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PERIODICITY PROPERTY WITH APPLICATIONS TO THE MODELING OF INTERACTIONS BETWEEN VERY INTENSE LASER BEAMS AND ELECTRONS OR ATOMS

Alexandru POPA

National Institute for Laser, Plasma and Radiation Physics, Laser Department, P.O. Box MG-36, RO-077125, Bucharest, Romania
E-mail: alexandru.popa@inflpr.ro

In previous papers we proved a general property of the relativistic interactions between very intense laser beams and (1) electron plasmas, (2) relativistic electron beams or (3) atoms. We proved that all the physical quantities, which are involved in the generation of harmonics, in the interactions (1), (2), (3), are periodic functions of only one variable, which is the phase of the incident electromagnetic field. The property is valid also for the physical quantities that describe the radiation damping effect, using the Landau and Lifschitz model, for the above interactions. It follows that all the expressions that describe the angular and spectral distributions of the radiations generated by interactions (1) and (2) and the harmonic spectrum resulted from interaction (3) are composite functions of only one variable. This property simplifies strongly the modeling of the interactions (1), (2), (3) and leads to a very good agreement with the experimental data from literature. We present now a short review of these results.

Key words: very intense laser beams, electron plasmas, relativistic electron beams, atoms, relativistic interactions.

1. INTRODUCTION

This paper belongs to a series of results establishing accurate models of relativistic interactions between very intense laser beams and electron plasmas, relativistic electron beams and atoms. We start by recalling briefly these results.

In a series of papers [1–5] we presented models of interactions between very intense laser beams and electron plasmas, relativistic electron beams and atoms. We proved that for a system composed of an electron interacting with an elliptic polarized electromagnetic field in the general case, when the initial velocity of the electron is arbitrary, the solution of the relativistic motion equation of the electron leads to expressions of velocity and acceleration of the electron that are periodic functions of the phase of the incident field. Introducing these functions in the Liènard-Wiechert equation, we find that the intensity of the electrical field and the intensity of the beam of the radiation resulted from this interaction are also functions of the phase of the incident field. In the case of the interaction between very intense laser field and relativistic electrons we proved that a similar analysis, made in the inertial system in which the initial velocity of the electron is zero, taking into account that the phase of the field is a relativistic invariant, leads to the same periodicity property. We proved also that the intensities of the harmonics generated by the interaction between very intense laser field and atoms, which is analyzed in the frame of the well-known three sequence mechanism [6], are periodic functions of the phase of the incident field. In a recent paper [5] we proved that radiation damping parameters in the interaction between very intense laser beams and electrons obey the same property.

This periodicity property has a practical importance because it leads to a calculation method of the angular and spectral distributions of the intensities of the radiations generated by above interactions. This method uses composite functions of only one variable, which can be easily calculated using simple software. The analysis is made in the International System of Units.

2. SOLUTIONS OF RELATIVISTIC MOTION EQUATIONS OF ELECTRON, AS FUNCTIONS OF THE FIELD PHASE

We consider a system composed of an electron interacting with an elliptic polarized electromagnetic field in the general case, when the components of the field are $\overline{E}_L = E_{M1}\cos\eta\overline{i} + E_{M2}\sin\eta\overline{j}$ and $\overline{B}_L = -B_{M2}\sin\eta\overline{i} + B_{M1}\cos\eta\overline{j}$, with $\eta = \omega_L t - \left|\overline{k}_L\right| z + \eta_i$ and $\left|\overline{k}_L\right| c = \omega_L$, where $\overline{i}$, $\overline{j}$, and $\overline{k}$ are versors of the *ox*, *oy* and *oz* axes, E_{M1}, E_{M2}, B_{M1}, and B_{M2} are the amplitudes of the oscillations of the electromagnetic field components, ω_L and $\overline{k}_L$ are, respectively, the angular frequency and wave vector of the electromagnetic field, c is the light velocity and η_i is an arbitrary initial phase. The following relations are also valid: $E_{M1} = cB_{M1}$, $E_{M2} = cB_{M2}$ and $c\overline{B}_L = \overline{k} \times \overline{E}_L$. We consider the following initial conditions in the most general case, when the components of the electron velocities have arbitrary values:

$$t = 0,\ x = y = z = 0,\ v_x = v_{xi},\ v_y = v_{yi},\ v_z = v_{zi} \text{ and } \eta = \eta_i \tag{1}$$

The system of relativistic equations of electron motion can be written in the following form [1]:

$$\mathrm{d}\gamma\beta_x/\mathrm{d}t = -a_1\omega_L\left(1-\beta_z\right)\cos\eta \tag{2}$$

$$\mathrm{d}\gamma\beta_y/\mathrm{d}t = -a_2\omega_L\left(1-\beta_z\right)\sin\eta \tag{3}$$

$$\mathrm{d}\gamma\beta_z/\mathrm{d}t = -\omega_L\left(a_1\beta_x\cos\eta + a_2\beta_y\sin\eta\right), \tag{4}$$

where

$$a_1 = eE_{M1}/\left(mc\omega_L\right),\ \ a_2 = eE_{M2}/\left(mc\omega_L\right), \tag{5}$$

$$\gamma = \left(1-\beta_x^2-\beta_y^2-\beta_z^2\right)^{-\frac{1}{2}}, \tag{6}$$

and $\beta_x = v_x/c$, $\beta_y = v_y/c$, $\beta_z = v_z/c$.

We multiply (2), (3), and (4), respectively, by β_x, β_y, and β_z. Taking into account Eq. (6), their sum leads to

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\omega_L\left(a_1\beta_x\cos\eta + a_2\beta_y\sin\eta\right). \tag{7}$$

From (4) and (7) we obtain $\mathrm{d}\left(\gamma\beta_z\right)/\mathrm{d}t = \mathrm{d}\gamma/\mathrm{d}t$. We integrate this relation with respect to time between 0 and t, taking into account the initial conditions (1), and obtain $\gamma - \gamma_i = \gamma\beta_z - \gamma_i\beta_{zi}$. Using the expression of η, we obtain

$$1-\beta_z = \frac{1}{\omega_L}\cdot\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{f_0}{\gamma}, \tag{8}$$

with

$$f_0 = \gamma_i\left(1-\beta_{zi}\right), \tag{9}$$

where $\beta_{xi} = v_{xi}/c$, $\beta_{yi} = v_{yi}/c$, $\beta_{zi} = v_{zi}/c$ and $\gamma_i = \left(1-\beta_{xi}^2-\beta_{yi}^2-\beta_{zi}^2\right)^{-\frac{1}{2}}$.

We integrate (2) with respect to time between 0 and t, taking into account (8) and the initial conditions (1) and obtain

$$\beta_x = f_1/\gamma \quad \text{with} \quad f_1 = -a_1\left(\sin\eta - \sin\eta_i\right) + \gamma_i\beta_{xi}. \tag{10}$$

Similarly, the integration of (3), taking into account (1) and (8), leads to

$$\beta_y = f_2/\gamma \quad \text{with} \quad f_2 = -a_2\left(\cos\eta_i - \cos\eta\right) + \gamma_i\beta_{yi}\,. \tag{11}$$

We substitute the expressions of β_x, β_y and β_z, respectively, from (10), (11), and (8) into (6) and obtain the expression of γ:

$$\gamma = \frac{1}{2f_0}\left(1 + f_0^2 + f_1^2 + f_2^2\right). \tag{12}$$

From (8) we have

$$\beta_z = f_3/\gamma \quad \text{with} \quad f_3 = \gamma - f_0\,. \tag{13}$$

From (7), (10), and (11) we have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{\omega_L}{\gamma}\left(a_1 f_1 \cos\eta + a_2 f_2 \sin\eta\right). \tag{14}$$

From Eqs. (2), (8), (10), and (14), we obtain:

$$\dot{\beta}_x = \omega_L g_1 \quad \text{with} \quad g_1 = -\left(a_1 f_0/\gamma^2\right)\cos\eta + \left(f_1/\gamma^3\right)\left(a_1 f_1\cos\eta + a_2 f_2 \sin\eta\right), \tag{15}$$

where the dot represents the derivation with respect to time. Similarly, we have:

$$\dot{\beta}_y = \omega_L g_2 \quad \text{with} \quad g_2 = -\left(a_2 f_0/\gamma^2\right)\sin\eta + \left(f_2/\gamma^3\right)\left(a_1 f_1\cos\eta + a_2 f_2 \sin\eta\right), \tag{16}$$

and

$$\dot{\beta}_z = \omega_L g_3 \quad \text{with} \quad g_3 = -\left(f_0/\gamma^3\right)\left(a_1 f_1\cos\eta + a_2 f_2 \sin\eta\right). \tag{17}$$

The analysis of relations (10–17) shows that the quantities β_x, β_y, β_z, γ, $\dot{\gamma}$, $\dot{\beta}_x$, $\dot{\beta}_y$, and $\dot{\beta}_z$ are periodic functions of only one variable, which is η. This property is valid for all the functions which result from our calculation. It leads to a strong simplification of all the calculations because the processing of the composite functions of one variable can be made with simple software [1]. All the calculations done in the papers [1–5], including the plots shown in figures of those papers are made with the aid of MATHEMATICA 7 programs. A program of this type is presented in the Supplementary Material EPAPS of Ref. [1].

3. INTERACTIONS BETWEEN VERY INTENSE LASER BEAMS AND ELECTRON PLASMAS, RELATIVISTIC ELECTRON BEAMS OR ATOMS, DESCRIBED BY COMPOSITE FUNCTIONS OF ONE VARIABLE

3.1. Interactions with electrons and electron plasmas

The periodicity of the electromagnetic field generated by the electron motion under the action of the electromagnetic field results from the Liènard-Wiechert relation [7]:

$$\bar{E} = \frac{-e}{4\pi\varepsilon_0 cR}\cdot\frac{\bar{n}\times\left[\left(\bar{n}-\bar{\beta}\right)\times\left(\mathrm{d}\bar{\beta}/\mathrm{d}t\right)\right]}{\left(1-\bar{n}\cdot\bar{\beta}\right)^3}, \tag{18}$$

where R is the distance from the electron to the observation point (the detector) and $\bar{n}$ is the versor of the direction electron-detector. In virtue of the significance of the quantities entering in the Liènard-Wiechert equation, it results that the field $\bar{E}$ corresponds to the time $t + R/c$ and we have $\bar{E} = \bar{E}\left(\bar{r} + R\bar{n}, t + R/c\right)$, where $\bar{r}$ is the position vector of the electron with respect to a system having its origin at the point defined by (1). The relation $R >> r$ is overwhelmingly fulfilled [8]. Using spherical coordinates for which θ is the azimuthal angle between the $\bar{n}$ and $\bar{k}$ versors and ϕ is the polar angle in the plane *xy*, the components of the versor $\bar{n}$ can be written $n_x = \sin\theta\cos\phi$, $n_y = \sin\theta\sin\phi$, and $n_z = \cos\theta$.

Introducing in (18) the components of $\overline{n}$, $\overline{\beta}$, and $\mathrm{d}\overline{\beta}/\mathrm{d}t$, we obtain the following expression of the intensity of the scattered electric field:

$$\overline{E} = \frac{K}{F_1^3}\left(h_1 \overline{i} + h_2 \overline{j} + h_3 \overline{k}\right) \quad \text{with} \quad K = \frac{-e\omega_L}{4\pi\varepsilon_0 cR}, \tag{19}$$

where

$$h_1 = F_2\left(n_x - f_1/\gamma\right) - F_1 g_1,\ h_2 = F_2\left(n_y - f_2/\gamma\right) - F_1 g_2,\ h_3 = F_2\left(n_z - f_3/\gamma\right) - F_1 g_3, \tag{20}$$

$$F_1 = 1 - n_x f_1/\gamma - n_y f_2/\gamma - n_z f_3/\gamma \quad \text{and} \quad F_2 = n_x g_1 + n_y g_2 + n_z g_3. \tag{21}$$

Since f_1, f_2, γ, f_3, g_1, g_2, and g_3 are periodic functions of η, in virtue of (18–21) it follows that the components of the electromagnetic field and the intensity of the scattered beam are periodic functions of η. The components of the field $\overline{E}$ can be developed in Fourier series, and the expression of the field, normalized to K, becomes:

$$\begin{aligned}\frac{\overline{E}}{K} = {} & f_{1c0}\,\overline{i} + \left(\sum_{j=1}^{\infty} f_{1sj} \sin j\eta + \sum_{j=1}^{\infty} f_{1cj} \cos j\eta\right)\overline{i} + \\ & + f_{2c0}\,\overline{j} + \left(\sum_{j=1}^{\infty} f_{2sj} \sin j\eta + \sum_{j=1}^{\infty} f_{2cj} \cos j\eta\right)\overline{j} + \\ & + f_{3c0}\,\overline{k} + \left(\sum_{j=1}^{\infty} f_{3sj} \sin j\eta + \sum_{j=1}^{\infty} f_{3cj} \cos j\eta\right)\overline{k},\end{aligned} \tag{22}$$

where

$$f_{\alpha c0} = \frac{1}{2\pi}\int_0^{2\pi} \frac{h_\alpha}{F_1^3}\mathrm{d}\eta,\quad f_{\alpha sj} = \frac{1}{\pi}\int_0^{2\pi} \frac{h_\alpha}{F_1^3}\sin j\eta\,\mathrm{d}\eta,\ \text{and}\quad f_{\alpha cj} = \frac{1}{\pi}\int_0^{2\pi} \frac{h_\alpha}{F_1^3}\cos j\eta\,\mathrm{d}\eta \tag{23}$$

with $\alpha = 1,\ 2,\ 3$.

The quantity

$$\overline{E}_j = \overline{E}_{js} + \overline{E}_{jc}, \tag{24}$$

where

$$\overline{E}_{js} = K\left(f_{1sj}\,\overline{i} + f_{2sj}\,\overline{j} + f_{3sj}\,\overline{k}\right)\sin j\eta \quad \text{and} \quad \overline{E}_{jc} = K\left(f_{1cj}\,\overline{i} + f_{2cj}\,\overline{j} + f_{3cj}\,\overline{k}\right)\cos j\eta \tag{25}$$

is the intensity of the *j*-th harmonic of the electric field. This harmonic is obtained by the addition of two plane fields having the same values of the angular frequency and wave vector magnitude, which are $\omega_j = j\omega_L$ and $\overline{k}_j = j\overline{k}_L$. From here we obtain the expressions of angular frequency, wavelength of the *j*-th component of the scattered radiation and energy of the quanta of the scattered radiation, in an arbitrary direction, which are $\omega_j = j\omega_L$, $\lambda_j = \lambda_L/j$, and $W_j = j\omega_L\hbar$.

The average of the intensity of the total scattered radiation, denoted by I_{av}, is given by the relation

$$I_{av} = \varepsilon_0 c \frac{1}{2\pi}\int_0^{2\pi} \overline{E}^2 \mathrm{d}\eta. \tag{26}$$

With the aid of (22), (25) and relations $\int_0^{2\pi} \cos m\eta \cos n\eta\,\mathrm{d}\eta = \pi\delta_{mn}$, $\int_0^{2\pi} \sin m\eta \sin n\eta\,\mathrm{d}\eta = \pi\delta_{mn}$ and $\int_0^{2\pi} \sin m\eta \cos n\eta\,\mathrm{d}\eta = 0$, where m and n are integer numbers, the expression of the average intensity of the total scattered radiation, normalized to $\varepsilon_0 c K^2$, which is denoted by $\underline{I}_{av}$, becomes

$$\underline{I}_{av} = \frac{I_{ac}}{\varepsilon_0 c K^2} = f_{1c0}^2 + f_{2c0}^2 + f_{3c0}^2 + \frac{1}{2}\sum_{j=1}^{\infty}\left(f_{1sj}^2 + f_{1cj}^2 + f_{2sj}^2 + f_{2cj}^2 + f_{3sj}^2 + f_{3cj}^2 \right). \tag{27}$$

The term $f_{1c0}^2 + f_{2c0}^2 + f_{3c0}^2$ corresponds to a constant component of the scattered field. The quantity

$$\underline{I}_j = \frac{1}{2}\left(f_{1sj}^2 + f_{1cj}^2 + f_{2sj}^2 + f_{2cj}^2 + f_{3sj}^2 + f_{3cj}^2 \right) \tag{28}$$

is the average intensity of the *j*-th harmonic of the scattered field.

We have shown that these relations can be used in the case of the interactions between very intense laser beams and electron plasmas, when the initial kinetic energy of the electrons is of the order of the energy of the electrons, acquired after the above threshold ionization [4].

The periodicity property is important because it makes possible to express physical quantities, like $\overline{E}$, $\underline{I}_{av}$ and $\underline{I}_j$, as composite functions of one variable that assume the form $f(\eta) = f\left(f_1\left(f_2\left(f_3\left(\dots f_n(\eta)\right)\right)\right)\right)$. This simplifies strongly the calculation, because in this case it is not necessary to write explicitly $f(\eta)$, since its calculation reduces simply to successive calculations of functions $f_n, \dots f_3$, f_2, f_1, and f, operations that can be performed numerically very fast and accurately. In our case the initial data are a_1, a_2, β_{xi}, β_{yi}, β_{zi}, β_{xi}, η_i, θ, ϕ, and *j*. The quantities $\overline{E}$, $\underline{I}_{av}$, and $\underline{I}_j$ are composite functions of f_1, f_2, γ, f_3, g_1, g_2, and g_3, which in their turn are periodic functions of η. The quantities $\overline{E}$, $\underline{I}_{av}$, and $\underline{I}_j$ are calculated with the aid of a MATHEMATICA 7 program, which is given in the Supplementary Material EPAPS [1].

With the aid of previous relations we have calculated the angular and spectral distributions of the radiations generated at the interactions between very intense laser beams and electron plasmas, which are in good agreement with numerous experimental data from literature [1, 4].

3.2. Interactions with relativistic electron beams

We consider the head-on collision between a laser beam and a relativistic electron beam [1], when the components of the field are $\overline{E}_L = E_{M1}\cos\eta\,\overline{i} + E_{M2}\sin\eta\,\overline{j}$ and $\overline{B}_L = -B_{M2}\sin\eta\,\overline{i} + B_{M1}\cos\eta\,\overline{j}$, with $\eta = \omega_L t - \left|\overline{k}_L\right| z + \eta_i$ and $\left|\overline{k}_L\right| c = \omega_L$. The initial conditions are:

$$t = 0,\ x = y = z = 0,\ v_x = v_y = 0,\ v_z = -\left|V_0\right| \ \text{and}\ \ \eta = \eta_i . \tag{29}$$

The analysis is similar to that presented previously for the interaction between electromagnetic field and electron. It is made in five steps, in the inertial system *S′* in which the velocity of the electron is zero.

On the first step, the Lorentz relations lead to the components of the electromagnetic field in the system *S′*, taking into account that the phase is relativistic invariant, namely, $\eta = \eta'$.

On the second step, the solution of the relativistic motion equations in the S′ system leads to relations of the form:

$$\overline{\beta}' = \overline{\beta}'(\sin\eta, \cos\eta, \text{const.}) \ \text{and}\ \ \mathrm{d}\overline{\beta}'/\mathrm{d}t = \overline{a}(\sin\eta, \cos\eta, \text{const.}), \tag{30}$$

where $\text{const.} = \left(a_1, a_2, \left|\overline{V}_0\right|, \eta_i\right)$. It follows that electron velocity and acceleration are periodic functions of the field phase.

On the third step, we introduce $\overline{\beta}'$ and $\mathrm{d}\overline{\beta}'/\mathrm{d}t$ in the Liènard-Wiechert equation and obtain

$$\overline{E}' = \overline{E}'(\sin\eta, \cos\eta, \text{const.}, \theta', \phi'), \tag{31}$$

where θ' and ϕ' are the spherical coordinates of the direction in which the radiation is emitted, in the *S′* system.

On the fourth step, the Fourier series development leads to the spectral components of the electrical field:

$$\overline{E}'_j = \overline{E}'_j \left(\sin j\eta, \cos j\eta, \text{const.}, \theta', \phi'\right). \tag{32}$$

On the fifth step, the Lorentz relations for transformation from *S'* system to the laboratory system *S*, leads to the expressions of the electrical field and intensity of the beam of the radiation generated by electron motion, $\overline{E}$ and $\underline{I}_j$, in the *S* system.

We have used these relations to calculate the spectral and angular distributions of the backscattered radiations resulted by head-on collisions between laser and relativistic electron beams, which are in good agreement with the experimental data from literature. These relations predict the possibility of obtaining photonic sources, having quanta energies greater than 1 MeV [2].

3.3. Interactions with atoms

In a recent paper [4] we have presented an accurate method for the calculation of the spectrum of the harmonics generated by the interaction between very intense laser beams and atoms, in the frame of the three sequences mechanism [6]. These sequences are as follows: 1) The multiphoton absorption, followed by leaving the atom by tunneling; 2) Oscillation in the ionization domain, in which interaction between electron and nucleus is neglected; 3) Returning of the electron in the vicinity of the atom where it transfers the kinetic energy to the electromagnetic field by electric dipole transition.

We have shown that two dominant effects are involved in the harmonic generation. First, the behavior of the system in the second phase can be approximated by classical motion of the electron and the solution of the electron motion is identical to that from Section 2. Second, the emission of the harmonics is due to the electric dipole transition. An accurate rate of electric dipole transition, was calculated by Lewenstein *et al*. [9], in the frame of the theory of Bethe and Salpeter [10].

Using the three sequences mechanism, we have deduced a method to calculate the harmonic spectrum. Our relations describe accurately, in good agreement with the experiment, important features of the harmonic spectrum and the evolution of its shape with the increase of the laser beam intensity.

4. CONCLUSIONS

In this paper we have shown that a general periodicity property is valid in the case of interactions between very intense laser field and electrons, relativistic electron beams or atoms. More specifically, the electron velocity and acceleration, and the components of the electromagnetic field, generated by these interactions, are periodic functions of only one variable, which is the phase of the incident electromagnetic field. A consequence of this property is the strong simplification of the modeling of these interactions, because the functions of only one variable can be easy calculated, using simple software. Our calculation method is in agreement with numerous experimental data from literature, as it was shown in our previous papers.

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REFERENCES

1. A. POPA, *Periodicity property of relativistic Thomson scattering with application to exact calculations of angular and spectral distributions of the scattered field*, Phys. Rev. A, **84**, 023824 (1–14), 2011.
2. A. POPA, *Polarization effects in collisions between very intense laser beams and relativistic electrons*, Laser and Particle Beams, **30**, pp. 591–603, 2012.
3. A. POPA, *Dependence of Harmonic Spectrum Shape on Laser Beam Intensity in Interaction with Atoms*, IEEE J. Quantum. Elect., **49**, pp. 522–527, 2013.
4. A. POPA, *Accurate Calculation of High Harmonics Generated by Interactions Between Very Intense Laser Fields and Electron Plasmas*, IEEE Trans. of Plasma Science, **41**, pp. 2246–2250, 2013.

5. A. POPA, *Accurate calculation of radiation damping parameters in the interaction between very intense laser beams and relativistic electron beams*, Laser and Particle Beams, **32**, pp. 477–486, 2014.
6. P.B. CORKUM, *Plasma Perspective on Strong-Field Multiphoton Ionization*, Phys. Rev. Lett., **71**, 1994–1997, 1993.
7. L.D. LANDAU, E.M. LIFSCHITZ, *The Classical Theory of Fields*, Pergamon Press, London, 1959.
8. J.D. JACKSON, *Classical Electrodynamics*, John Wiley, New York, 1999.
9. M. LEWENSTEIN *et al*., *Theory of high-harmonic generation by low-frequency laser fields*, Phys. Rev. A, **49**, 2117–2132, 1994.
10. H.A. BETHE, E.E. SALPETER, *Quantum Mechanics of One and Two Electrons Atom*, Academic Press, New York, 1957.

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