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APPROXIMATE ANALYTIC SOLUTIONS OF A NONLINEAR ELASTIC WAVE EQUATIONS WITH THE ANHARMONIC CORRECTION

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In this article we consider the propagation equation of the longitudinal elastic wave in the presence of the volume forces, taking into account the shear phenomena. In order to find an approximate analytic solutions of the governing systems we apply Optimal Variational Method (OVM). This approach involves the presence of some initially unknown convergence-control parameters which are optimally determined. An excellent agreement is found between approximate and numerical solutions.

Key words: optimal variational method, nonlinear partial differential equations, thin elastic plate.

1. INTRODUCTION

In spite of the great deal of work of nonlinear phenomena, the wave equation with the volume forces correction has still received little attention in the continuum limit. But the mathematical modeling of physical processes leads to nonlinear partial differential equations (PDE) whose analytic solutions are hard to find. A powerful general technique for analyzing nonlinear PDEs is given by the classical Lie symmetry method [1], [2]. Recently, in 2009 Mustafa and Massoud [3] apply the Lie symmetry method to obtain several exact solutions in an explicit form, analyzing a nonlinear elastic wave equation for longitudinal deformations with third-order anharmonic corrections to the elastic energy. Approximate symmetries and prolongation technique were used to carry out symmetry analysis of some interesting cases of such nonlinear wave equations by Alfinito et al. in [4]. The equation of motion for longitudinal deformation with the third-order anharmonic terms has also been analyzed in [5] by using quadratures and asymptotic series, where it was concluded that anharmonic corrections to the elastic energy are likely to lead to solutions involving time-dependent singularities at finite times. A similarity analysis of a nonlinear wave equation in elasticity was studied in [6] where were obtained the exact solution via the method of invariants taking into account the third-order anharmonic corrections to the elastic energy.

In general, such problem are not amenable to exact treatment and approximate techniques must be resorted to. Among these, the perturbation methods are in common use. The perturbation methods are, in principle, intended to solve problems involving a small parameter [7]. However, in science and engineering there exist many nonlinear problems in which parameters are not small. The application of the perturbation methods have been extended to oscillations with strong nonlinearities. Some extensions of the Lindstedt-Poincare perturbation method or the incremental harmonic balance method have been proposed by Cheung et al. [8] for strongly nonlinear systems. An approach which combines the harmonic balance method and linearization of nonlinear oscillations equation was presented in [9]. There also exist a wide range of literature dealing with approximate determination of periodic solutions for nonlinear problems by using a mixture of methodologies: the optimal homotopy asymptotic method [10], an equivalent linearization method [12], the linearized Galerkin and artificial parameter techniques [13]. In 1999, the variational iteration method was proposed by J. H. He [14]. A variational principle for the nonlinear oscillations by constructing the Hamiltonian was studied in [15].

In the present work we propose a novel variational approach to nonlinear partial differential equation. Our procedure involves an original construction of Lagrangian which is used to construct approximations to solve nonlinear problems with particular emphasis on elastic wave equation with the volume forces correction. The construction of an Lagrangian for dynamical systems with more general Newtonian forces are nowadays applicable only to systems with force derivable from a potential function (namely conservative systems).

2. MATHEMATICAL MODEL OF NONLINEAR ELASTIC WAVE EQUATIONS WITH THE VOLUME FORCES CORRECTION

The terms of order higher than second in the strain tensor must be considered in the elastic energy for large values of the elastic deformations. These higher-order terms generate nonlinear equations of motion and they are usually called anharmonic corrections to the wave equation. But the superposition of the solutions does not hold anymore for anharmonic corrections, in general, and the elastic waves exhibit the combined-frequency phenomenon and temporal resonances. In the literature, the contribution of the nonlinear terms in this equation is treated as a small perturbation to the plane waves. From the elasticity theory [16, 17, 18] it is known that the nonlinear contributions to elasticity appear first through the full expression

$$\gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right), \quad i,\ j = 1,2,3 \tag{1}$$

of the strain tensor and secondly through higher-order terms in the elastic energy, where $\vec{u} = (u_1, u_2, u_3)$ is the displacement vector at position of coordinates x_i, $i = 1,2,3$.

If t_{ij} denotes the stress tensor and λ and μ represents the Lamé's material physical constants then the Hooke's law for elastic solids can be written in the form

$$t_{ij} = [\lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\gamma_{kl}, \quad i,\ j = 1,2,3 \tag{2}$$

where δ_{ij} means the Kronecker's symbol.

The nonlinear equations of motion of the elastic solids under the action of the volume forces $\overline{f}^* = (f_1^*, f_2^*, f_3^*)$ are written as:

$$\sum_{j=1}^{3} \frac{\partial t_{ij}}{\partial x_j} + \rho_0 f_i^* = \rho_0 \frac{\partial^2 u_i}{\partial t^2}, \ i = 1,2,3, \ (x_1, x_2, x_3) \in \Omega \times \mathrm{R}_+^2, t \geq 0, \tag{3}$$

where $\Omega = [a, b]$ is the bounded domain and ρ_0 is the density of medium. We assume that the medium is isotropic i.e. λ, μ and ρ_0 are constants.

Firstly, we note that the Ox_1 axis coincides with the propagation direction of the longitudinal elastic wave, i.e.:

$$\begin{gathered} u_1(x_1, t) = u(x, t),\ u_2(x_1, t) = u_2(x, t), u_3(x_1, t) = u_3(x, t), \\ f_1^* = f^*(x, t), \quad f_2^* = f_3^* = 0. \end{gathered} \tag{4}$$

Taking into account Eqs. (1), (2) and (4), Eq. (3) can be written as [15], [16]:

$$v_1^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) + f_1 = \frac{\partial^2 u}{\partial t^2}, \ v_2^2 \frac{\partial^2 u_2}{\partial x^2} = \frac{\partial^2 u_2}{\partial t^2}, \ v_2^2 \frac{\partial^2 u_3}{\partial x^2} = \frac{\partial^2 u_3}{\partial t^2}. \tag{5}$$

where

$$f_1 = f^* + v_1^2\left(\frac{\partial u_2}{\partial x}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial u_3}{\partial x}\frac{\partial^2 u_3}{\partial x^2}\right), v_1^2 = \frac{\lambda + 2\mu}{\rho_0}, \quad v_2^2 = \frac{\mu}{\rho_0},$$
$$f^*(x,t) = 2\cos(2\pi x)\sin(2\pi v_2 t) - 2\sin(4\pi x)\sin^2(4\pi v_1 t) - \\ -(4\pi v_1)^2\cos^2(4\pi v_2 t)\sin(8\pi x). \tag{6}$$

Equation $(5)_1$ is named the propagation equation of the longitudinal elastic wave in the presence of the volume forces. In Eq. $(6)_2$ v_1 and v_2 are longitudinal and transversal elastic velocity, respectively.

In the following, we study the nonlinear longitudinal elastic wave of Eq. $(5)_1$ with the free boundary conditions of stress, i.e.

$$\frac{\partial u}{\partial x}(0,t) = 0, \ \frac{\partial u}{\partial x}(1,t) = 0, \quad a = 0, b = 1, \ t \in [0,1]. \tag{7}$$

The initial conditions are assumed to be of the form:

$$u(x,0) = v_0(x), \quad \frac{\partial u}{\partial t}(x,0) = w_0(x), \quad x \in [0,1], \tag{8}$$

where $v_0(x)$ and $w_0(x)$ are known continuous functions.

For linear partial differential equations $(5)_2$ and $(5)_3$, we suppose that the boundary and initial conditions are respectively:

$$\frac{\partial u_2}{\partial x}(0,t) = 0, \ \frac{\partial u_2}{\partial x}(1,t) = 0, \quad u_2(x,0) = \cos(4\pi x), \quad \frac{\partial u_2}{\partial t}(x,0) = 0, \tag{9}$$

$$\frac{\partial u_3}{\partial x}(0,t) = 0, \frac{\partial u_3}{\partial x}(1,t) = 0, \ u_3(x,0) = \cos(4\pi x), \ \frac{\partial u_3}{\partial t}(x,0) = 0, \tag{10}$$

with the solutions:

$$u_2(x,t) = \cos(4\pi v_2 t)\cos(4\pi x), \ u_3(x,t) = \cos(4\pi v_2 t)\cos(4\pi x), \tag{11}$$

such that Eq. $(6)_1$ becomes:

$$f_1(x,t) = 2\cos(2\pi x)\sin(2\pi v_2 t) - \sin(4\pi x) + \cos(4\pi v_1 t)\sin(4\pi x) \tag{12}$$

3. BASIC IDEAS OF OVM AND SOLUTIONS

In order to develop an application of the OVM, we consider the general form of a nonlinear partial differential equation:

$$E(x,t,u(x,t),u_x(x,t),u_t(x,t)) = 0, \tag{13}$$

here $u_x(x,t)$ and $u_t(x,t)$ denote the partial derivatives with respect to x and t, respectively of the function $u(x,t)$.

The variational principle for Eq. (13) can be established if there exists a functional

$$J = \int_a^b \int_{t_1}^{t_2} L(x,t,u(x,t),u_x(x,t),u_t(x,t))\mathrm{d}t\mathrm{d}x, \tag{14}$$

which admits as extremals the solutions of Eq.(13), where L is the Lagrangian of the system (13), $x \in [a,b]$, $t \in [t_1,t_2]$.

This problem is based on the study of the conditions under which there exists a functional

$L(x,t,u(x,t),u_x(x,t),u_t(x,t))$ such that Euler-Ostrogradski's equation of functional (14) coincide with the Eq. (13) i.e.:

$$\frac{\partial L}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right)=E(x,t,u,u_x,u_t). \tag{15}$$

In our procedure, we assume that the approximate solutions of Eq. (13) depends of a number of parameters C_1, $C_2,...,C_s$:

$$\bar{u}=\bar{u}(x,t,C_1,...,C_s), \tag{16}$$

such that the action functional given by Eq. (14) becomes

$$J(C_1,...,C_s)=\int_a^b\int_{t_1}^{t_2}L(x,t,\bar{u}(x,t,C_i),\bar{u}_x(x,t,C_i),\bar{u}_t(x,t,C_i))\mathrm{d}t\mathrm{d}x,\ \ i=1,2,..,s. \tag{17}$$

The parameters C_i (which will be named convergence-control parameters) which appear in Eq. (17) can be determined optimally, applying the Ritz method [10], [19]:

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=...=\frac{\partial J}{\partial C_s}=0. \tag{18}$$

We mention that the approximate solution $\bar{u}$ given by Eq. (16) is chosen such that the boundary and initial conditions are fulfilled. The expression of the approximate solution (16) is not unique.

Remark. The Ritz method [11] of obtaining such "average" solutions can be derived from calculus of variations by seeking functions that minimize a certain integral

$$J=\int_{t_0}^{t_1}F(\dot{x},x,t)\mathrm{d}t. \tag{19}$$

Consider a function of the form

$$\bar{x}(t)=C_1\Psi_1(t)+C_2\Psi_2(t)+...+C_n\Psi_n(t),$$

where the $\Psi_k(t)$ are prescribed functions, all satisfy the initial / boundary conditions. If $\bar{x}(t)$ is now introduced for $x(t)$, then $J=J(C_1,C_2,...,C_n)$ and necessary conditions for J to be minimum are given by Eqs. (18). This gives n equations of the form

$$\frac{\partial J}{\partial C_k}=\int_{t_0}^{t_1}\left(\frac{\partial F}{\partial \bar{x}}-\frac{\partial F}{\partial \bar{x}}\dot{\Psi}_k\right)\mathrm{d}t=0,$$

for determining the n unknown constants C_k. Integrating the last equation, we have:

$$\frac{\partial J}{\partial C_k}=\frac{\partial F}{\partial \bar{x}}\Psi_k\Big|_{t_0}^{t_1}+\int_{t_0}^{t_1}\left(\frac{\partial F}{\partial \bar{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \bar{x}}\right)\right)\Psi_k\mathrm{d}t=0.$$

The first term is zero because Ψ_n must satisfy the initial/boundary conditions. The expression in brackets under the integral in the second term is Euler's equations. The condition given in Eqs. (18) then reduce to

$$\int_{t_0}^{t_1}E(\bar{x})\Psi_k\mathrm{d}t=0,\ \ k=1,\ 2,\ ...,\ n,$$

where $E(x)$ is the differential equation of motion (13).

Unlike the Ritz method, within our procedure the prescribed functions should not satisfy the initial / boundary conditions. On the other hand, the approximate solution $\bar{x}$ is substituted directly into the functional (15), not in the differential equation of motion.

The validity of the proposed approach is illustrate on the Eq. $(5)_1$ with the conditions given by Eqs. (7) and (8). In our case, the Lagrangian of Eq. $(5)_1$ can be written as:

$$L(x,t,u(x,t),u_x(x,t),u_t(x,t)) = v_1^2\left(\frac{1}{2}u_x^2+\frac{1}{6}u_x^3\right) - f_1 u - \frac{1}{2}u_t^2. \tag{19}$$

If we consider $s=5$ in Eq. (16) and $v_0=\cos(2\pi x)$, $w_0=0$ in Eq. (8), then the approximate solution of the Eq. $(5)_1$ can be written as:

$$\begin{gathered}\bar{u}(x,t)=\cos 2\pi x + C_1\cos(2\pi x)(1-\cos(2\pi v_2 t))+\\ +C_2\cos(2\pi x)(1-\cos(6\pi v_2 t))+C_3\cos(2\pi x)(1-\cos(8\pi v_2 t))+\\ +C_4\cos(2\pi x)(1-\cos(4\pi v_1 t))+C_5\cos(2\pi x)(1-\cos(8\pi v_1 t)).\end{gathered} \tag{20}$$

Also, we can choose this approximate solution in the forms:

$$\bar{u}(x,t)=\cos 2\pi x + (C_1 + C_2\cos 2\pi x)[C_3(1-\cos 2\pi v_2 t)+C_4(1-\cos 6\pi v_2 t)] \tag{21}$$

or

$$\begin{gathered}\bar{u}(x,t)=\cos 2\pi x + C_1\cos(2\pi x)(1-\cos(2\pi v_2 t))+\\ +C_2\cos(2\pi x)(1-\cos(6\pi v_2 t))+C_3\cos(4\pi x)(1-\cos(8\pi v_2 t))+\\ +C_4\cos(6\pi x)(1-\cos(4\pi v_1 t)).\end{gathered} \tag{22}$$

and so on.

The parameters C_1, C_2, C_3, C_4, C_5 which appear in Eq. (20) are obtained from Eqs. (18):

$$\frac{\partial J}{\partial C_1}=\frac{\partial J}{\partial C_2}=\frac{\partial J}{\partial C_3}=\frac{\partial J}{\partial C_4}=\frac{\partial J}{\partial C_5}=0. \tag{23}$$

4. NUMERICAL EXAMPLES

In order to show the validity and accuracy of the OVM, we consider Eqs. $(5)_1$ and

$$v_1=0.02,\quad v_2=1,\quad x\in[0,\ 1],\quad t\in[0,\ 1]. \tag{24}$$

From Eq. (23), in the conditions (24), we obtain the following results:

$$\begin{gathered}C_1=0.00320517,\quad C_2=-0.00097268,\quad C_3=-0.00059360,\\ C_4=-879.94559641,\quad C_5=226.21331816.\end{gathered} \tag{25}$$

The approximate solutions of of Eq. $(5)_1$ in the conditions (7) and (8) with $v_0=\cos(2\pi x)$, $w_0=0$ becomes:

$$\begin{gathered}\bar{u}(x,t)=\cos(2\pi x)-879.946(1-\cos(0.251327t))\cos(2\pi x)+\\ +226.213(1-\cos(0.502655t))\cos(2\pi x)+0.00320518(1-\cos(2\pi t))\cos(2\pi x)-\\ -0.00097268(1-\cos(6\pi t))\cos(2\pi x)-0.000593606(1-\cos(8\pi t))\cos(2\pi x).\end{gathered} \tag{26}$$

Figures 1and 2 present a comparisons between the present solution (26) and numerical results obtained using the Wolfram Mathematica 6.0 software for $x\in[0,1]$ and $t\in[0,1]$.

Figures 3, 4 and 5 present a comparisons between the present solution (26) for different values of x from the domain $[0,1]$ and three different random values of t: $\frac{1}{5}$, $\frac{1}{2}$ and $\frac{7}{10}$ respectively and numerical results. It is easier to emphasize, the accuracy of the obtained results in comparison with the numerical results. It can be seen that the periodic solutions obtained by our procedure is in very good agreement with numerical results.t is easier to emphasize, the accuracy of the obtained result in comparison with the numerical results.

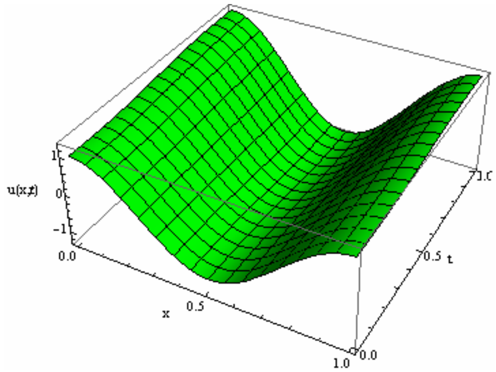

Fig. 1 – The numerical solution of Eq. $(5)_1$ for $v_1 = 0.02$, $v_2 = 1$.

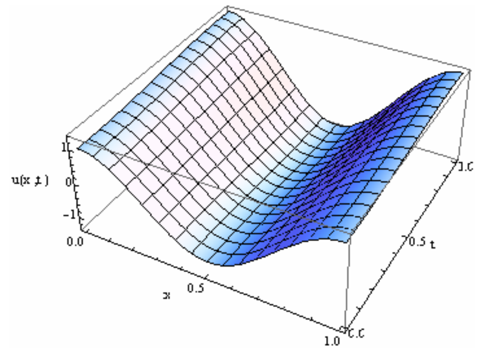

Fig. 2 – The approximate solution of Eq. (26) for $v_1 = 0.02$, $v_2 = 1$.

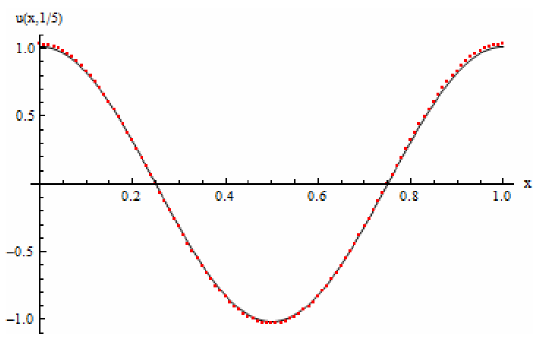

Fig. 3 – Comparison between the numerical solution of Eq. $(5)_1$ and approximate solution (26) for $v_1 = 0.02$, $v_2 = 1$, $t_1 = 1/5$:
------------- numerical solution;
…................ approximate solution

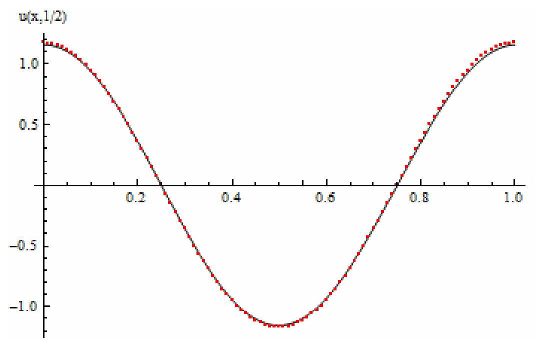

Fig. 4 – Comparison between the numerical solution of Eq. $(5)_1$ and approximate solution (26) for $v_1 = 0.02$, $v_2 = 1$, $t_2 = 1/2$:
------------- numerical solution;
.... approximate solution

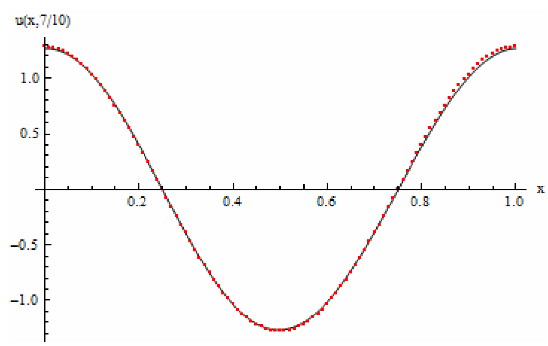

Fig. 5 – Comparison between the numerical solution of Eq. $(5)_1$ and approximate solution (26) for $v_1 = 0.02$, $v_2 = 1$, $t_3 = 7/10$: -------- numerical solution;…........ approximate solution

5. CONCLUSIONS

In this paper we proposed and used Optimal Variational Method to determine an analytic approximate solutions to a nonlinear problems related to elastic wave equations with anharmonic correction. The proposed procedure is valid even if the nonlinear equation does not contain any small parameter. Our construction is based of the construction of the approximate solution depending of some convergence-control parameters C_i which are optimally determined. Actually, the capital strength of the proposed procedure is its fast convergence. Our procedure is very effective and accurate for nonlinear approximations converging rapidly to exact solutions and provides a convenient way to control the convergence of approximate periodic solution.

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