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MATHEMATICS

SOME PROPERTIES OF CAYLEY GRAPHS OF CANCELLATIVE SEMIGROUPS

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Abstract. In [S. Panma *et al*., *Characterizations of Clifford semigroup digraphs*, Discrete Math. **306** (12) (2006), 1247–1252] a characterization of Cayley graphs of Clifford semigroups is given. In this paper, first we characterize Cayley graphs of cancellative semigroups, and we give a criterion to check whether a digraph is a Cayley graph of a cancellative semigroup. Also Kelarev and Praeger gave necessary and sufficient conditions for Cayley graphs of semigroups to be vertex-transitive. Then, some authors gave descriptions for all vertex-transitive Cayley graphs of some special classes of semigroups. In this note similar descriptions for all vertex-transitive Cayley graphs of cancellative semigroups are given.

Key words: Cayley graph, vertex-transitive graph, cancellative semigroup.

1. INTRODUCTION

Let S be a semigroup and $C \subseteq S$. Recall that the *Cayley graph* $Cay(S,C)$ of S with the *connection set* C is defined as the digraph with vertex set S and edge set $E(Cay(S,C)) = \{\ (s,\ cs):\ s \in\ S,\ c \in\ C\ \}$.

Cayley graphs of groups have been extensively studied and some interesting results have been obtained (see for example, [21]). Also, the Cayley graphs of semigroups have been considered by some authors (see for example, [2–4, 6–20, 22–23]).

One of the interesting subjects in the study of Cayley graphs of semigroups is considering how the results obtained for the Cayley graphs of groups work in case of semigroups. It is known that the Cayley graphs of groups are *vertex-transitive*; in the sense that for every two vertices u, v there exists a graph automorphism f such that $(u)f = v$ [21]. Also, Kelarev and Praeger in [11] characterized vertex- transitive Cayley graphs of semigroups S for which all principal left ideals of the subsemigroup generated by the connection set C are finite. In fact Kelarev and Praeger gave necessary and sufficient conditions for Cayley graphs of semigroups to be vertex-transitive. Then, some authors gave descriptions for all vertex-transitive Cayley graphs of some special classes of semigroups. In this note similar descriptions for all vertex-transitive Cayley graphs of cancellative semigroups are given.

In [24] a characterization of Cayley graphs of groups was presented by Sabidussi. A characterization of Cayley graphs of Clifford semigroups was presented by Panma *et al.* [22]. Also some combinatorial properties of some classes of semigroups were considered [3–4, 6–22]. In this note, we extend Sabidussi's Theorem and we present a characterization of Cayley graphs of cancellative semigroups.

2. PRELIMINARIES

In this section, we give some preliminaries needed in the sequel on graphs, semigroups, and Cayley graphs of semigroups. For more information on graphs, we refer to [1], and for semigroups see [5].

Recall that a *digraph* (directed graph) $\Gamma = (V, E)$ is a non-empty set $V = V(\Gamma)$ of *vertices*, together with a binary relation $E = E(\Gamma)$ on V. For every vertex u in Γ, let $N_\Gamma^+(u) = \{v \in \Gamma : (u,v) \in E(\Gamma)\}$,

$N_{\Gamma}^{-}(u)=\{v\in\Gamma:(v,u)\in E(\Gamma)\}$, $d_{\Gamma}^{-}(u)=|N_{\Gamma}^{-}(u)|$ and $d_{\Gamma}^{+}(u)=|N_{\Gamma}^{+}(u)|$. The numbers $d_{\Gamma}^{-}(u)$ and $d_{\Gamma}^{+}(u)$ are called in-degree and out-degree of u, respectively. Throughout this paper, by a graph we mean a digraph without multiple edges (possibly with loops). By the *underlying undirected graph* of Γ we mean a graph with the same set of vertices V and an undirected edge $\{u,v\}$ for each directed edge (u,v) of Γ. The graph Γ is said to be *connected* if its underlying undirected graph is connected. By a *connected component* of a digraph Γ we mean any component of the underlying graph of Γ.

Let (V_1,E_1) and (V_2,E_2) be digraphs. A mapping $\varphi:V_1\longrightarrow V_2$ is called a *(digraph) graph homomorphism* if $(u,v)\in E_1$ implies $((u)\varphi,(v)\varphi)\in E_2$, and is called a *(digraph) graph isomorphism* if it is bijective and both of φ and φ^{-1} are graph homomorphisms. Also it is called a *graph monomorphism* if it is one-to-one. A graph homomorphism $\varphi:(V,E)\longrightarrow(V,E)$ is called an *endomorphism*, and a graph isomorphism on $\Gamma=(V,E)$ is said to be an *automorphism*. We denote the set of all endomorphisms on the graph Γ by $End(\Gamma)$, the set of all monomorphisms on the graph Γ by $Mon(\Gamma)$, and the set of all automorphisms on Γ by $Aut(\Gamma)$.

Let S be a semigroup, and C be a subset of S. The *Cayley graph* $Cay(S,C)$ of S *relative to* C is defined as the graph with vertex set S and edge set $E(Cay(S,C))$ consisting of those ordered pairs (s,t) such that $cs=t$ for some $c\in C$. The set C is called the *connection set* of $Cay(S,C)$ (see [7]).

The following proposition, known as Sabidussi's Theorem, gives a criterion for a digraph to be a Cayley graph of a group.

PROPOSITION 2.1. [24]. *A finite digraph $\Gamma=(V,E)$ is a Cayley graph of a group G if and only if the automorphism group $Aut(\Gamma)$ contains a subgroup Δ isomorphic to G such that for every two vertices $u,v\in V$ there exists a unique $\sigma\in\Delta$ such that $(u)\sigma=v$.*

For a Cayley graph $Cay(S,C)$, we denote $End(Cay(S,C))$ by $End_C(S)$, and $Aut(Cay(S,C))$ by $Aut_C(S)$. An element $f\in End_C(S)$ is called a *color-preserving endomorphism* if $cx=y$ implies $c(x)f=(y)f$ for every $x,y\in S$ and $c\in C$. The set of all color-preserving endomorphisms of $Cay(S,C)$ is denoted by $ColEnd_C(S)$, and the set of all color-preserving automorphisms of $Cay(S,C)$ by $ColAut_C(S)$.

The Cayley graph $Cay(S,C)$ is said to be vertex-transitive (*automorphism-vertex transitive)* or $Aut_C(S)$*-vertex-transitive* if, for every two vertices $x,y\in S$, there exists $f\in Aut_C(S)$ such that $(x)f=y$. The notions of $ColAut_C(S)$*-vertex-transitive*, $ColEnd_C(S)$*-vertex-transitive*, and $End_C(S)$*-vertex-transitive* for Cayley graphs are defined similarly.

As a corollary of Proposition 2.1, we have that *every Cayley graph of a group is vertex transitive*. Also in the sequel we apply the following lemma.

LEMMA 2.2. [11, Lemma 6.1]. *Let S be a semigroup, and C be a subset of S.*

i. If $Cay(S,C)$ is $End_C(S)$-vertex-transitive, then $CS=S$;
ii. If Cay*(S,C) is* $ColEnd_C(S)$*-vertex-transitive, then cS=S for each* $c\in C$.

The following propositions describe semigroups S and subsets C of S, satisfying a certain finiteness condition, such that the Cayley graph $Cay(S,C)$ is $ColAut_C(S)$-vertex-transitive ($Aut_C(S)$-vertex-transitive).

PROPPOSITION 2.3. [11, Theorem 2.1]. *Let S be a semigroup, and C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal left ideals of the subsemigroup $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S,C)$ is $ColAut_C(S)$-vertex-transitive if and only if the following conditions hold:*

i. $cS = S$, for all $c \in C$;
ii. $\langle C\rangle$ is isomorphic to a direct product of a right zero semigroup and a group;
iii. $|\langle C\rangle s|$ *is independent of the choice of* $s \in S$.

PROPOSITION 2.4. [11, Theorem 2.2]. *Let S be a semigroup, and C be a subset of S such that all principal left ideals of the subsemigroup $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S,C)$ is $Aut_C(S)$-vertex-transitive if and only if the following conditions hold:*

i. $CS = S$;
ii. $\langle C\rangle$ *is a completely simple semigroup*;
iii. *The Cayley graph* $Cay(\langle C\rangle, C)$ *is* $Aut_C(\langle C\rangle)$ *-vertex-transitive*;
iv. $|\langle C\rangle s|$ *is independent of the choice of* $s \in S$.

Recall that a *right zero semigroup* (*left zero semigroup*) is a semigroup S satisfying the identity $xy = y$ (respectively, $xy = x$), for all $x, y \in S$. An element s in semigroup S is called *left cancellable* (*right cancellable*) if $sr = st$ ($rs = ts$), for $r,t \in S$, implies $r = t$, and is called *cancellable* if S is left cancellable and right cancellable. The semigroup S is called *left cancellative*, *right cancellative* or *cancellative* if all elements of S are left cancellable, right cancellable or cancellable, respectively.

Also, recall that a semigroup is said to be *left simple* (*right simple*) if it has no proper left (right) ideals. A semigroup is called a *left group* (*right group*) if it is left (right) simple and right (left) cancellative. It is known that a semigroup is a right (left) group if and only if it is isomorphic to the direct product of a group and a right (left) zero semigroup (see [6]). A semigroup is *completely simple* if it has no proper ideals and has an idempotent element which is minimal with respect to the partial order $e \leq f \Leftrightarrow e = ef = fe$ on idempotent elements.

Let S be a monoid and $A \neq \varnothing$ be a set. If we have a mapping $\mu : A \times S \to A$, by $\mu(a,s) = as$ such that

a) $a1 = a$,
b) $a(st) = (as)t$, for $a \in A,\ s,t \in S$,

we call A a *right S-act*, which is denoted by A_S, and say S acts on A. Also we say S acts strongly faithfully on A if for $s,t \in S$ the equality $as = at$, for some $a \in A$, implies that $s = t$. We note that some authors call this property a *free action* of the semigroup S.

LEMMA 2.5. [11, Lemma 5.1]. *Let S be a semigroup with a subset C, let $g \in S$ and let C_g be the set of all vertices v of the Cayley graph $Cay(S,C)$ such that there exists a directed path from g to v. Then C_g is equal to the right coset $\langle C\rangle g$.*

LEMMA 2.6. [11, Lemma 5.2, Corollary 5.3]. *Let S be a semigroup with a subset C such that $\langle C\rangle$ is completely simple, and $CS = S$. Then, every connected component of the Cayley graph $Cay(S,C)$ is strongly connected, and for every $v \in S$, the component containing v is equal to $\langle C\rangle v$. Also, if $\langle C\rangle$ is isomorphic to a right group, then the right $\langle C\rangle$-cosets are the connected components of $Cay(S,C)$.*

3. CHARACTERIZATION OF CAYLEY GRAPHS OF CANCELLATIVE MONOIDS

Sabidussi in [20] presented a characterization of Cayley graphs of groups. We note that every finite cancellative semigroup is a group. In this section, we characterize Cayley graphs of cancellative monoids.

LEMMA 3.1. *Let Γ be a digraph. If $Mon(\Gamma)$ has a subsemigroup S such that it acts strongly faithfully on $V(\Gamma)$, then S is a cancellative semigroup.*

Proof. To prove S is a cancellative semigroup, let $\sigma_1,\sigma_2,\sigma_3 \in S$ such that $\sigma_1\sigma_2 = \sigma_1\sigma_3$. So, for every $v \in V(\Gamma)$, $(v)\sigma_1\sigma_2 = (v)\sigma_1\sigma_3$. Let $u = (v)\sigma_1$. Thus $(u)\sigma_2 = (u)\sigma_3$. Since S acts strongly faithfully on $V(\Gamma)$, we get that $\sigma_2 = \sigma_3$. Therefore S is left cancellative. Now we prove that S is right cancellative. For this purpose we assume that $\sigma_2\sigma_1 = \sigma_3\sigma_1$. Similarly to the above we get that $(v)\sigma_2\sigma_1 = (v)\sigma_3\sigma_1$, for every $v \in V(\Gamma)$. Since σ_1 is one-to-one, we conclude that $(v)\sigma_2 = (v)\sigma_3$, for every $v \in V(\Gamma)$. Hence $\sigma_2 = \sigma_3$. Therefore S is cancellative. ■

THEOREM 3.2. *Let Γ be a digraph such that the out-degree of all vertices of Γ are finite and equal to each other. Then Γ is isomorphic to a Cayley graph of a cancellative monoid if and only if $Mon(\Gamma)$ has a submonoid T such that T acts strongly faithfully on $V(\Gamma)$, and there exists $u \in V(\Gamma)$ such that $uT = \{u.\sigma = (u)\sigma \mid \sigma \in T\} = V(T)$.*

Proof. $(\Rightarrow)$ Let $\Gamma = Cay(S,C)$, where S is a cancellative monoid and $C \subseteq S$. First we note that every vertex s of $Cay(S,C)$ is joined to exactly $|C|$ vertices of Γ which are $\{cs \mid c \in C\}$, since S is a cancellative semigroup. Because the out-degree of all vertices of Γ are finite and equal to each other, we get that $|C| < +\propto$. Now we take $T = \{\rho_s \mid s \in S\}$, where $\rho_s : S \to S$ is defined by $(x)\rho_s = xs$, for $x \in S$. Let $s \in S$. We know that S is a cancellative semigroup and so $x = y$ if and only if $(x)\rho_s = xs = ys = (y)\rho_s$. Thus ρ_s is one-to-one. Now we prove that ρ_s is a graph homomorphism. For this purpose, we consider an arbitrary edge (x,y) in $E(Cay(S,C))$. Hence there exists $c \in C$ such that $y = cx$. Therefore $ys = cxs$. Thus $(y)\rho_s = c(x)\rho_s$. Hence $((x)\rho_s,(y)\rho_s) \in E(Cay(S,C))$ and so ρ_s is a graph homomorphism.

By the definition of ρ_s where $s \in S$, we get that $\rho_{s_1}\rho_{s_2} = \rho_{s_1s_2} \in T$, for $s_1, s_2 \in S$. So T is a semigroup. Since S is a monoid, S has an identity 1. Hence ρ_1 is the identity of T and so T is a monoid. Now we show that T acts strongly faithfully on $V(\Gamma)$. To prove it, let $(u)\rho_s = (u)\rho_{s'}$, for $s, s' \in S$ and $u \in V(\Gamma)$. So $us = us'$. Since S is cancellative, $\rho_s = \rho_{s'}$. Therefore T acts strongly faithfully on $V(\Gamma)$.

To complete the proof of the necessary part, we take $u = 1$, where 1 is the identity element of S. Then for every $s \in V(\Gamma) = S$, since $(1)\rho_s = s$, we conclude that $1.T = \{(1)\rho_s \mid \rho_s \in T\} = V(\Gamma)$.

$(\Leftarrow)$ Suppose that there exist $T \le Mon(\Gamma)$ and $u \in V(\Gamma)$ such that $uT = V(\Gamma)$. So, for every $v \in V(\Gamma)$, there exists $\sigma_v \in T$ such that $(u)\sigma_v = v$. Now we prove that for each $v \in V(\Gamma)$, there exists a unique $\sigma_v \in \Gamma$ such that $(u)\sigma_v = v$. Let $\sigma_1,\sigma_2 \in T$ and $(u)\sigma_1 = (u)\sigma_2 = v$. Hence strongly faithfully property of T implies that $\sigma_1 = \sigma_2$. So $\sigma_v \in T$ is unique. By Lemma 3.1, we conclude that T is a cancellative monoid. Let $C = \{\sigma_v \in T \mid v = (u)\sigma_v, v \in N^+_\Gamma(u)\}$. We claim that $\Gamma \cong Cay(T,C)$. For this purpose we define $\psi : \Gamma \longrightarrow Cay(T,C)$ by $(v)\psi = \sigma_v$, where $v \in V(\Gamma)$. By the above discussion ψ is well-defined. We claim that ψ is a graph isomorphism. Since $(v)\psi = \sigma_v = \sigma_{v'} = (v')\psi$ implies that $v = (u)\sigma_v = (u)\sigma_{v'} = v'$, we conclude that $\sigma_v = \sigma_{v'}$ and so ψ is one-to-one. To prove ψ is onto, we consider $\sigma \in T$. We know that $(u)\sigma \in V(\Gamma)$. Thus there exists $v'' \in V(\Gamma)$ such that $v'' = (u)\sigma$. By the above discussion we know that $\sigma = \sigma_{v''}$. Thus $(v'')\psi = \sigma_{v''} = \sigma$. Therefore ψ is onto. Finally we prove that ψ is a graph isomorphism. To prove ψ preserves adjacency, we consider $(v,v') \in E(\Gamma)$. Since $\sigma_v \in T \leq Mon(\Gamma)$ and $(u,x) \in E(\Gamma)$, for each $x \in N_\Gamma^+(u)$, then $\{((u)\sigma_v,(x)\sigma_v) \mid x \in N_\Gamma^+(u)\} \subseteq E(\Gamma)$. We claim that, for every $y \in N_\Gamma^+(v)$, there exists $x \in N_\Gamma^+(u)$ such that $y = (x)\sigma_v$. We know that $\{(v,(x)\sigma_v) \mid x \in N_\Gamma^+(u)\} \subseteq E(\Gamma)$, since $(u,x) \in E(\Gamma)$. Because σ_v is one-to-one, we get that if $x_1, x_2 \in N_\Gamma^+(u)$, and $x_1 \neq x_2$, then $(x_1)\sigma_v \neq (x_2)\sigma_v$. Also we know that $|N_\Gamma^+(u)| = |N_\Gamma^+(v)| < +\infty$. Hence $N_\Gamma^+(v) = \{(x)\sigma_v \mid x \in N_\Gamma^+(u)\}$. Thus since $(v,v') \in E(\Gamma)$, $v' = (x_0)\sigma_v$, for some $x_0 \in N_\Gamma^+(u)$. Also we know that $x_0 = (u)\sigma_{x_0}$. Therefore $v' = (u)\sigma_{x_0}\sigma_v$. Hence $(u)\sigma_{v'} = (u)\sigma_{x_0}\sigma_v$. Since T acts strongly faithfully on $V(\Gamma)$, we conclude that $\sigma_{v'} = \sigma_{x_0}\sigma_v$. Hence $(\sigma_v,\sigma_{v'}) \in E(Cay(T,C))$. Therefore $((v)\psi,(v')\psi) \in E(Cay(T,C))$. Thus ψ preserves adjacency. Now we prove that ψ preserves non-adjacency. For this purpose we consider an arbitrary edge $((v)\psi,(v')\psi) = (\sigma_v,\sigma_{v'}) \in E(Cay(T,C))$. Hence there exists $\sigma_{v''} \in C$ such that $\sigma_{v'} = \sigma_{v''}\sigma_v$. By definition of C, we get that $v'' \in N_\Gamma^+(u)$. Thus $(u)\sigma_{v'} = (u)\sigma_{v''}\sigma_v$. So $v' = (v'')\sigma_v$. Since $(u,v'') \in E(\Gamma)$ and $\sigma \in Mon(\Gamma)$, we conclude that $((u)\sigma_v,(v'')\sigma_v) \in E(\Gamma)$. Thus $(v,v') \in E(\Gamma)$. Therefore ψ preserves non-adjacency. So ψ is a graph isomorphism. Hence $\Gamma \cong Cay(T,C)$. Therefore Γ is isomorphic to a Cayley graph of a cancellative semigroup. ■

Using the above theorem, Sabidussi's Theorem and the fact that every finite cancellative semigroup is a group, we have the following corollary.

COROLLARY 3.3. *Let Γ be a finite graph. Then the monomorphism set of Γ has a submonoid T such that T acts strongly faithfully on $V(\Gamma)$ and there exists $u \in T$ such that $uT = \{u.\sigma = (u)\sigma \mid \sigma \in T\} = V(\Gamma)$ if and only if the automorphism group $Aut(\Gamma)$ contains a subgroup Δ such that $|\Delta| = |V(\Gamma)|$ and for every two vertices $u, v \in V(\Gamma)$ there exists a unique $\sigma \in \Delta$ such that $(u)\sigma = v$.*

4. COLOR-PRESERVING AUTOMORPHISM VERTEX TRANSITIVITY

Kelarev and Praeger in [11] described all semigroups S and all subsets C of S, satisfying a certain finiteness condition, all principal left ideals of $\langle C \rangle$ are finite, such that the Cayley graph $Cay(S,C)$ is $ColAut_C(S)$ – vertex-transitive. Under their condition, every component of $Cay(S,C)$ is finite. In this section it is shown that if S is a cancellative semigroup, then some of the conditions of Proposition 2.3 will be satisfied.

LEMMA 4.1. *Let S be a cancellative semigroup, and suppose that S has no identity. Then there is no pair of elements $e, a \in S$ such that $ea = a$ or $ae = a$.*

Proof. On the contrary suppose that there exists a pair $e, a \in S$ such that $ea = a$. Then $aea = a^2$. So $ae = a$, since S is cancellative. Now for every element $b \in S$, $bea = ba$. Hence $be = b$. Thus $beb = b^2$. So $eb = b$. Therefore S has an identity which is a contradiction. ∎

LEMMA 4.2. *Let S be a cancellative semigroup and there exists $c \in S$ such that $cS = S$. Then S is a monoid.*

Proof. On the contrary suppose that S is not a monoid. Since $cS = S$ and $c \in S$, we conclude that there exists $t \in S$ such that $ct = c$ which is a contradiction by Lemma 4.1. ∎

LEMMA 4.3. *Let S be a cancellative semigroup and there exists $c \in S$ such that $cS = S$. Then S is a monoid and c has an inverse in S.*

Proof. Since cS=S, by Lemma 4.2, we conclude that S is a monoid. Since $cS = S$, there exists $c' \in S$ such that $1 = cc'$. So $cc'c = c$. Hence $c'c = 1$ since S is a cancellative semigroup. Therefore c' is the inverse of c in S. ∎

COROLLARY 4.4. *Let S be a cancellative semigroup, $\varnothing \neq C \subseteq S$ and $Cay(S, C)$ be $ColEnd_C(S)$-vertex-transitive. Then S is a monoid and every element of C has an inverse in S.*

Proof. By Lemmas 2.2 and 4.3, we get the result. ∎

THEOREM 4.5. *Let S be a cancellative semigroup and let C be a nonempty subset of S such that $C = C^{-1}$. Then the following statements are equivalent*

i. $Cay(S, C)$ *is* $ColAut_C(S)$*-vertex-transitive;*

ii. $Cay(S, C)$ *is* $ColEnd_C(S)$*-vertex-transitive;*

iii. $cS = S$, *for all* $c \in C$, *and* $\langle C \rangle$ *is a group.*

Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) By Lemma 2.2, we conclude that $cS = S$, for all $c \in C$. Since $C = C^{-1}$, we conclude that $\langle C \rangle$ is a group.

(iii) ⇒ (i) Since $cS = S$, $CS = S$. Also by hypothesis we know that $\langle C \rangle$ is a group which is a completely simple semigroup. Hence by Lemma 2.6, we conclude that the right $\langle C \rangle$-cosets are the connected components of $Cay(S, C)$.

Choose distinct elements $s, s' \in S$. We will define a mapping $\varphi : S \to S$ such that $(s)\varphi = s'$, and we show that $\varphi \in ColAut_C(S)$. If $\langle C \rangle s = \langle C \rangle s'$, then we define a mapping $\lambda : \langle C \rangle s \to \langle C \rangle s'$ by

$$(x)\lambda = \begin{cases} \alpha s', & \text{if} \quad x = \alpha s, \quad \text{for} \quad \text{some} \quad \alpha \in \langle C \rangle; \\ x, & \text{if} \quad x \notin \langle C \rangle s, \end{cases}$$

which is a graph automorphism. If $\langle C \rangle s \neq \langle C \rangle s'$, then for every $x \in S$, we define

$$(x)\varphi = \begin{cases} \alpha s', & \text{if} \quad x = \alpha s, \quad \text{for} \quad \text{some} \quad \alpha \in \langle C \rangle \\ \alpha s, & \text{if} \quad x = \alpha s', \quad \text{for} \quad \text{some} \quad \alpha \in \langle C \rangle \\ x, & \text{if} \quad x \notin \langle C \rangle s \cup \langle C \rangle s' \end{cases}$$

Now we note that for $s_1, s_2 \in S$, if $\alpha_1 s = s_1 = s_2 = \alpha_2 s$ (or $\alpha_1 s' = s_1 = s_2 = \alpha_2 s'$), then $\alpha_1 = \alpha_2$ and so $\alpha_1 s' = \alpha_2 s'$ (or $\alpha_1 s = \alpha_2 s$). Also if $s_1 = s_2 \notin \langle C\rangle s \bigcup \langle C\rangle s'$, then it is obvious that $(s_1)\varphi = (s_2)\varphi$. Therefore we conclude that φ is well-defined and one-to-one. By the definition of φ, we get that φ is onto. Obviously $(s_1)\varphi = s_2$.

Now we show that φ preserves adjacency and non-adjacency. For this purpose take an arbitrary edge $(t, ct) \in E(Cay(S,C))$, for $t \in S$ and $c \in C$. If $t = \alpha s \in \langle C\rangle s$ where $\alpha \in \langle C\rangle$, then $ct = c\alpha s \in \langle C\rangle s$. So $(ct)\varphi = (c\alpha s)\varphi = c\alpha s'$. Also we know that $(t)\varphi = \alpha s'$. Therefore $(ct)\varphi = (c\alpha s)\varphi = c\alpha s' = c(t)\varphi$ and so $((t)\varphi, (ct)\varphi) \in E(Cay(S,C))$. Similarly if $t = \alpha s' \in \langle C\rangle s'$, we get that $((t)\varphi, (ct)\varphi) \in E(Cay(S,C))$. Also if $x \notin \langle C\rangle s \bigcup \langle C\rangle s'$, then $((t)\varphi, c(t)\varphi) = (t, ct) \in E(Cay(S,C))$. Hence φ preserves adjacency. Similarly we can conclude that φ preserves non-adjacency. Also by the above argument we can conclude that φ preserves color, too. Thus $\varphi \in ColAut_C(S)$. Therefore $Cay(S,C)$ is $ColAut_C(S)$-vertex-transitive. ■

THEOREM 4.6. *Let S be a cancellative semigroup, and let $C \neq \varnothing$ be a subset such that $\langle C\rangle$ is finite. Then the following statements are equivalent*

i. $Cay(S,C)$ *is* $Aut_C(S)-$ *vertex-transitive;*
ii. $Cay(S,C)$ *is* $End_C(S)-$ *vertex-transitive;*
iii. $CS = S$.

Proof. (i) ⇒ (ii) It is obvious.

(ii) ⇒ (iii) By Lemma 2.2, we get the result.

(iii) ⇒ (i) Since $\langle C\rangle$ is a finite cancellative semigroup, we get that $\langle C\rangle$ is a group. So $|\langle C\rangle s|$ is independent of the choice of $s \in S$ and $\langle C\rangle$ is a completely simple semigroup. Also since $\langle C\rangle$ is a group by Proposition 2.1 we know that every Cayley graph of a group is vertex-transitive and so $Cay(\langle C\rangle, C)$ is $Aut_C(\langle C\rangle)-$vertex-transitive. Therefore by Proposition 2.4, we conclude that $Cay(S,C)$ is $Aut_C(S)-$ vertex-transitive. ■

About strongly connected we have the following results.

LEMMA 4.7. *Let S be a semigroup and $C \subseteq S$. If $Cay(S,C)$ is strongly connected, then $S = \langle C\rangle$.*

Proof. We know that $\langle C\rangle \subseteq S$. Let $c \in C$. Since $Cay(S,C)$ is strongly connected, for every vertex $s \in S$, there exists a path between c and s. So there exist $c_1, \dots, c_t \in C$ such that $s = c_1 \dots c_t c$. Hence $s \in \langle C\rangle$. Therefore $S \subseteq \langle C\rangle$. Thus $S = \langle C\rangle$. ■

THEOREM 4.8. *Let S be a cancellative semigroup and C be a subset of S. If $Cay(S,C)$ is strongly connected, then S is a group.*

Poof. By Lemma 4.7, we conclude that $S = \langle C\rangle$. Now we claim that S has an identity. On the contrary suppose S does not have any identity. By Lemma 2.5 we conclude that C_s, the set of all vertices v of $Cay(S,C)$ such that there exists a directed path from s to v, is equal to $\langle C\rangle s$. On the other hand since $Cay(S,C)$ is strongly connected, $\langle C\rangle s = S = \langle C\rangle$. Hence there exists $\alpha \in \langle C\rangle$ such that $\alpha s = s$

which is a contradiction by Lemma 4.1. Therefore S has an identity e. Now we note that $C_s = \langle C \rangle s = \langle C \rangle$, for every $s \in S = \langle C \rangle$. Hence for every $s \in S$ there exists a $s' \in \langle C \rangle$ such that $s's = e$. Similarly there exists s'' such that $s''s' = e$. Hence $ss' = ess' = s''s'ss' = s''es' = s''s' = e$. Thus every element of S has an inverse. Therefore S is a group. ■

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