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A CHEBYSHEV-LAGUERRE-GAUSS-RADAU COLLOCATION SCHEME FOR SOLVING A TIME FRACTIONAL SUB-DIFFUSION EQUATION ON A SEMI-INFINITE DOMAIN

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We propose a new efficient spectral collocation method for solving a time fractional sub-diffusion equation on a semi-infinite domain. The shifted Chebyshev-Gauss-Radau interpolation method is adapted for time discretization along with the Laguerre-Gauss-Radau collocation scheme that is used for space discretization on a semi-infinite domain. The main advantage of the proposed approach is that a spectral method is implemented for both time and space discretizations, which allows us to present a new efficient algorithm for solving time fractional sub-diffusion equations.

Key words: time fractional sub-diffusion equation, semi-infinite domain, Chebyshev-Gauss-Radau collocation scheme, Laguerre-Gauss-Radau collocation scheme, Caputo derivatives.

1. INTRODUCTION

Several computational problems in diverse research areas are considered on semi-infinite domains. The earthquake engineering field and underwater acoustic problems can be modeled as partial differential equations on semi-infinite domains. Spectral methods provide a computational approach that become popular during the last decade [1]. They have gained new popularity in automatic computations for a wide class of physical problems in fluid and heat flows. Recently, spectral methods were used to numerically solve problems on semi-infinite domains [2–7]; in such a case, the choice of the basis functions for a truncated series expansion of the solution depends on orthogonal systems of infinitely differentiable global functions defined on the half line. In recent years there has been a high level of interest of employing spectral methods for numerically solving many types of integral and differential equations, due to their ease of applying them for both finite and infinite domains [1, 8, 9, 10]. Spectral methods not only have exponential rates of convergence but also have a high level of accuracy. There are three main types of spectral methods namely collocation [11, 12, 13], tau [14, 15], and Galerkin [16, 17, 18] methods.

Fractional differential equations (FDEs) model many phenomena in several fields such as fluid mechanics, chemistry, biology, viscoelasticity, engineering, finance and physics [19–29]. Most FDEs do not have exact analytic solutions, so approximation techniques must be used. Finite element methods were presented in [30–33] to obtain the numerical solutions of FDEs. Meanwhile, the numerical treatment based on finite difference methods for FDEs was proposed in [34–36]. Moreover, several spectral algorithms were also designed for FDEs, see for example [37]. Time fractional sub-diffusion equation on a semi-infinite domain is studied in this paper using a new collocation method. The collocation method has a wide range of applications, due to its ease of use and adaptability in various problems [38–40].

The aim of this article is to extend the application of shifted Chebyshev-Gauss-Radau interpolation method in combination with generalized Laguerre-Gauss-Radau collocation scheme for the numerical treatment of the time fractional sub-diffusion equations on semi-infinite domains. The shifted Chebyshev-Gauss-Radau interpolation method is adapted for time discretization and the Laguerre-Gauss-Radau

collocation scheme is used for space discretization on a semi-infinite domain. The underlined scheme provides a system of algebraic equations. Finally, we demonstrate the accuracy of this new method by presenting a test example.The outline of this paper is as follows. In Sec. 2 we present few relevant properties of generalized Laguerre-Gauss-Radau interpolation, Chebyshev-Gauss-Radau interpolation, and fractional integrals. The mentioned scheme is implemented for the time-fractional diffusion model in Sec. 3. A test example is given in Sec. 4. Finally, some concluding remarks are given in the last section.

2. ORTHOGONAL POLYNOMIALS AND FRACTIONAL INTEGRALS

In this section we recall some relevant properties of the generalized Laguerre polynomials, Chebyshev polynomials, and the fractional integrals of these polynomials [4, 37, 41]. Now, let $\Lambda = (0,\infty)$ and $w^{(\alpha)}(x) = x^{\alpha}e^{-x}$ be a weight function on Λ in the usual sense. Define

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{w^{(\alpha)}} < \infty\},$$

equipped with the following inner product and norm

$$(u,v)_{w^{(\alpha)}} = \int_{\Lambda} u(x)v(x)w^{(\alpha)}(x)\mathrm{d}x, \ \|v\|_{w^{(\alpha)}} = (v,v)^{\frac{1}{2}}_{w^{(\alpha)}}.$$

Next, let $L_i^{(\alpha)}(x)$ be the generalized Laguerre polynomials of degree i. According to [42] for $\alpha > -1$, we have

$$\partial_x L_i^{(\alpha)}(x) = -L_{i-1}^{(\alpha+1)}(x),$$

$$L_{i+1}^{(\alpha)}(x) = \frac{1}{i+1}[(2i+\alpha+1-x)L_i^{(\alpha)}(x) - (i+\alpha)L_{i-1}^{(\alpha)}(x)], \quad i = 1,2,\ldots,$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = 1+\alpha-x$. The set of generalized Laguerre polynomials is a $L^2_{w^{(\alpha)}}(\Lambda)-$ orthogonal system, namely

$$\int_0^{\infty} L_j^{(\alpha)}(x)L_k^{(\alpha)}(x)w^{(\alpha)}(x)\mathrm{d}x = h_k\delta_{jk}, \tag{1}$$

where δ_{jk} is the Kronecher function and $h_k = \dfrac{\Gamma(k+\alpha+1)}{k!}$.

The analytical form of generalized Laguerre polynomials of degree i on the interval $\Lambda \equiv (0,\infty)$, is given by (see e.g. [43])

$$L_i^{(\alpha)}(x) = \sum_{k=0}^{i}(-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)\,(i-k)!\,k!}x^k, \quad i = 0,1,\cdots. \tag{2}$$

The special value

$$D^q L_i^{(\alpha)}(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)!(i-j-q)!} L_j^{(\alpha)}(0), \tag{3}$$

where $L_j^{(\alpha)}(0) = \dfrac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)j!}$, will be of important use later, for treating the initial conditions of the given FDEs.

Let $u(x) \in L^2_{w^{(\alpha)}}(\Lambda)$, then $u(x)$ may be expressed in terms of generalized Laguerre polynomials as

$$u(x)=\sum_{j=0}^{\infty}a_jL_j^{(\alpha)}(x),\ \ a_j=\frac{1}{h_k}\int_0^{\infty}u(x)L_j^{(\alpha)}(x)w^{(\alpha)}(x)\mathrm{d}x,\ \ j=0,1,2,\cdots. \tag{4}$$

In particular applications, the generalized Laguerre polynomials up to degree $N+1$ are considered. Then we have

$$u_N(x)=\sum_{j=0}^{N}a_jL_j^{(\alpha)}(x). \tag{5}$$

The well-known Chebyshev polynomials are defined on the interval $[-1,\ 1]$, by

$$T_k(t)=\cos(k\arccos(t)),\quad k\geq 0. \tag{6}$$

The Chebyshev polynomials satisfy the following relations

$$T_k(\pm 1)=(\pm 1)^k,\quad T_k(-t)=(-1)^kT_k(t). \tag{7}$$

Let $w^c(t)=\dfrac{1}{\sqrt{1-t^2}}$, then we define the weighted space $L^2_{w^c}$. The inner product and the norm of $L^2_{w^c}$ with respect to the weight function are defined as follows:

$$(u,v)_{w^c}=\int_{-1}^{1}u(x)v(x)w^c(t)\mathrm{d}t,\ \ \|u\|_{w^c}=(u,u)_{w^c}^{\frac{1}{2}}. \tag{8}$$

The set of Chebyshev polynomials forms a complete $L^2_{w^c}$ -orthogonal system, and

$$\|T_k\|^2{}_{w^c}=h_k^c=\begin{cases}\dfrac{\varsigma_k}{2}\pi, & k=j,\\ 0, & k\neq j,\end{cases}\qquad \varsigma_0=2,\quad \varsigma_k=1,\quad k\geq 1. \tag{9}$$

For Chebyshev-Gauss-Radau quadrature formula [1],

$$t_{N,j}=\cos\left(\frac{2\pi j}{2N+1}\right),\quad \varpi^c_{N,j}=\begin{cases}\dfrac{\pi}{2N+1}, & j=0;\\ \dfrac{2\pi}{2N+2}, & j=1,\cdots,N,\end{cases} \tag{10}$$

where $t_{N,j}$ ($0\leq j\leq N$) and $\varpi^c_{N,j}$ ($0\leq j\leq N$) are the nodes and the corresponding Christoffel numbers of the Chebyshev-Gauss-Radau quadrature formula on the interval $[-1,\ 1]$, respectively.

Now, we introduce the following discrete inner product and norm

$$(u,v)_{w^c}=\sum_{j=0}^{N}u(t_{N,j})v(t_{N,j})\varpi^c_{N,j},\qquad \|u\|_{w^c}=(u,u)_{w^c}^{\frac{1}{2}}. \tag{11}$$

In order to use Chebyshev polynomials $T_n(t)$ on the interval $(0,L)$ we define the so-called shifted Chebyshev polynomials by introducing the change of variable $t=\dfrac{2t}{L}-1$. Let the shifted Chebyshev polynomials $T_n(\dfrac{2t}{L}-1)$ be denoted by $T_{L,n}(t)$. The analytic form of the shifted Chebyshev polynomials $T_{L,n}(t)$ of degree n is given by

$$T_{L,n}(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k-1)!\,2^{2k}}{(n-k)!\,(2k)!\,L^k}t^k, \tag{12}$$

where $T_{L,n}(0) = (-1)^n$ and $T_{L,n}(L) = 1$. The orthogonality condition is

$$\int_0^L T_{L,m}(t)T_{L,n}(x)w_L(x)\mathrm{d}x = \delta_{mn}h_n^L, \tag{13}$$

where $w_L(t) = \dfrac{1}{\sqrt{Lt-t^2}}$ and $h_n^L = \dfrac{c_n}{2}\pi$, with $c_0 = 2$, $c_i = 1$, $i \ge 1$.

Any function $u(t)$, square integrable in $(0, L)$, may be expressed in terms of shifted Chebyshev polynomials as

$$u(t) = \sum_{j=0}^{\infty} a_j T_{L,j}(t), \tag{14}$$

where the coefficients a_j are given by

$$a_j = \frac{1}{h_j^L}\int_0^L u(t)T_{L,j}(t)w_L(t)\mathrm{d}t, \quad j = 0, 1, 2, \cdots. \tag{15}$$

There are several definitions of the fractional integration of order $\nu > 0$, and not necessarily equivalent to each other. Riemann-Liouville and Caputo fractional definitions are the two most used from all the other definitions of fractional calculus that have been introduced recently.

Definition 2.1. The fractional integral of order $\nu \ge 0$ according to Riemann-Liouville is given by

$$\begin{aligned} J^\nu f(x) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-\zeta)^{\nu-1} f(\zeta)\mathrm{d}\zeta, \ \nu > 0, \ x > 0, \\ J^0 f(x) &= f(x), \end{aligned} \tag{16}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}\mathrm{d}x$ is the gamma function.

Definition 2.2. The Caputo fractional derivatives of order ν is defined as

$${}^cD^\nu f(x) = \frac{1}{\Gamma(m-\nu)}\int_0^x (x-\zeta)^{m-\nu-1}\frac{\mathrm{d}^m}{\mathrm{d}t^m}f(\zeta)\mathrm{d}\zeta, \ m-1 < \nu \le m, \ x > 0, \tag{17}$$

where m is the ceiling function of ν.

3. FULLY COLLOCATION METHOD

The main objective of this section is to develop the collocation method to solve numerically the time fractional sub-diffusion equations on semi-infinite domain in the following form

$${}^cD_t^\nu u(x,t) = a_1 D_{xx}u(x,t) + H(x,t), \quad (x,t) \in [0,\infty)\times[0,L], \tag{18}$$

subject to the initial conditions

$$\begin{aligned} u(0,t) &= g_1(t), \quad D_x u(x,t)|_{x=0} = g_2(t), \\ u(x,0) &= g_3(x), \quad (x,t) \in [0,\infty)\times[0,L], \end{aligned} \tag{19}$$

where a_1 is constant, while $H(x,t)$, $g_1(t)$, $g_2(t)$ and $g_3(x)$ are given functions. Here, we use the set of generalized Laguerre ($x_{N,r}^{(\alpha_1)}$) and shifted Chebyshev ($t_{L,M,s}$) Gauss-Radau points for the space and time

approximation, respectively. The aim of this work is to consider the advantage of the collocation point distribution in a specified domain. Now, we outline the main steps of the collocation method for solving the previous time fractional sub-diffusion equations on a semi-infinite domain. Assume we approximate the solution as a finite double expansion of the form,

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_i^{(\alpha_1)}(x) T_{L,j}(t) = \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \end{aligned} \tag{20}$$

where $f_0^{i,j}(x,t) = L_i^{(\alpha_1)}(x) T_{L,j}(t)$. Then the spatial partial derivatives $D_x u(x,t)$ and $D_{xx} u(x,t)$ may be written as

$$D_x u(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x (L_i^{(\alpha_1)}(x)) T_{L,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t) \tag{21}$$

$$D_{xx} u(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} D_{xx} L_i^{(\alpha_1)}(x) T_{L,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t), \tag{22}$$

where $f_1^{i,j}(x,t) = D_x (L_i^{(\alpha_1)}(x)) T_{L,j}(t)$ and $f_2^{i,j}(x,t) = D_{xx} L_i^{(\alpha_1)}(x) T_{L,j}(t)$.

Furthermore, the approximation of the time fractional derivative ${}^c D^\nu$ can be computed as

$$\begin{aligned} {}^c D_t^\nu u(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_i^{(\alpha_1)}(x)\, {}^c D_t^\nu T_{L,j}(t) = \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_i^{(\alpha_1)}(x) T_{L,j}^{(\nu)}(t) = \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t), \end{aligned} \tag{23}$$

where

$$T_{L,j}^{(\nu)}(t) = j\sum_{k=0}^{j}(-1)^{j-k} \frac{(j+k-1)!\, 2^{2k}}{(j-k)!\,(2k)!\, L^k} \frac{\Gamma(k+1)}{\Gamma(k+1-\nu)} t^{k-\nu}, \quad f_3^{i,j}(x,t) = L_i^{(\alpha_1)}(x) T_{L,j}^{(\nu)}(t).$$

Now, adopting (20–23), enable one to write (18–19) in the form:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t) = a_1 \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t) + H(x,t), \quad (x,t) \in [0,\infty)\times[0,L]. \tag{24}$$

From the initial conditions immediately, we get

$$\begin{aligned} u(0,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,t) = g_1(t), \\ D_x u(x,t)|_{x=0} &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(0,t) = g_2(x), \\ u(x,0) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,0) = g_3(x). \end{aligned} \tag{25}$$

The functions $f_1^{i,j}(x,t)$, $f_2^{i,j}(x,t)$ and $f_3^{i,j}(x,t)$, can be explicitly obtained by using the information included in Section 2. Now Eq. (24), yields $M(N-1)$ algebraic equations in $(M+1)(N+1)$ unknown expansion coefficients, $a_{i,j}$, $i=0,\cdots,M$, $j=0,\cdots,N$,

$$\sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j} a_{i,j} = H(x_{N,r}^{(\alpha_1)}, t_{L,M,s}), \quad r=1,\cdots,N-1; \quad s=1,\cdots,M, \tag{26}$$

where $F_{r,s}^{i,j} = f_3^{i,j}(x_{N,r}^{(\alpha_1)}, t_{L,M,s}) - a_1 f_2^{i,j}(x_{N,r}^{(\alpha_1)}, t_{L,M,s})$, and the initial conditions give

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0, t_{M,s}) = g_1(t_{M,s}), \quad r=1,\cdots,N-1, \tag{27}$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(0, t_{M,s}) = g_2(t_{M,s}), \quad s=0,\cdots,M, \tag{28}$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{N,r}^{(\alpha_1)}, 0) = g_3(x_{N,r}^{(\alpha_1)}), \quad s=0,\cdots,M, \tag{29}$$

and this in turn, yields $(M+1)(N+1)$ algebraic equations, namely

$$\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j} a_{i,j} = H(x_{N,r}^{(\alpha_1)}, t_{L,M,s}), \quad r=1,\cdots,N-1; \quad s=1,\cdots,M,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0, t_{M,s}) = g_1(t_{M,s}), \quad r=1,\cdots,N-1,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(0, t_{M,s}) = g_2(t_{M,s}), \quad s=0,\cdots,M,\\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{N,r}^{(\alpha_1)}, 0) = g_3(x_{N,r}^{(\alpha_1)}), \quad s=0,\cdots,M.
\end{aligned} \tag{30}$$

The resulting system of algebraic equations (30) is then solved by any standard solver.

4. NUMERICAL SIMULATION

In this section, we present a numerical example to show the accuracy and applicability of the proposed method. Consider the following fractional sub-diffusion equation

$${}^{c}D_t^{\gamma} u = D_{xx} u + t^3 e^{-x}\left(\frac{6t^{-\gamma}}{\Gamma(4-\gamma)} - 1\right), \quad (x,t) \in [0,\infty) \times [0,T], \tag{31}$$

with the initial conditions

$$u(0,t) = t^3, \quad D_x u(x,t)|_{x=0} = -t^3, \quad u(x,0) = 0, \quad (x,t) \in [0,\infty) \times [0,T], \tag{32}$$

and the exact solution given by $u(x,t) = t^3 e^{-x}$, $(x,t) \in [0,\infty) \times [0,T]$.

For the numerical implementations, we consider the domain $[0,20] \times [0,1]$. From Table 1, we see the highly accurate results based on maximum absolute errors using this method. Meanwhile, we list absolute errors for several points in Table 2. In Fig. 1, we see the behavior of absolute error of numerical solution at $t = 0.5$, with $N = 24$, $M = 16$, and $\alpha = 3$, $\gamma = 0.5$.

Table 1

Maximum absolute errors for problem (31) where $\gamma = 0.5$

	(N,M)				
α	(16,16)	(16,20)	(20,16)	(20,20)	(24,16)
1	5.18×10^{-2}	5.18×10^{-2}	4.95×10^{-3}	4.96×10^{-3}	2.49×10^{-4}
2	5.03×10^{-2}	5.03×10^{-2}	1.18×10^{-3}	4.62×10^{-4}	1.04×10^{-4}
3	2.80×10^{-2}	2.80×10^{-2}	1.18×10^{-3}	1.18×10^{-3}	4.61×10^{-5}

Table 2

Absolute error at $N = 24$, $M = 16$, $\gamma = 0.5$, $\alpha = 1$ of the proposed problem

x	t	$E(x,t)$	x	t	$E(x,t)$	x	t	$E(x,t)$
2	0.1	3.37631×10^{-9}	2	0.5	8.0603×10^{-9}	2	1	5.04649×10^{-9}
4		5.86684×10^{-9}	4		2.56594×10^{-9}	4		4.64242×10^{-9}
6		1.21766×10^{-8}	6		1.11927×10^{-8}	6		6.09881×10^{-8}
8		2.06907×10^{-9}	8		2.25883×10^{-10}	8		1.59624×10^{-8}
10		2.91471×10^{-9}	10		1.43967×10^{-8}	10		3.6075×10^{-8}
12		1.31183×10^{-9}	12		4.62057×10^{-8}	12		2.26885×10^{-7}
14		3.74811×10^{-8}	14		3.16998×10^{-7}	14		2.5135×10^{-6}
16		8.67033×10^{-8}	16		1.20567×10^{-6}	16		1.01073×10^{-5}
18		1.98039×10^{-7}	18		2.21201×10^{-6}	18		210591×10^{-5}
20		2.01704×10^{-7}	20		2.96254×10^{-6}	20		1.73909×10^{-5}

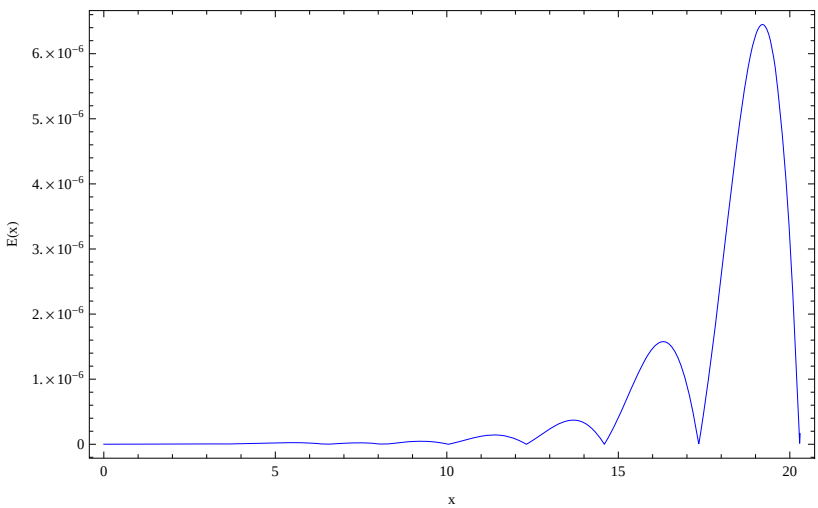

Fig. 1 – x-direction curve of absolute error of problem (31).

5. CONCLUSIONS

In this paper, we have developed and implemented a new numerical algorithm to solve time fractional sub-diffusion equations on semi-infinite domains. The numerical results given in this work demonstrate the good accuracy of this algorithm. Moreover, the algorithm introduced in this paper can be well suited for

handling more general linear and nonlinear fractional partial differential equations. A numerical example was given to demonstrate the applicability and the validity of the proposed algorithm.

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REFERENCES

1. C. CANUTO, M.Y. HUSSAINI, A. QUARTERONI, T.A. ZANG, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, New York, 2006.
2. J.P. BOYD, *Chebyshev and Fourier spectral methods*, 2nd ed. New York, Dover, 2000.
3. E.H. DOHA, A.H. BHRAWY, R.M. HAFEZ, R.V. GORDER, *Jacobi rational-Gauss collocation method for Lane-Emden equations of astrophysical significance*, Nonlinear Anal. Model. Contr., **19**, *4*, pp. 537–550, 2014.
4. A.H. BHRAWY *et al.*, *New spectral techniques for systems of fractional differential equations using fractional-order generalized Laguerre orthogonal functions*, Fract. Calc. Appl. Anal., **17**, *4*, pp. 1137–1157, 2014.
5. E.H. DOHA, A.H. BHRAWY, D. BALEANU, R.M. HAFEZ, *A new Jacobi rational-Gauss collocation method for numerical solution of generalized pantograph equations*, Appl. Numer. Math., **77**, pp. 43–54, 2014.
6. B.Y. GUO, *Gegenbauer approximation and its applications to differential equations on the whole line*, J. Math. Anal. Appl., **226,** pp. 180–206, 1998.
7. A.H. BHRAWY, A.A. AL-ZAHRANI, Y.A. ALHAMED, D. BALEANU, *A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations*, Rom. J. Phys., **59**, pp. 646–657, 2014.
8. E.H. DOHA, A.H. BHRAWY, R.M. HAFEZ, M.A. ABDELKAWY, *A Chebyshev-Gauss-Radau scheme for nonlinear hyperbolic system of first order,* Appl. Math. Inform. Sci., **8**, *2*, pp. 535–544, 2014.
9. E.H. DOHA, A.H. BHRAWY, M.A. ABDELKAWY, R.A. VAN GORDER, *Jacobi-Gauss-Lobatto collocation method for the numerical solution of 1+1 nonlinear Schrödinger equations*, J. Comput. Phys., **261**, pp. 244–255, 2014.
10. J.R. GRAEF, L. KONG, M. WANG, *A Chebyshev spectral method for solving Riemann–Liouville fractional boundary value problems*, Appl. Math. Comp., **241**, pp. 140–150, 2014.
11. R.K. SAEED, J.S. HASSAN, *Solving singular integral equations by using collocation method*, Math. Sci. Lett., **3**, *3*, pp. 185–187, 2014.
12. M. UDDIN, *RBF-PS method and Fourier pseudospectral method for solving stiff nonlinear partial differential equations*, Math. Sci. Lett., **2**, *1*, pp. 55–61, 2013.
13. A.H. BHRAWY, D. BALEANU, *A spectral Legendre-Gauss-Lobatto collocation method for a space-fractional advection diffusion equation with variable coefficients*, Rep. Math. Phys., **72**, pp. 219–233, 2013.
14. A.H. BHRAWY *et al.*, *A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations*, J. Comput. Phys., doi.org/10.1016/j.jcp.2014.03.039, 2014.
15. H. KHALIL, R.A. KHAN, *The use of Jacobi polynomials in the numerical solution of coupled system of fractional differential equations*, Intern. J. Comp. Math., doi:10.1080/00207160.2014.945919, 2014.
16. E.H. DOHA, A.H. BHRAWY, *An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method*, Comput. Math. Appl., **64**, pp. 558–571, 2012.
17. H. WANG, X. ZHANG, *A high-accuracy preserving spectral Galerkin method for the Dirichlet boundary-value problem of variable-coefficient conservative fractional diffusion equations*, J. Comput. Phys., **281**, pp. 67–81, 2015.
18. M.A. EL-KHATEB, M.E. AL-HOHALY, H.S. HUSSIEN, *Spectral Galerkin method for optimal control problems governed by integral and integro- differential equations*, Math. Sci. Lett., **1**, *1*, pp. 33–42, 2012.
19. R. GARRAPPA, M. POPOLIZIO, *On the use of matrix functions for fractional partial differential equations*, Math. Comput. Simul., **81**, pp. 1045–1056, 2011.
20. J.W. KIRCHNER, X. FENG, C. NEAL, *Fractal stream chemistry and its implications for containant transport in catchments*, Nature, **403**, pp. 524–526, 2000.
21. M.A. AKINLAR, A. SECER, A. SECER, M. BAYRAM, *Numerical solution of fractional Benney equation*, Appl. Math. Infor. Sci. **8**, *4*, pp. 1633–1637, 2014.
22. H.A.A. EL-SAKA, *The fractional-order SIR and SIRS epidemic models with variable population size*, Math. Sci. Lett. **2**, *3*, pp. 195–200, 2013.
23. M. GIONA, H.E. ROMAN, *Fractional diffusion equation for transport phenomena in random media*, Physica A, **185**, pp. 87–97, 1992.
24. J. F. GÓMEZ AGUILAR, D. BALEANU, *Solutions of the telegraph equations using a fractional calculus approach*, Proc. Romanian Acad. A, **15**, pp. 27–34, 2014.
25. V.L. CHINCHANE, D.B. PACHPATTE, *New fractional inequalities involving Saigo fractional integral operator*, Math. Sci. Lett. **3**, *3*, pp. 133–139, 2014.
26. A.A.M. ARAFA, S.Z. RIDA, M. KHALIL, *Solutions of fractional order model of childhood diseases with constant vaccination strategy*, Math. Sci. Lett. **1**, pp. 17–23, 2012.

27. G. W. WANG, T. Z. XU, *The improved fractional sub-equation method and its applications to nonlinearfractional partial differential equations*, Rom. Rep. Phys., **66**, pp. 595–602, 2014.
28. X.J. YANG, D. BALEANU, J.H. HE, *Transport equations in fractal porous media within fractional complex transform method*, Proc. Romanian Acad. A, **14**, pp. 287–292, 2013.
29. D. ROSTAMY, M. ALIPOUR, H. JAFARI, D. BALEANU, *Solving multi-term orders fractional differential equations by operational matrices of BPs with convergence analysis*, Rom. Rep. Phys. **65**, pp. 334–349, 2013.
30. J. MA, J. LIU, Z. ZHOU, *Convergence analysis of moving finite element methods for space fractional differential equations*, J. Comput. Appl. Math., **255**, pp. 661–670, 2014.
31. Y. JIANG, J. MA, *High-order finite element methods for time-fractional partial differential equations*, J.Comput. Appl. Math., **235**, *11*, pp. 3285–3290, 2011.
32. H. ZHANG, F. LIU, V. ANH, *Galerkin finite element approximation of symmetric space-fractional partial differential equations*, Appl. Math. Comput., **217**, *6*, pp. 2534–2545, 2010.
33. L. LI, D. XU, M. LUO, *Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation*, J. Comput. Phys., **255**, pp. 471–485,2013.
34. M.M. MEERSCHAERT, C. TADJERAN, *Finite difference approximations for two-sided space-fractional partial differential equations*, Appl. Numer. Math., **56**, *1*, pp. 80–90, 2006.
35. Z. DING, A. XIAO, M. LI, *Weighted finite difference methods for a class of space fractional partial differential equations with variable coefficients*, J. Comput. Appl. Math., **233**, *8*, pp. 1905–1914, 2010.
36. H. WANG, N. DU, *Fast alternating-direction finite difference methods for three-dimensional space-fractional diffusion equations*, J. Comput. Phys., **258**, pp. 305–318, 2014.
37. A.H. BHRAWY, M.A. ZAKY, *A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations*, J. Comput. Phys., in Press, 2014.
38. Z. XIAO-YONG, L. JUNLIN, *Convergence analysis of Jacobi pseudo-spectral method for the volterra delay integro-differential equations*, Appl Math. Infor. Sci., **9**, pp. 135–145, 2015.
39. M. SARBOLAND, A. AMINATAEI, *An efficient numerical scheme for coupled nonlinear Burgers' equations*, Appl. Math. Infor. Sci., **9**, pp. 245–255, 2015.
40. E.H. DOHA, A.H. BHRAWY, D. BALEANU, M.A. ABDELKAWY, *Numerical treatment of coupled nonlinear hyperbolic Klein-Gordon equations*, Rom. J. Phys., **59**, pp. 247–264, 2014.
41. B.-Y. GUO, X.-Y. ZHANG, *A new generalized Laguerre spectral approximation and its applications*, J. Comput. Appl. Math., **181**, pp. 342–363, 2005.
42. D.K. DIMITROV, F. MARCELLAN, F.R. RAFAELI, *Monotonicity of zeros of Laguerre-Sobolev-type orthogonal polynomials,* J. Math. Anal. Appl., **368**, pp. 80–89, 2010.
43. G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., **23**, Amer. Math. Soc., Providence, RI, 4th edition, 1975.

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