink solutions given by

$$u(x,y,t)=\frac{2\sum_{i=1}^{3}k_i\mathrm{e}^{k_ix+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y-\left(\frac{4k_i(\alpha^2k_i^2-3\alpha\beta k_i+3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1+\sum_{i=1}^{3}\mathrm{e}^{k_ix+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y-\left(\frac{4k_i(\alpha^2k_i^2-3\alpha\beta k_i+3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{24}$$

This shows that the (2+1)-dimensional Gardner equation (7) gives N-kink solutions for finite N, where $N\geq 1$. Based on the obtained results, the general N-kink solutions can be set in the form

$$u(x,y,t)=\frac{2\sum_{i=1}^{N}k_i\mathrm{e}^{k_ix+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y-\left(\frac{4k_i(\alpha^2k_i^2-3\alpha\beta k_i+3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1+\sum_{i=1}^{N}\mathrm{e}^{k_ix+\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha}y-\left(\frac{4k_i(\alpha^2k_i^2-3\alpha\beta k_i+3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{25}$$

3. MULTIPLE SINGULAR KINK SOLUTIONS

We will next derive multiple singular kink solutions for the (2+1)-dimensional Gardner equation

$$u_t+6\beta uu_x+u_{xxx}-\frac{3}{2}\alpha^2u^2u_x+3\sigma^2\partial_x^{-1}u_{yy}-3\alpha\sigma\partial_x^{-1}u_y=0, \tag{26}$$

that becomes

$$v_{xt}+6\beta v_xv_{xx}+v_{xxxx}-\frac{3}{2}\alpha^2v_x^2v_{xx}+3\sigma^2v_{yy}-3\alpha\sigma v_{xx}v_y=0, \tag{27}$$

upon using the potential

$$u(x,y,t)=v_x(x,y,t). \tag{28}$$

The dispersion relation c_i, the dispersion variables, the coefficient R, and the constraint condition for the coefficients r_i remain the same, given by

$$c_i=k_i^3+\frac{3\sigma^2r_i^2}{k_i}, \tag{29}$$

$$\theta_i=k_ix+r_iy-(k_i^3+\frac{3\sigma^2r_i^2}{k_i})t, \tag{30}$$

$$R=\frac{2}{\alpha}, \tag{31}$$

and

$$r_i=\frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha},\quad i=1,2,\cdots,N. \tag{32}$$

respectively.

The simplified Hirota's method admits the use of the auxiliary function $f(x,y,t)$ for the singular kink solution by

$$f(x,y,t)=1-\mathrm{e}^{\theta}. \tag{33}$$

Using the potential (28) gives the single singular kink solution by

$$u(x,y,t) = -\frac{2k_1 \mathrm{e}^{k_1 x + \frac{k_1(2\beta-\alpha k_1)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_1^2 - 3\alpha\beta k_1 + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 - \mathrm{e}^{k_1 x + \frac{k_1(2\beta-\alpha k_1)}{\sigma\alpha} y - \left(\frac{4k_1(\alpha^2 k_1^2 - 3\alpha\beta k_1 + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{34}$$

For the two kink solutions we set

$$f(x,y,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2}. \tag{35}$$

This in turn gives the two singular kink solutions given by

$$u(x,y,t) = -\frac{2\sum_{i=1}^{2} k_i \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 - \sum_{i=1}^{2} \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{36}$$

For the three kink solutions, we set

$$f(x,y,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} \tag{37}$$

Proceeding as before, we find that the three kink solutions are given by

$$u(x,y,t) = -\frac{2\sum_{i=1}^{3} k_i \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 - \sum_{i=1}^{3} \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{38}$$

This shows that the (2+1)-dimensional Gardner equation (26) gives N-kink solutions for finite N, where $N \geq 1$. Based on the obtained results, the general singular *N*-kink solutions can be set in the form

$$u(x,y,t) = -\frac{2\sum_{i=1}^{N} k_i \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}}{\alpha\left(1 - \sum_{i=1}^{N} \mathrm{e}^{k_i x + \frac{k_i(2\beta-\alpha k_i)}{\sigma\alpha} y - \left(\frac{4k_i(\alpha^2 k_i^2 - 3\alpha\beta k_i + 3\beta^2)}{\alpha^2}\right)t}\right)}. \tag{39}$$

4. DISCUSSION

We studied in this work the (2+1)-dimensional Gardner equation proposed by Konopelchenko and Dubrovsky in [2, 3]. We employed the simplified form of Hirota's direct method to derive multiple kink solutions and multiple singular kink solutions. This equation is integrable and can be reduced to other integrable equations.

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