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NEW (3+1)-DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS WITH BURGERS AND SHARMA-TASSO-OLVER EQUATIONS CONSTITUTING THE MAIN PARTS

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In this work, we introduce three new (3+1)-dimensional nonlinear evolution equations. The Burgers equation constitutes the main part for the first two equations, and the Sharma-Tasso-Olver equation constitutes the main part for the third equation. We determine multiple front wave solutions for each model by using the simplified Hirota's method and the Cole–Hopf transformation method.

Key words: nonlinear (3+1)-dimensional equation, Burgers equation, Sharma-Tasso-Olver equation.

1. INTRODUCTION

The Burgers equation [1] that occurs in modelling of gas dynamics and traffic flow, is given by

$$w_t + \beta w w_x + w_{xx} = 0, \tag{1}$$

with recursion operator Φ given by

$$\Phi(w) = \partial_x + w + w_x \partial_x^{-1}, \tag{2}$$

where ∂_x denotes the total derivative with respect to x, and ∂_x^{-1} is its integration operator.

It is well known that the Burgers equation is the lowest order approximation for the one-dimensional propagation of weak shock waves in a fluid [1]. It is one of the fundamental model equations in fluid mechanics [1]. The Burgers equation demonstrates the coupling between dissipation effect of w_{xx} and the convection process of ww_x. Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion [1–5]. It is also used to describe the structure of shock waves, traffic flow, and acoustic transmission. Burgers equation is completely integrable. The wave solutions of Burgers equation are single and multiple-front solutions.

The double nonlinear dispersive Sharma-Tasso-Olver (STO) equation reads [2–5]

$$w_t + \alpha(w^3)_x + \frac{3}{2}\alpha(w^2)_{xx} + \alpha w_{xxx} = 0, \tag{3}$$

where α is a real parameter and $w(x,t)$ is the unknown function that depends on the temporal variable t and the spatial variable x. The STO equation contains both linear dispersive term u_{xxx} and the double nonlinear terms $(u^3)_x$ and $(u^2)_{xx}$. Olver [4] showed that this equation is one of evolution equations that possesses infinitely many symmetries. This equation is well known as a model equation describing the propagation of nonlinear dispersive waves in inhomogeneous media [2–5]. The STO equation (3) attracted a considerable size of research work by mathematicians and physicists due to its appearance in scientific applications. The STO equation was first derived as an example of odd members of Burgers hierarchy by Tasso [2]. It also appears as an evolution equation possessing infinitely many symmetries [4]. In [2], Tasso proved that this equation possesses the bi-Hamiltonian formulation and the generalized Poisson bracket.

A variety of powerful methods such as the simple symmetry reduction procedure, Bäcklund transformation method, Darboux transformation, Painlevé integrability, etc, were used to achieve more insights through the structures of the solutions [6–22]. The simple symmetry reduction procedure is repeatedly used in [3] to obtain infinitely many symmetries and exact solutions. Wang *et al*. [11] examined the soliton fission and fusion thoroughly by using the standard truncated Painlevé analysis, the Hirota's bilinear method, and the Bäcklund transformation method. Moreover, the STO Eq. (3) was treated analytically by using the tanh method, the extended tanh method, and solitary wave ansatze involving hyperbolic and exponential functions [7–19].

The studies of completely integrable equations are flourishing by researchers because these equations describe the real features in a variety of science, technology, and engineering fields. Mathematicians have been investing their works to develop and apply new methods for solving these integrable equations, whereas physicists usually look for the dynamical behavior of the physical systems. Many powerful methods were developed and used. The algebraic-geometric method [7–8], the inverse scattering method, the Bäcklund transformation method, the Darboux transformation method, the Hirota bilinear method [12], and others were thoroughly used to derive the multiple-soliton solutions of these equations. In addition, a basic property of integrable equations is that they possess infinitely many conservation laws. The existence of infinitely many conservation laws for soliton equations further confirms their integrability.

The (3+1)-dimensional nonlinear evolution equation

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x\partial_x^{-1}w_y)_x = 0, \tag{4}$$

was studied in [6–9], to establish *N*-soliton solutions, where the existence of multi-soliton solutions often indicates its integrability. However, other methods, such as the Lax pair, the Darboux transformation, Painlevé property, and other methods are essential to confirm the integrability of any equation.

The inverse operator ∂_x^{-1} is defined by

$$(\partial_x^{-1} f)(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t, \tag{5}$$

under the decaying condition at infinity. Note that $\partial_x\partial_x^{-1} = \partial_x^{-1}\partial_x = 1$.

The (3+1)-dimensional integrable equation (4) was first introduced in [7–8] in the study of the algebraic-geometrical solutions. We can easily show that the relationship of the (3+1)-dimensional equation (4) and the Korteweg-de Vries (KdV) equation is very strong [6]. This is due to the fact that the (3+1)-dimensional equation (4) possesses the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{6}$$

as its main term $2w_t + w_{xxx} - 2ww_x$ under the transformations

$$\begin{aligned} w(x,t) &\to u(x',t'),\\ x' &\to \frac{1}{\sqrt{3}}x,\\ t' &\to \frac{1}{6\sqrt{3}}t. \end{aligned} \tag{7}$$

Based on this, the (3+1)-dimensional equation (4) may be used to study shallow-water waves and short waves in nonlinear dispersive media.

In this work, we will establish three new (3+1)-dimensional equations where the main part of (4), located in the middle term will be replaced by the Burgers equation for the first two models, and by the STO equation for the last one. Consequently, the three new (3+1)-dimensional equations are given by

$$3w_{xz} - (w_t + \beta ww_x + w_{xx})_y + \gamma(w_x\partial_x^{-1}w_y)_x = 0, \tag{8}$$

$$3w_{xz} - (w_t + \beta ww_x + w_{xx})_y + \gamma(w_x\partial_x^{-1}w_y)_x + w_{yz} = 0, \tag{9}$$

and

$$\delta w_{xz} - (w_t + \alpha(w^3)_x + \frac{3}{2}\alpha(w^2)_{xx} + \alpha w_{xxx})_y + w_{yz} = 0, \tag{10}$$

Notice that Eq. (9) contains an additional term, namely, w_{yz} when compared to Eq. (8).

Our aim is to study the aforementioned established equations (8)–(10). We also plan to derive multiple front wave solutions for each new model. The simplified form of the Hirota's method, that is well known now and can be found in [12–22], will be used to achieve this goal. The Cole-Hopf transformation method will also be used to handle Eq. (10).

2. MODEL I: BURGERS EQUATION CONSTITUTES THE MAIN PART

In this section we will study the (3+1)-dimensional nonlinear equation

$$3w_{xz} - (w_t + \beta w w_x + w_{xx})_y + \gamma(w_x \partial_x^{-1} w_y)_x = 0. \tag{11}$$

We first remove the integral term in (11) by introducing the potential

$$w(x,y,z,t) = u_x(x,y,z,t), \tag{12}$$

to carry (11) to the equation

$$3u_{xxz} - (u_{xt} + \beta u_x u_{xx} + u_{xxx})_y + \gamma(u_{xx} u_y)_x = 0. \tag{13}$$

Substituting

$$u(x,y,z,t) = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z - \omega_i t, \tag{14}$$

into the linear terms of (13), and solving the resulting equation for ω_i we obtain the dispersion relation

$$\omega_i = k_i^2 - \frac{3k_i s_i}{r_i}, \quad i = 1, 2, \cdots N, \tag{15}$$

and hence θ_i becomes

$$\theta_i = k_i x + r_i y + s_i z - \left(k_i^2 - \frac{3k_i s_i}{r_i}\right)t. \tag{16}$$

To determine R, we substitute

$$u(x,y,z,t) = R\big(\ln f(x,y,z,t)\big), \tag{17}$$

where

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + s_1 z - \left(k_1^2 - \frac{3k_1 s_1}{r_1}\right)t}, \tag{18}$$

into Eq. (13) and solve to find that

$$R = \frac{2}{\beta - \gamma}. \tag{19}$$

This gives a constraint condition that, for front wave solutions to exist, then

$$\beta \neq \gamma \tag{20}$$

This in turn gives

$$u(x,y,z,t) = \frac{2}{\beta - \gamma} \ln\left(1 + e^{k_1 x + r_1 y + s_1 z - \left(k_1^2 - \frac{3k_1 s_1}{r_1}\right)t}\right). \tag{21}$$

Consequently, the single front wave solution for the (3+1)-dimensional nonlinear evolution equation (11) is given by

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{k_1\mathrm{e}^{k_1x+r_1y+s_1z-\left(k_1^2-\frac{3k_1s_1}{r_1}\right)t}}{1+\mathrm{e}^{k_1x+r_1y+s_1z-\left(k_1^2-\frac{3k_1s_1}{r_1}\right)t}}\right),\tag{22}$$

obtained upon using the potential defined in (12).

We noticed that the single front wave solution exists where the parameters k_1, r_1, and s_1 are left as free parameters. However, this is not the case for the two and three front wave solutions, where

$$r_i=k_i,\quad i=1,2,\cdots,N,\tag{23}$$

is a necessary condition in order the multiple front wave solutions to exist.

To determine the two front wave solutions, we use the auxiliary function $f(x,y,z,t)$ in the form

$$f(x,y,z,t)=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2},\tag{24}$$

where the wave variable $\theta_i,\quad i\geq 1$ in this case is given by

$$\theta_i=k_ix+k_iy+s_iz-\left(k_i^2-3s_i\right)t,\quad i=1,2,\cdots,N,\tag{25}$$

and consequently, the auxiliary function becomes

$$f(x,y,z,t)=1+\mathrm{e}^{k_1x+k_1y+s_1z-(k_1^2-3s_1)t}+\mathrm{e}^{k_2x+k_2y+s_2z-(k_2^2-3s_2)t}.\tag{26}$$

This in turn gives

$$u(x,y,z,t)=\frac{2}{\beta-\gamma}\ln\left(1+\mathrm{e}^{k_1x+k_1y+s_1z-(k_1^2-3s_1)t}+\mathrm{e}^{k_2x+k_2y+s_2z-(k_2^2-3s_2)t}\right).\tag{27}$$

Consequently, the two front wave solutions for the (3+1)-dimensional nonlinear evolution equation (11) is given by

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{k_1\mathrm{e}^{k_1x+k_1y+s_1z-\left(k_1^2-3s_1\right)t}+k_2\mathrm{e}^{k_2x+k_2y+s_2z-\left(k_2^2-3s_2\right)t}}{1+\mathrm{e}^{k_1x+k_1y+s_1z-(k_1^2-3s_1)t}+\mathrm{e}^{k_2x+k_2y+s_2z-(k_2^2-3s_2)t}}\right),\tag{28}$$

obtained upon using the potential defined in (12).

To determine the three front wave solutions, we use the auxiliary function

$$f(x,y,z,t)=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\mathrm{e}^{\theta_3},\tag{29}$$

and proceed as before. Thus we obtain the three front wave solutions

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{\sum_{i=1}^{3}k_i\mathrm{e}^{k_ix+k_iy+s_iz-\left(k_i^2-3s_i\right)t}}{1+\sum_{i=1}^{3}\mathrm{e}^{k_ix+k_iy+s_iz-\left(k_i^2-3s_i\right)t}}\right).\tag{30}$$

For N-front wave solutions, where N is finite, we set the generalized front wave solutions in the form

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{\sum_{i=1}^{N}k_i\mathrm{e}^{k_ix+k_iy+s_iz-\left(k_i^2-3s_i\right)t}}{1+\sum_{i=1}^{N}\mathrm{e}^{k_ix+k_iy+s_iz-\left(k_i^2-3s_i\right)t}}\right).\tag{31}$$

For comparison reasons, it was found in [21] that the dispersion relation for the standard Burgers equation (1) is given by

$$\omega_i = k_i^2, \tag{32}$$

and the coefficient of the Cole-Hopf transformation was found as

$$R = \frac{2}{\beta}, \quad \beta \neq 0. \tag{33}$$

Moreover, in [21] it was found that the generalized formula for the N multiple front wave solutions is given by

$$w(x,t) = \frac{2}{\beta}\left(\frac{\sum_{i=1}^{N} k_i \mathrm{e}^{k_i x - k_i^2 t}}{1 + \sum_{i=1}^{N} \mathrm{e}^{k_i x - k_i^2 t}}\right). \tag{34}$$

Comparing (31) with (34), we observe that the multiple front wave solutions for the (3+1)-dimensional equation (11) exists provided that $\beta \neq \gamma$. However, for (1) such a constraint does not exist, where $\beta \neq 0$ is a necessary case for the nonlinearity of the Burgers equation.

3. MODEL II: BURGERS EQUATION AS A MAIN PART AND ADDITIONAL TERM

In this section we will study the (3+1)-dimensional nonlinear equation

$$3w_{xz} - (w_t + \beta w w_x + w_{xx})_y + \gamma(w_x \partial_x^{-1} w_y)_x + w_{yz} = 0, \tag{35}$$

where an additional term, namely w_{yz} is added to the first model (11).

To remove the integral term in (35) we use the potential

$$w(x,y,z,t) = u_x(x,y,z,t), \tag{36}$$

that will carry (35) to the equation

$$3u_{xxz} - (u_{xt} + \beta u_x u_{xx} + u_{xxx})_y + \gamma(u_{xx} u_y)_x + u_{xyz} = 0. \tag{37}$$

Substituting

$$u(x,y,z,t) = \mathrm{e}^{\theta_i}, \quad \theta_i = k_i x + r_i y + s_i z - \omega_i t, \tag{38}$$

into the linear terms of (37), and solving the resulting equation for ω_i we obtain the dispersion relation

$$\omega_i = k_i^2 - \frac{3k_i s_i}{r_i} - s_i, \quad i = 1,2,\cdots N, \tag{39}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x + r_i y + s_i z - \left(k_i^2 - \frac{3k_i s_i}{r_i} - s_i\right)t. \tag{40}$$

To determine R, we use as before

$$u(x,y,z,t) = R(\ln f(x,y,z,t)), \tag{41}$$

where

$$f(x,y,z,t) = 1 + \mathrm{e}^{k_1 x + r_1 y + s_1 z - \left(k_1^2 - \frac{3k_1 s_1}{r_1} - s_1\right)t}, \tag{42}$$

and from Eq. (37) we find that

$$R=\frac{2}{\beta-\gamma}. \tag{43}$$

This gives, as in the first model, that for front wave solutions to exist, then

$$\beta\neq\gamma. \tag{44}$$

This in turn gives

$$u(x,y,z,t)=\frac{2}{\beta-\gamma}\ln\left(1+e^{k_1x+r_1y+s_1z-\left(k_1^2-\frac{3k_1s_1}{r_1}-s_1\right)t}\right). \tag{45}$$

Consequently, the single front wave solution for the (3+1)-dimensional nonlinear evolution equation (35) is given by

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{k_1e^{k_1x+r_1y+s_1z-\left(k_1^2-\frac{3k_1s_1}{r_1}-s_1\right)t}}{1+e^{k_1x+r_1y+s_1z-\left(k_1^2-\frac{3k_1s_1}{r_1}-s_1\right)t}}\right), \tag{46}$$

obtained upon using the potential defined in (36).

We noticed that the single front wave solution exists where the parameters k_1, r_1, and s_1 are left as free parameters. However, this is not the case for the multiple front wave solutions, where

$$r_i=k_i,\quad i=1,2,\cdots,N, \tag{47}$$

should be used in order the multiple front wave solutions to exist.

To determine the two front wave solutions, we use the auxiliary function $f(x,y,z,t)$ in the form

$$f(x,y,z,t)=1+e^{\theta_1}+e^{\theta_2}, \tag{48}$$

where the wave variable $\theta_i,\quad i\geq 1$ in this case is given by

$$\theta_i=k_ix+k_iy+s_iz-\left(k_i^2-4s_i\right)t,\quad i=1,2,\cdots,N, \tag{49}$$

and thus the auxiliary function becomes

$$f(x,y,z,t)=1+e^{k_1x+k_1y+s_1z-(k_1^2-4s_1)t}+e^{k_2x+k_2y+s_2z-(k_2^2-4s_2)t}. \tag{50}$$

This in turn gives

$$u(x,y,z,t)=\frac{2}{\beta-\gamma}\ln\left(1+e^{k_1x+k_1y+s_1z-(k_1^2-4s_1)t}+e^{k_2x+k_2y+s_2z-(k_2^2-4s_2)t}\right). \tag{51}$$

Consequently, the two front wave solutions for the (3+1)-dimensional nonlinear evolution equation (35) is given by

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{k_1e^{k_1x+k_1y+s_1z-\left(k_1^2-4s_1\right)t}+k_2e^{k_2x+k_2y+s_2z-\left(k_2^2-4s_2\right)t}}{1+e^{k_1x+k_1y+s_1z-(k_1^2-4s_1)t}+e^{k_2x+k_2y+s_2z-(k_2^2-4s_2)t}}\right), \tag{52}$$

obtained upon using the potential defined in (36).

To determine the three front wave solutions, we use the auxiliary function

$$f(x,y,z,t)=1+e^{\theta_1}+e^{\theta_2}+e\theta_3, \tag{53}$$

and proceed as before. Thus we obtain the three front wave solutions

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{\sum_{i=1}^{3}k_i e^{k_i x+k_i y+s_i z-\left(k_i^2-4s_i\right)t}}{1+\sum_{i=1}^{3}e^{k_i x+k_i y+s_i z-\left(k_i^2-4s_i\right)t}}\right). \tag{54}$$

For N-front wave solutions, where N is finite, we set the generalized front wave solutions in the form

$$w(x,y,z,t)=\frac{2}{\beta-\gamma}\left(\frac{\sum_{i=1}^{N}k_i e^{k_i x+k_i y+s_i z-\left(k_i^2-4s_i\right)t}}{1+\sum_{i=1}^{N}e^{k_i x+k_i y+s_i z-\left(k_i^2-4s_i\right)t}}\right). \tag{55}$$

4. MODEL III: THE STO EQUATION CONSTITUTES THE MAIN PART

In this section we will study the (3+1)-dimensional nonlinear equation

$$\delta w_{xz}-(w_t+\alpha(w^3)_x+\frac{3}{2}\alpha(w^2)_{xx}+\alpha w_{xxx})_y+w_{yz}=0, \tag{56}$$

where the Sharma-Tasso-Olver equation constitutes its main part in the middle term, and δ, unlike the previous two models, is any real number.

We first substitute

$$w(x,y,z,t)=e^{\theta_i},\quad \theta_i=k_i x+r_i y+s_i z-\omega_i t, \tag{57}$$

into the linear terms of (56), and solving the resulting equation for ω_i we obtain the dispersion relation

$$\omega_i=\alpha k_i^3-\frac{\delta k_i s_i}{r_i}-s_i,\quad i=1,2,\cdots N, \tag{58}$$

and hence the wave variable θ_i becomes

$$\theta_i=k_i x+r_i y+s_i z-\left(\alpha k_i^3-\frac{\delta k_i s_i}{r_i}-s_i\right)t. \tag{59}$$

To determine R, we use the Cole-Hopf transformation

$$w(x,y,z,t)=R\big(\ln f(x,y,z,t)\big)_x, \tag{60}$$

where

$$f(x,y,z,t)=1+e^{k_1 x+r_1 y+s_1 z-\left(\alpha k_1^3-\frac{\delta k_1 s_1}{r_1}-s_1\right)t}, \tag{61}$$

into Eq. (56) and solve to find that $R=1$. It is obvious that R does not depend on the parameters α and δ.

Using the previous results, the single front wave solution for the (3+1)-dimensional nonlinear evolution equation (56) is given by

$$w(x,y,z,t)=\frac{k_1 e^{k_1 x+r_1 y+s_1 z-\left(\alpha k_1^3-\frac{\delta k_1 s_1}{r_1}-s_1\right)t}}{1+e^{k_1 x+r_1 y+s_1 z-\left(\alpha k_1^3-\frac{\delta k_1 s_1}{r_1}-s_1\right)t}} \tag{62}$$

We noticed that the single front wave solution exists where the parameters k_1, r_1, and s_1 are left as free parameters. However, this is not the case for the two and three front wave solutions, where

$$r_i = k_i, \quad i = 1, 2, \cdots, N, \tag{63}$$

is a necessary condition in order the multiple front wave solutions to exist.

To determine the two front wave solutions, we use the auxiliary function $f(x, y, z, t)$ in the form

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2}, \tag{64}$$

where the wave variable $\theta_i, \quad i \ge 1$ in this case is given by

$$\theta_i = k_i x + k_i y + s_i z - \left(\alpha k_i^3 - (\delta+1)s_i\right)t, \quad i = 1, 2, \cdots, N, \tag{65}$$

and consequently, the auxiliary function becomes

$$f(x, y, z, t) = 1 + e^{k_1 x + k_1 y + s_1 z - (\alpha k_1^3 - (\delta+1)s_1)t} + e^{k_2 x + k_2 y + s_2 z - (\alpha k_2^3 - (\delta+1)s_2)t}. \tag{66}$$

This in turn gives the two front wave solution for the (3+1)-dimensional nonlinear evolution equation (56) by

$$w(x, y, z, t) = \frac{k_1 e^{k_1 x + k_1 y + s_1 z - (\alpha k_1^3 - (\delta+1)s_1)t} + k_2 e^{k_2 x + k_2 y + s_2 z - (\alpha k_2^3 - (\delta+1)s_2)t}}{1 + e^{k_1 x + k_1 y + s_1 z - (\alpha k_1^3 - (\delta+1)s_1)t} + e^{k_2 x + k_2 y + s_2 z - (\alpha k_2^3 - (\delta+1)s_2)t}}. \tag{67}$$

To determine the three front wave solutions, we use the auxiliary function

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{68}$$

and proceed as before. Thus we obtain the three front wave solutions

$$w(x, y, z, t) = \frac{\sum_{i=1}^{3} k_i e^{k_i x + k_i y + s_i z - \left(\alpha k_i^3 - (\delta+1)\right)t}}{1 + \sum_{i=1}^{3} e^{k_i x + k_i y + s_i z - \left(\alpha k_i^3 - (\delta+1)\right)t}}. \tag{69}$$

For N-front wave solutions, where N is finite, we set the generalized front wave solutions in the form

$$w(x, y, z, t) = \frac{\sum_{i=1}^{N} k_i e^{k_i x + k_i y + s_i z - \left(\alpha k_i^3 - (\delta+1)\right)t}}{1 + \sum_{i=1}^{3} e^{k_i x + k_i y + s_i z - \left(\alpha k_i^3 - (\delta+1)\right)t}}. \tag{70}$$

For comparison reasons, it was found in [22] that the dispersion relation for the standard STO equation (3) is given by

$$\omega_i = \alpha k_i^3, \tag{71}$$

and the coefficient of the Cole-Hopf transformation was found as $R = 1$.

Moreover, in [22] it was found that the generalized formula for the N multiple front wave solutions is given by

$$w(x, t) = \left(\frac{\sum_{i=1}^{N} k_i e^{k_i x - \alpha k_i^3 t}}{1 + \sum_{i=1}^{N} e^{k_i x - \alpha k_i^3 t}} \right). \tag{72}$$

Comparing (70) with (72), the dispersion relations of (3) and the nonlinear equation (56) are distinct, and this is normal due to the presence of additional variables y and z.

4. DISCUSSION

In the standard (3+1)-dimensional nonlinear evolution equation developed in [7], the integrable Korteweg-de Vries equation constitutes its main part. We used this fact to build up three new (3+1)-dimensional equations, where we used the integrable Burgers equation to constitute the main part for each of the first two equations, whereas we substituted the Sharma-Tasso-Olver equation to replace the Korteweg-de Vries equation and to constitute its main part. Multiple front wave solutions were formally derived by using the simplified Hirota's method and the Cole-Hopf transformation method. A comparison was conducted on the derived solutions for the newly established equations and the standard Burgers and Sharma-Tasso-Olver equations.

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