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RISK-NEUTRAL DENSITIES IN ENTROPY THEORY OF STOCK OPTIONS USING LAMBERT FUNCTION AND A NEW APPROACH

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In the theory of option pricing the risk neutral probabilities plays very important role. The application of entropy in Finance can be regarded as the extension of both information entropy and the probability entropy. Les Gulko [8] applied Entropy Pricing Theory (EPT) for pricing stock options and introduced an alternative framework of Black-Scholes model for pricing European stock option. In this article we present a solution of maximum entropy problem based on new entropy [22], to obtain risk-neutral density using Lambert function and a new type of approach.

Key words: option pricing, black-scholes model, entropy pricing theory, entropy measures.

1. INTRODUCTION

The application of entropy in Finance can be regarded as the extension of both information entropy and probability entropy. Since last two decades, it has become a very important tool for the methods of portfolio selection and asset pricing. The famous Black-Scholes model [1] assumes the condition of no arbitrage which implies the universe of risk-neutral probabilities. The uniqueness of these risk-neutral probabilities is very crucial. The stock price process is controlled by Geometric Brownian Motion (GBM) in Black and Scholes model and in this framework stochastic calculus is vital. The Entropy Pricing Theory (EPT) was introduced by Les Gulko as an alternative method for the construction of risk-neutral probabilities without relying on stochastic calculus [8, 9]. The Principle of Maximum Entropy (MEP) was used to estimate the distribution of an asset from a set of option prices [12]. Beside this work the maximum entropy principle was used to retrieve the risk-neutral density of future stock risks or other asset risks [20]. The Renyi entropy [19] generalizes the frequently used Shannon entropy [20] and it has been used for option price calibration [6]. Some more extensions for the use of entropy in the general class of chains with complete connection can be found in [10, 11]. Recently Preda et al. used Tsallis and Kaniadakis entropy measures for the case of semi-Markov regime switching interest rate models [18]. For maximum entropy distribution of asset returns, application of entropy maximization problems, and others can be found in [3, 7, 15, 16, 17].

In In this article we use new type of entropy introduced in [22] to find the risk-neutral densities using the framework of EPT for stock options, by defining a new approach. In Section 2 we present the introduction of EPT and formulation of our problems, and then we further develop the structure to obtain our new results. In Section 3 we present our first approach with Lambert function to get risk-neutral density of stock options. In Section 4 we introduce the weighted Shafee entropy and the expected utility-weighted Shafee entropy (EU-WSE). We extend our results in this framework. In Section 5 we present an alternative approach. Section 6 concludes our results.

2. PROBLEM FORMULATION

In this section we use the concept of EPT [8, 9]. The term market belief is vital in option pricing and the current price of any risky asset indicates this belief. The future picture of the market up (down) reflects a state of maximum possible uncertainty, therefore market belief for the future performance of an efficient price is characterized by maximum uncertainty. Consider a risky asset on time interval $[0,T]$. Let Y_T be the asset price process of S_T at the future time T, G the state space, a subset of the real line $\mathbf{R}$, $g(S_T)$ the probability densities on P, $f(S_T)$ the efficient market belief and $H(g)$ the index of market uncertainty about Y_T. The function $H(g)$ is defined on the set of beliefs $g(S_T)$, therefore the efficient market belief $f(S_T)$ maximizes $H(g)$. We can determine $f(S_T)$ given $H(g)$ with some relevant information about current price of S. The index of the market uncertainty about Y_T as a Shannon entropy can be written as:

$$H(g) = -\mathrm{E}^g\left[\ln g(Y_T)\right].$$

In the above equation $H(g)$ is a functional defined on $g(S_T)$, $f(S_T)$, which maximizes $H(g)$, is called the entropy of random variable Y_T and used to measure the degree of uncertainty of $g(S_T)$. The maximum entropy characterizes the market beliefs regardless of the subjective risk preferences and it is useful to find the risk neutral beliefs in incomplete arbitrage free markets. The maximum entropy market belief $f(S_T)$ as a solution to the maximum entropy problem can be written as follows:

$$f = \arg\max\left\{H(g), g \in G\right\}.$$

In 2007 non additive entropy was proposed [22]. It gives the general form that is non-extensive like Tsallis, but is linearly dependent on component entropies. The mathematical description of the new entropy functions was in discrete case and for our problem we consider the continuous cases as an analogue of the discrete case:

$$H_q(g) = -\mathrm{E}^g\left[g(Y_T)^{q-1}\ln g(Y_T)\right], \;\; q>0, q \neq 1.$$

The Lambert function has become an important tool since its beginning with Lambert in 1768 and Euler in 1779. The Lambert function W is a multivalued complex function which is defined as the solution of the equation $W(z)e^{W(z)} = z$, where z is a complex number. If z is a real number such that $z \geq -e^{-1}$, then $W(z)$ becomes a real function with two possible real branches taking values in $(-\infty,-1]$ and $[-1,\infty)$.

LEMMA 2.1 [12]. *Let a, b and c be any fixed complex numbers. Then the solution of the equation $z + ab^z = c$, with $z \in \mathbf{C}$, is $z = c - \frac{1}{\log(b)} W\left(ab^c \log(b)\right)$, where W is the Lambert function and $\mathbf{C}$ denotes the set of the complex numbers.*

Casquilho et al. [4] used the expected utility-weighted entropy (EU-WE) framework under a 1-parameter generalization of Shannon formula focused on an ecological and economic application at the landscape level. Following this approach the EU-WE framework, if u is a positive utility application on G, then we can define mathematically for the case of Weighted-Shannon entropy the EU-EW framework as:

$$\mathrm{E}^g\left[u(Y_\mathrm{T})\right] - \mathrm{E}^g\left[u(Y_\mathrm{T})\ln g(u(Y_\mathrm{T}))\right].$$

3. THE FRAME OF SHAFEE ENTROPY

We consider the analogue of the discrete case of non additive entropy defined by [22]. We present the solution of risk-neutral density using Lambert function. Also we assume that the Lambert function is well defined for the quantities which have been considered. We suppose that all expectations are also well defined and underlying optimization problems admit solutions for some continuous cases. More details can be found in [2, 5, 14].

THEOREM 3.1. *Consider the entropy maximization problem*

$$\begin{aligned} &\max -\mathrm{E}^g\left[g^{q-1}(Y_T)\ln g(Y_T)\right] \\ \text{subject to}\quad &\mathrm{E}^g\left[I_{\{Y_T>0\}}\right]=1, \\ &\mathrm{E}^g\left[\varphi_i(Y_T)\right]=a_i,\ i=\overline{1,n}, \end{aligned}$$

where $\varphi_i:\mathbf{R}\to\mathbf{R}$ *and* $a_1,\ldots,a_n$ *are given real values. Then the solution of the problem is given by:*

$$g(S_T)=\left[\frac{qW\left(\frac{\left(\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)(1-q)}{q}\exp\left\{-\left(\frac{1-q}{q}\right)\right\}\right)}{\left(\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)(1-q)}\right]^{\frac{1}{1-q}},$$

where $\alpha,\beta_1,\ldots,\beta_n$ *are the Lagrange multipliers and* W *is the Lambert function.*

Proof. We can write the Lagrangian L and using calculus of variations for optimization of functional (see for example Luenberger, 1969, Cover and Thomas, 1991, Borwein, 2003) we find

$$L=\mathrm{E}^g\left[g^{q-1}(Y_T)\ln g(Y_T)\right]-\alpha\left(\mathrm{E}^g\left[I_{\{Y_T>0\}}\right]-1\right)-\sum_{i=1}^{n}\beta_i\left(\mathrm{E}^g\left[\varphi_i(Y_T)\right]-a_i\right).$$

Then we obtain that g has to satisfy the relation

$$qg^{q-1}(S_T)\ln g(S_T)+\frac{g^q(S_T)}{g(S_T)}+\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)=0,$$

which is equivalent to

$$q\ln g(S_T)+1=-\left(\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\right)g^{1-q}(S_T).$$

Let us suppose $y=g^{1-q}(S_T)$ and $\gamma(S_T)=\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)\ \gamma(S_T)=\alpha+\sum_{i=1}^{n}\beta_i\varphi_i(S_T)$. Then $\ln g(S_T)=\frac{\ln y}{1-q}$ and we can write the above equation as $1+\frac{q}{1-q}\ln y=-\gamma(S_T)y$. We put $\frac{\gamma(S_T)(1-q)}{q}y=z$, i.e. $y=\frac{qz}{\gamma(S_T)(1-q)}$ and the last equation is equivalent to $ze^z=\frac{\gamma(S_T)(1-q)}{q}e^{-\left(\frac{1-q}{q}\right)}$. Using Lemma 2.1 we obtain $z=W\left(\frac{\gamma(S_T)(1-q)}{q}e^{-\left(\frac{1-q}{q}\right)}\right)$ and therefore

$$\frac{\gamma(S_T)(1-q)}{q} g^{1-q}(S_T) = W\left(\frac{\gamma(S_T)(1-q)}{q} e^{-\left(\frac{1-q}{q}\right)} \right).$$

Now we have obtained the form of $g(S_T)$ and the proof is complete.

4. WEIGHTED SHAFEE ENTROPY AND EU-WE FRAMEWORK

In this section we consider the Weighted-Shafee entropy and the case of Expected Utility-Weighted Shafee Entropy (EU-WSE) framework.

THEOREM 3.2. *Let the weighted entropy* $u(S_T) > 0, S_T > 0$. *We consider the entropy maximization problem*

$$\begin{aligned} &\max -\mathrm{E}^g\left[u(Y_T) g^{q-1}(Y_T) \ln g(Y_T)\right] \\ \text{subject to} \quad &\mathrm{E}^g\left[I_{\{Y_T>0\}}\right] = 1, \\ &\mathrm{E}^g\left[u(Y_T)\varphi_i(Y_T)\right] = a_i,\ i = \overline{1,n}, \end{aligned}$$

where $\varphi_i : \mathbf{R} \to \mathbf{R}$ *and* $a_1, \dots, a_n$ *are given real values. Then the solution of the problem is*

$$g(S_T) = \left[\frac{qu(S_T) W\left(\frac{\left(\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)\right)(1-q)}{qu(S_T)} \exp\left\{ -\left(\frac{1-q}{q} \right) \right\} \right)}{\left(\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)\right)(1-q)} \right]^{\frac{1}{1-q}},$$

where $\alpha, \beta_1, \dots, \beta_n$ *are the Lagrange multipliers and* W *is the Lambert function.*

Proof. The proof is similar to Theorem 3.1.

THEOREM 3.3. *Consider the case of EU-WSE under the constraints given in Theorem 3.2. Then the solution of risk-neutral density is given by:*

$$g(S_T) = \left[\frac{q}{1-q} \left(\frac{u(S_T)}{u(S_T) + \gamma(S_T)} \right) W\left(\frac{q-1}{q} \left(1 + \frac{\gamma(S_T)}{u(S_T)} \right) e^{\frac{q-1}{q}} \right) \right]^{\frac{1}{1-q}}.$$

Proof. In the framework of EU-WSE and using Theorem 3.2 we can write that g has to satisfy the following relation:

$$u(S_T) - u(S_T) g^{q-1}(S_T)\left(q \ln g(S_T) + 1\right) = -\left(\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T) \right).$$

Now we get

$$g^{q-1}(S_T)\left(q\ln g(S_T)+1\right)=1+\frac{\left(\alpha+\sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)\right)}{u(S_T)}.$$

Therefore

$$q\ln g(S_T)+1=\left(1+\frac{\left(\alpha+\sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)\right)}{u(S_T)}\right)g^{q-1}(S_T).$$

Let us suppose $y=g^{q-1}(S_T)$ and $\gamma(S_T)=\alpha+\sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)$. Since $\ln g(S_T)=1+\frac{\ln y}{1-q}$, we can write the above equation as $\frac{q}{1-q}\ln y+1=\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)y$, i.e. $\ln y+\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)y=\frac{q-1}{q}$, $\ln\left(y\cdot\exp\left\{\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)y\right\}\right)=\ln e^{\frac{q-1}{q}}$. We put $\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)y=z$. Then we can write:

$$y\cdot\exp\left\{\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)y\right\}=e^{\frac{q-1}{q}},\ \text{i.e.}\ ze^{z}=\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)e^{\frac{q-1}{q}}.$$

This is again a Lambert type equation and by using Lemma 2.1 we obtain:

$$z=W\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)e^{\frac{q-1}{q}}.$$

Now, using the value of z, we obtain $\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)y=W\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)\right)e^{\frac{q-1}{q}}$.

Since $y=\frac{qu(S_T)z}{(q-1)\left(u(S_T)+\gamma(S_T)\right)}$, therefore we obtain the solution of the optimization problem:

$$g(S_T)=\left[\frac{q}{1-q}\left(\frac{u(S_T)}{u(S_T)+\gamma(S_T)}\right)W\left(\frac{q-1}{q}\left(1+\frac{\gamma(S_T)}{u(S_T)}\right)e^{\frac{q-1}{q}}\right)\right]^{\frac{1}{1-q}}.$$

5. SECOND APPROACH

Now, considering the problem of Theorem 3.1, we present a new approach of solution by using the following Lemmas.

LEMMA 3.2. *If* $\psi=x^q\ln x$, $\psi:(0,\infty)\to(0,\infty)$, $q>0$, $q\neq 1$, *then* ψ', *the first derivative of* ψ, *admits an inverse and we consider*

$$(i)\quad \psi' : \left(0,\, e^{\frac{1-2q}{q(q-1)}}\right) \to \left(0,\, \frac{-qe^{\frac{1-2q}{q}}}{q-1}\right);$$

$$(ii)\quad \psi' : \left(0, e^{\frac{1-2q}{q(q-1)}}\right) \to \left(\frac{-qe^{\frac{1-2q}{q}}}{q-1},\, \infty\right).$$

Proof. The proof is immediate as we see ψ' is strictly increasing.

LEMMA 3.3. *If* $\psi = x^q \ln x, \psi : (0,\infty) \to (0,\infty),\ q \in (0,1),$ *then* ψ *is convex on* $\left(0,\, e^{\frac{1-2q}{q(q-1)}}\right)$ *and concave on* $\left(e^{\frac{1-2q}{q(q-1)}},\, \infty\right)$*. Also, for* $q > 1$*,* ψ *is convex.*

Proof. The proof of Lemma 3.3 follows Lemma 3.2.

THEOREM 3.4. *Consider the entropy maximization problem:*

$$\max -\mathrm{E}^g\left[g^{q-1}(Y_T)\ln g(Y_T)\right],$$

with same constraints as in Theorem 3.1. Then we have the solution of risk-neutral density as:

$$g(S_T) = (\psi')^{-1}\left(\alpha + \sum_{i=1}^{n} \beta_i \varphi_i(S_T)\right).$$

Proof. We can write the Lagrangian L, and using calculus of variations for optimization of functional (see for example Luenberger, 1969; Cover and Thomas, 1991; Borwein, 2003) we get

$$L = \mathrm{E}^g\left[g^{q\text{-}1}(Y_T)\ln g(Y_T)\right] - \alpha\left(\mathrm{E}^g\left[I_{\{Y_T>0\}}\right]-1\right) - \sum_{i=1}^{n}\beta_i\left(\mathrm{E}^g\left[\varphi_i(Y_T)\right]-a_i\right).$$

Therefore g has to satisfy the following relation

$$(qg^q(S_T)\ln g(S_T))' - \alpha - \sum_{i=1}^{n}\beta_i\varphi_i(S_T) = 0,$$

Now we suppose $g(S_T) = x$, therefore $g^q(S_T)\ln g(S_T) = x^q \ln x$. Using Lemma 3.1we can write the solution of risk-neutral density $\mathrm{g}(S_T)$ as: $\psi'\left(g(S_T)\right) = \alpha + \sum_{i=1}^{n}\beta_i\varphi_i(S_T)$, i.e.

$$g(S_T) = (\psi')^{-1}\left(\alpha + \sum_{i=1}^{n} \beta_i \varphi_i(S_T)\right).$$

THEOREM 3.5. *For the case of weighted entropy we can write risk-neutral density for Theorem 3.4 as:*

$$g(S_T) = (\psi')^{-1}\left(\frac{\alpha + \sum_{i=1}^{n}\beta_i u(S_T)\varphi_i(S_T)}{u(S_T)}\right).$$

THEOREM 3.6. *Using the framework of EU-WSE we have solution for the risk-neutral density for the problem of theorem 3.4 as follows:*

$$g(S_T) = (\psi')^{-1}\left(1 - \frac{\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)}{u(S_T)}\right).$$

Proof. Following previous Theorem 3.1 we write:

$$L = \mathrm{E}^g\left[u(Y_T)\, g^{q-1}(Y_T) \ln g(Y_T)\right] - \alpha\left(\mathrm{E}^g\left[I_{\{Y_T>0\}}\right] - 1\right) - \sum_{i=1}^{n} \beta_i\left(\mathrm{E}^g\left[u(Y_T)\varphi_i(Y_T)\right] - a_i\right).$$

Using the EU-WSE framework, g has to satisfy the following relation:

$$u(S_T) - u(S_T)(g^q(S_T)\ln g(S_T))' - \alpha - \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T) = 0,$$

which is equivalent to:

$$1 - (g^q(S_T)\ln g(S_T))' = \frac{\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)}{u(S_T)}.$$

Thus we get

$$(g^q(S_T)\ln g(S_T))' = 1 - \frac{\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)}{u(S_T)}.$$

For $g(S_T) = x$ and we have $g^q(S_T)\ln g(S_T) = x^q \ln x$. Now using the Lemma 3.1 we may write the solution of risk-neutral density $\mathrm{g}(S_T)$ as:

$$g(S_T) = (\psi')^{-1}\left(1 - \frac{\alpha + \sum_{i=1}^{n} \beta_i u(S_T)\varphi_i(S_T)}{u(S_T)}\right).$$

6. CONCLUSIONS

In this article we have presented two different approaches to obtain risk-neutral densities using entropy pricing theory of stock options. The problem of extracting implied volatilities from market price of the options has always attained the concentration of researchers in option pricing but this is single statistic which can be extracted and depends on the option pricing model. The problem of getting risk-neutral density implies a comprehensive package without specifying any model has become crucial and entropy pricing theory is an alternative structure to solve such problems.

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