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A NEW SPECTRAL ALGORITHM FOR TIME-SPACE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH SUBDIFFUSION AND SUPERDIFFUSION

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Abstract. This paper reports a new spectral collocation algorithm for solving time-space fractional partial differential equations with subdiffusion and superdiffusion. In this scheme we employ the shifted Legendre Gauss-Lobatto collocation scheme and the shifted Chebyshev Gauss-Radau collocation approximations for spatial and temporal discretizations, respectively. We focus on implementing the new algorithm for two physical problems, namely, time fractional modified anomalous subdiffusion and fractional nonlinear superdiffusion equations. The numerical results obtained by using this algorithm have been compared with another numerical scheme in order to demonstrate the high accuracy and efficiency of the proposed method.

Key words: time and space fractional diffusion equation, subdiffusion and superdiffusion, legendre Gauss-Lobatto and Chebyshev Gauss-Radau quadratures.

1. INTRODUCTION

In recent years there has been a high level of interest of employing *spectral methods* for numerically solving many types of integral and differential equations, due to their ease in implementation over finite and infinite domains [1–6]. The speed of convergence is one of the great advantages of spectral methods. Besides, spectral methods have exponential rates of convergence; they also have high level of accuracy. Spectral methods have been classified to three types namely, collocation [7, 8], tau [9, 10] and Galerkin [11, 12] methods. Fractional differential equations (FDEs) [13–18] model many phenomena in several fields such as fluid mechanics, chemistry, biology, viscoelasticity, engineering, finance, and physics [19–23]. Most FDEs do not have exact solutions, so approximative methods and numerical techniques have been proposed and developed to find the solutions of such equations. Finite element methods have been presented in [24–26] to obtain the numerical solutions of fractional differential equations. Numerical treatments based on finite difference methods were developed for solving FDEs [27–29]. Recently, several spectral algorithms were designed and developed for numerical solutions of FDEs, see for example [30–34]. The *collocation method* has a wide range of applications, due to its ease of use and adaptability in various problems. We focus basically on proposing a new spectral collocation method for two well-known FDEs from mathematical physics, besides demonstrating the accuracy of this proposed collocation method. The main objective of this article is to propose a new collocation method for the numerical solutions of two types of fractional partial differential equations (FPDEs), namely, the modified anomalous fractional subdiffusion and the fractional nonlinear superdiffusion equations. The proposed method is based upon the shifted Legendre Gauss-Lobatto collocation scheme for spatial discretization in conjunction with Chebyshev Gauss-Radau collocation scheme for temporal discretization. Therefore, we present a fully collocation scheme for solving such problems. The problem is then reduced to a system of algebraic equations. Finally, the accuracy of the proposed method is demonstrated by a test problem, which corresponds to a physically meaningful case. To the best of our knowledge, there are no results on the collocation method for solving nonlinear fractional subdiffusion or superdiffusion equations. This paper is organized as follows. We present few relevant properties of fractional derivatives and shifted Jacobi polynomials in Sec. 2. The proposed scheme is investigated and implemented for the time fractional diffusion model in Sec. 3. Section 4 is devoted to solve time-space fractional superdiffusion equation. A numerical simulation is presented in Sec. 5. Finally, some concluding remarks are given in the last section.

2. FRACTIONAL CALCULUS AND JACOBI POLYNOMIALS

There are several definitions of a fractional integration of order $\nu > 0$, and not necessarily equivalent to each other [35]. Riemann-Liouville and Caputo fractional definitions are the two most used from all the other definitions of fractional calculus.

Definition 2.1. The integral of order $\nu \geq 0$ (fractional) according to Riemann-Liouville is given by

$$\begin{aligned} J^{\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\mathrm{d}t, \ \nu > 0, \ x > 0, \\ J^0 f(x) &= f(x), \end{aligned} \tag{1}$$

The operator J^{ν} satisfies the following properties

$$J^{\nu} J^{\mu} f(x) = J^{\nu+\mu} f(x), \ J^{\nu} J^{\mu} f(x) = J^{\mu} J^{\nu} f(x), \ J^{\nu} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)} \ x^{\beta+\nu}. \tag{2}$$

Definition 2.2. The Riemann-Liouville fractional derivatives of order ν is defined as

$$D^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(\int_0^x (x-t)^{m-\nu-1} f(t)\mathrm{d}t \right), \ m-1 < \nu \leq m, \ x > 0, \tag{3}$$

where m is the ceiling function of ν.

Definition 2.3. The Caputo fractional derivatives of order ν is defined as

$${}^{c}D^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\mathrm{d}t, \ m-1 < \nu \leq m, \ x > 0. \tag{4}$$

Here, we remind the reader of some useful properties of shifted Jacobi polynomials that are most relevant to spectral approximations [36, 37]. It includes Legendre and Chebyshev polynomials as two special cases. The q th derivative of Jacobi polynomial $(P_j^{(\theta,\vartheta)}(x))$, can be obtained from

$$D^{(q)} P_j^{(\theta,\vartheta)}(x) = \frac{\Gamma(j+\theta+\vartheta+q+1)}{2^q \Gamma(j+\theta+\vartheta+1)} P_{j-q}^{(\theta+q,\vartheta+q)}(x). \tag{5}$$

The set of $P_k^{(\theta,\vartheta)}(x)$ consists a complete $L^2_{w^{(\theta,\vartheta)}}$ -orthogonal system with

$$\| P_k^{(\theta,\vartheta)} \|^2_{w^{(\theta,\vartheta)}} = h_k = \frac{2^{\theta+\vartheta+1} \Gamma(k+\theta+1) \ \Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)\Gamma(k+1) \ \Gamma(k+\theta+\vartheta+1)}. \tag{6}$$

We denote by $P_{\ell,k}^{(\theta,\vartheta)}(x) = P_k^{(\theta,\vartheta)}(\frac{2x}{\ell} - 1)$, $\ell > 0$, the shifted Jacobi polynomial of degree k defined on the interval $[0, \ell]$. In virtue of (5), we deduce that

$$D^q P_{\ell,k}^{(\theta,\vartheta)}(0) = \frac{(-1)^{k-q} \Gamma(k+\vartheta+1)(k+\theta+\vartheta+1)_q}{\ell^q \Gamma(k-q+1)\Gamma(q+\vartheta+1)}, \tag{7}$$

$$D^q P_{\ell,k}^{(\theta,\vartheta)}(\ell) = \frac{\Gamma(k+\theta+1)(k+\theta+\vartheta+1)_q}{\ell^q \Gamma(k-q+1)\Gamma(q+\theta+1)}, \tag{8}$$

$$D^m P_{\ell,k}^{(\theta,\vartheta)}(x) = \frac{\Gamma(m+k+\theta+\vartheta+1)}{\ell^m \Gamma(k+\theta+\vartheta+1)} P_{\ell,k-m}^{(\theta+m,\vartheta+m)}(x). \tag{9}$$

Next, let $w_\ell^{(\theta,\vartheta)}(x) = (\ell - x)^\theta x^\vartheta$, then we define the weighted space $L^2_{w_\ell^{(\theta,\vartheta)}}[0,\ell]$ in the usual way, with the following inner product and norm

$$(u,v)_{w_\ell^{(\theta,\vartheta)}} = \int_0^\ell u(x)v(x)w_\ell^{(\theta,\vartheta)}(x)dx, \quad \| v \|_{w_\ell^{(\theta,\vartheta)}} = (v,v)^{\frac{1}{2}}_{w_\ell^{(\theta,\vartheta)}}. \tag{10}$$

The set of shifted Jacobi polynomials forms a complete $L^2_{w_\ell^{(\theta,\vartheta)}}[0,\ell]$-orthogonal system. Due to (10), we obtain

$$\| P_{\ell,k}^{(\theta,\vartheta)} \|^2_{w_\ell^{(\theta,\vartheta)}} = \left(\frac{\ell}{2}\right)^{\theta+\vartheta+1} h_k^{(\theta,\vartheta)} = h_{\ell,k}^{(\theta,\vartheta)}. \tag{11}$$

We denote by $x_{N,j}^{(\theta,\vartheta)}, 0 \le j \le N,$ the nodes of the standard Jacobi-Gauss interpolation on the interval $[-1,1]$. Their corresponding Christoffel numbers are $\varpi_{N,j}^{(\theta,\vartheta)}, 0 \le j \le N.$ The nodes of the shifted Jacobi-Gauss interpolation on the interval $[0,\ell]$ are the zeros of $P_{\ell,N+1}^{(\theta,\vartheta)}(x)$, which we denote by $x_{\ell,N,j}^{(\theta,\vartheta)},\ 0 \le j \le N.$ Clearly $x_{\ell,N,j}^{(\theta,\vartheta)} = \frac{\ell}{2}(x_{N,j}^{(\theta,\vartheta)}+1)$, and their corresponding Christoffel numbers are $\varpi_{\ell,N,j}^{(\theta,\vartheta)} = (\frac{\ell}{2})^{\theta+\vartheta+1}\varpi_{N,j}^{(\theta,\vartheta)},\ 0 \le j \le N.$ Let $S_N(0,\ell)$ be the set of polynomials of degree at most N. Thanks to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi \in S_{2N+1}(0,\ell)$,

$$\begin{aligned}\int_0^\ell (\ell - x)^\theta x^\vartheta \phi(x)\mathrm{d}x &= \left(\frac{\ell}{2}\right)^{\theta+\vartheta+1} \int_{-1}^1 (1-x)^\theta (1+x)^\vartheta \phi\left(\frac{\ell}{2}(x+1)\right)\mathrm{d}x = \\ &= \left(\frac{\ell}{2}\right)^{\theta+\vartheta+1} \sum_{j=0}^N \varpi_{N,j}^{(\theta,\vartheta)} \phi\left(\frac{\ell}{2}(x_{N,j}^{(\theta,\vartheta)}+1)\right) = \sum_{j=0}^N \varpi_{\ell,N,j}^{(\theta,\vartheta)} \phi\left(x_{\ell,N,j}^{(\theta,\vartheta)}\right).\end{aligned} \tag{12}$$

The above integral is exact for $\phi \in S_{2N}(0,\ell)$ and $\phi \in S_{2N-1}(0,\ell)$ in cases of selecting $x_{\ell,N,j}^{(\theta,\vartheta)}, \varpi_{\ell,N,j}^{(\theta,\vartheta)}$ the zeros and weights of Jacobi Gauss-Radau and Jacobi Gauss-Lobatto quadratures, respectively.

3. TIME FRACTIONAL MODIFIED ANOMALOUS SUBDIFFUSION EQUATION

We propose a Legendre-Chebyshev collocation method and describe its implementation for the numerical solution of a time-fractional modified anomalous subdiffusion equation. We approximate the solution of such equation for spatial and temporal discretizations by adapting the Legendre Gauss-Lobatto [38] method in conjunction with Chebyshev Gauss-Radau collocation method [39]. Recently, some models have been used for describing processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator with a nonlinear source term [40, 41, 42]:

$$\partial_t u = (\rho_1 \partial_t^{1-\nu} + \rho_2 \partial_t^{1-\mu})\partial_x^2 u + R(u(x,t),x,t), \quad (x,t) \in [0,\ell]\times[0,\tau], \tag{13}$$

with the initial condition

$$u(x,0) = g_0(x), \quad x \in [0,\ell], \tag{14}$$

and the boundary conditions

$$u(0,t) = g_1(t), \quad u(\ell,t) = g_2(t), \quad t \in [0,\tau], \tag{15}$$

where $0 < \nu, \mu \le 1$, $\rho_1, \rho_2 \ge 0$ are real constants, the nonlinear source term $R(u(x,t),x,t) \in C^2[0,\ell]$ and $g_0(x)$, $g_1(t)$, $g_2(t)$ are given functions. The fractional time derivative operators $\partial_t^{1-\nu}$ and $\partial_t^{1-\mu}$ are defined in terms of the Riemann-Liouville fractional derivative. Now, we outline the main steps of applying the shifted Legendre Gauss-Lobatto interpolation points as collocation nodes for the spatial approximation, meanwhile the shifted Chebyshev Gauss-Radau interpolation points is investigated as collocation nodes for the temporal approximation. To this end, let us expand the approximate solution by means of Legendre and Chebyshev series in the form

$$u_{N,M}(x,t) \quad = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_{\ell,i}(x) T_{\tau,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \tag{16}$$

where

$$f_0^{i,j}(x,t) = L_{\ell,i}(x) T_{\tau,j}(t).$$

Furthermore, the approximation of the spatial partial derivative $\partial_x u_{N,M}(x,t)$ can be computed as

$$\partial_x u_{N,M}(x,t) \quad = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x (L_{\ell,i}(x)) T_{\tau,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t), \tag{17}$$

where $f_1^{i,j}(x,t) = \partial_x (L_{\ell,i}(x)) T_{\tau,j}(t)$. The approximation of the temporal partial derivative $\partial_t u_{N,M}(x,t)$ can be computed as

$$\partial_t u_{N,M}(x,t) \quad = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_{\ell,i}(x) \partial_t (T_{\tau,j}(t)) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t), \tag{18}$$

and

$$\partial_x^2 u_{N,M}(x,t) \quad = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x^2 (L_{\ell,i}(x)) T_{\tau,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_3^{i,j}(x,t), \tag{19}$$

with

$$f_2^{i,j}(x,t) = L_{\ell,i}(x) \partial_t (T_{\tau,j}(t)), \quad f_3^{i,j}(x,t) = \partial_x^2 (L_{\ell,i}(x)) T_{\tau,j}(t). \tag{1}$$

Based on the above treatment of temporal and spatial partial derivatives, the first component of the right hand side of (13), can be obtained explicitly by

$$\begin{aligned} \left(\rho_1 \partial_t^{1-\nu} + \rho_2 \partial_t^{1-\mu}\right) \partial_x^2 u(x,t) \quad &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x^2 (L_{\ell,i}(x)) \left(\rho_1 \partial_t^{1-\nu} + \rho_2 \partial_t^{1-\mu}\right) T_{\tau,j}(t) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_4^{i,j}(x,t), \end{aligned} \tag{20}$$

where

$$f_4^{i,j}(x,t) = \partial_x^2 (L_{\ell,i}(x)) \left(\rho_1 \partial_t^{1-\nu} + \rho_2 \partial_t^{1-\mu}\right) T_{\tau,j}(t).$$

The above relation is expressed explicitly by adapting the Riemann-Liouville fractional derivative of the power series of the shifted Chebyshev polynomial. Therefore, adopting (16–20), enables one to express (13) in the form:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_2^{i,j}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_4^{i,j}(x,t) + R\left(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(x,t), x, t\right), \quad (x,t)\in[0,\ell]\times[0,\tau]. \tag{21}$$

It remains to design an approximation for the initial and boundary conditions. We may approximate these conditions by means of Legendre and Chebyshev polynomials as

$$\begin{aligned} u(x,0) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(x,0) = g_0(x), \\ u(0,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(0,t) = g_1(t), \\ u(\ell,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(\ell,t) = g_2(t). \end{aligned} \tag{22}$$

Now, Eq. (21) is collocated at $(N-1)\times M$ of collocation nodes. Moreover, the initial-boundary conditions in (22) is also collocated at Legendre and Chebyshev collocation nodes. First, we obtain $M\times(N-1)$ algebraic equations for the unknown coefficients $a_{i,j}$, from

$$\begin{aligned} \sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j}\, a_{i,j} &= R\left(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(x,t), \zeta_{N,r}^{\ell}, \eta_{M,s}^{\tau}\right), \\ & r = 1,\cdots,N-1; \quad s = 1,\cdots,M, \end{aligned} \tag{23}$$

where $F_{r,s}^{i,j} = f_2^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}) - f_4^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau})$, while $\zeta_{N,r}^{\ell}$ and $\eta_{M,s}^{\tau}$ are the shifted Legendre Gauss-Lobatto and the shifted Chebyshev Gauss-Rad au quadratures nodes, respectively. Second, treating the initial condition at the Legendre Gauss-Lobatto nodes, leads to $N-1$ algebraic equations

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(\zeta_{N,r}^{\ell},0) = g_0(\zeta_{N,r}^{\ell}), \quad r = 1,\cdots,N-1. \tag{24}$$

Finally, the approximation of the boundary conditions, at the Chebyshev Gauss-Radau nodes, leads to $2M+2$ algebraic equations

$$\begin{aligned} \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(0,\eta_{M,s}^{\tau}) &= g_1(\eta_{M,s}^{\tau}), \\ \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(L,\eta_{M,s}^{\tau}) &= g_2(\eta_{M,s}^{\tau}), \quad s = 0,\cdots,M. \end{aligned} \tag{25}$$

This in turn, yields a system of $(M+1)(N+1)$ algebraic equations

$$\begin{aligned} \sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j}\, a_{i,j} &= R\left(\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}), \zeta_{N,r}^{\ell}, \eta_{M,s}^{\tau}\right), \\ & r = 1,\cdots,N-1; s = 1,\cdots,M, \\ \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(\zeta_{N,r}^{\ell},0) &= g_0(\zeta_{N,r}^{\ell}), \quad r = 1,\cdots,N-1, \\ \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(0,\eta_{M,s}^{\tau}) &= g_1(\eta_{M,s}^{\tau}), \quad s = 0,\cdots,M, \\ \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\, f_0^{i,j}(L,\eta_{M,s}^{\tau}) &= g_2(\eta_{M,s}^{\tau}), \quad s = 0,\cdots,M. \end{aligned} \tag{26}$$

The above system of nonlinear algebraic equations in the expansion coefficients may be solved numerically in a step-by-step manner by using Newton's iterative method.

4. TIME-SPACE FRACTIONAL NONLINEAR SUPERDIFFUSION EQUATION

In this section, we propose an efficient solution for the time-space fractional diffusion equation related to Riemann-Liouville and Caputo fractional derivatives with a nonlinear term [43]:

$$ {}^{c}\partial_t^{\nu} u = \rho_1 \partial_x^{1+\mu} u + \rho_2 u u_x + R(x,t), \quad (x,t) \in [0,\ell]\times[0,\tau], \tag{27}$$

subject to

$$u(x,0) = g_0(x), \quad x \in [0,\ell], \tag{28}$$

$$u(0,t) = g_1(t), \quad u(\ell,t) = g_2(t), \quad t \in [0,\tau], \tag{29}$$

where ρ_1 and ρ_2 are constants, while $R(x,t)$, $g_0(x)$, $g_1(t)$, and $g_2(t)$ are given functions. The approximate solution has a series of the form

$$u_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_{\ell,i}(x) T_{\tau,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \tag{30}$$

The Caputo fractional derivative of the approximate solution is given by

$${}^{c}\partial_t^{\nu} u_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_{\ell,i}(x)\, {}^{c}\partial_t^{\nu}(T_{\tau,j}(t)) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_5^{i,j}(x,t), \tag{31}$$

and ${}^{c}\partial_t^{\nu} T_{\tau,i}(t)$ may be expressed in an explicit form by adapting the Caputo fractional derivative for the power series of the shifted Chebyshev polynomial. Moreover, the Riemann-Liouville fractional partial derivative for space variable is obtained by

$$\partial_x^{1+\mu} u_{N,M}(x,t) \quad = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x^{1+\mu}(L_{\ell,i}(x)) T_{\tau,j}(t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_6^{i,j}(x,t), \tag{32}$$

where $f_6^{i,j}(x,t) = \partial_x^{1+\mu} L_{\ell,i}(x) T_{\tau,j}(t),$ and $\partial_x^{1+\mu} L_{\ell,i}(x)$ may be obtained in an explicit form by applying the Riemann-Liouville fractional derivative of the power series of the shifted Legendre polynomial. Therefore, the fully collocation scheme to (27–29), after employing (30–32), leads to a system of $(M+1)\times(N+1)$ algebraic equations

$$\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_5^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}) = \rho_2 \left(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}) \right)\left(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}) \right) + \\
&\quad + \rho_1 \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_6^{i,j}(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}) + R(\zeta_{N,r}^{\ell},\eta_{M,s}^{\tau}),\ r = 1,\cdots,N-1;\ s = 1,\cdots,M, \\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(\zeta_{N,r}^{\ell},0) = g_0(\zeta_{N,r}^{\ell}),\ r = 1,\cdots,N-1, \\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,\eta_{M,s}^{\tau}) = g_1(\eta_{M,s}^{\tau}),\ s = 0,\cdots,M, \\
&\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(\ell,\eta_{M,s}^{\tau}) = g_2(\eta_{M,s}^{\tau}),\ s = 0,\cdots,M.
\end{aligned} \tag{33}$$

Finally, the above system of nonlinear algebraic equations in the expansion coefficients may be solved by using Newton's iterative method.

Table 1

Maximum absolute errors using the present method and the CD scheme [44]

ν	$N = M$	Our scheme	CD scheme [44]
0.3	5	4.98×10^{-2}	1.34×10^{-4}
	10	1.22×10^{-6}	4.34×10^{-5}
	15	1.65×10^{-7}	1.38×10^{-5}
	20	5.33×10^{-8}	4.40×10^{-6}
0.7	5	3.98×10^{-2}	1.23×10^{-3}
	10	3.11×10^{-7}	5.11×10^{-4}
	15	3.81×10^{-8}	2.09×10^{-4}
	20	9.32×10^{-9}	8.57×10^{-5}

5. NUMERICAL RESULTS

In this section, we shall highlight the accuracy and the efficiency of our method. Consider the fractional subdiffusion equation with Neumann conditions [44]:

$$\partial_t u = \partial_t^{1-\nu}\partial_x^2 u + e^x t^{\nu+1}\left((\nu+2)(x-1)^2 x^2 - \frac{\left(x\left(x^3+6x^2+x-8\right)+2\right)\,\Gamma(\nu+3)t^{\nu}}{\Gamma(2\nu+2)}\right), \tag{34}$$

$$\partial_x u(0,t) = \partial_x u(1,t) = 0, \quad u(x,0) = 0, \quad (x,t)\in[0,1]\times[0,1]. \tag{35}$$

The exact solution is given by

$$u(x,t) = e^x x^2(1-x)^2 t^{\nu+2}, \quad (x,t)\in[0,1]\times[0,1]. \tag{36}$$

In Table 1, we show and compare the maximum absolute errors using the Legendre Gauss-Lobatto method in conjunction with Chebyshev Gauss-Radau collocation method (our scheme) and those that have been presented by Ren *et al*. [44] by implementing the compact difference (CD) scheme [44]. From this table, we conclude that the present method is more accurate than the CD scheme [44]. It is also observed that the maximum absolute error is very small, despite the relatively small number of grid points used. This numerical experiment demonstrates the utility of the method.

6. CONCLUSIONS

In this paper, we have proposed an efficient and accurate algorithm based on the Legendre Gauss-Lobatto method in combination with Chebyshev Gauss-Radau collocation method to obtain the numerical solutions of time-space fractional partial differential equations (FPDEs) with subdiffusion and superdiffusion. The method is based upon reducing the mentioned problem into a system of algebraic equations in the expansion coefficient of the solution. We have outlined the application of Legendre and Chebyshev collocation methods for solving FPDEs. In principle, this algorithm may be extended to related problems, such as to coupled nonlinear FPDEs. One might also consider time-space complex fractional Schrödinger equations and two-sided space FPDEs. We should note that, as a numerical method, we are restricted to solving problems over a finite domain. Hence, this method is particularly well suited for boundary value problems with finite spatial domains. We hope to extend the proposed method using generalized Laguerre polynomials for spatial discretization for problems on a half-line [30]. Moreover, this method may be extended to the two-dimensional case for similar problems.

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REFERENCES

1. C. CANUTO *et al.*, *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, New York, 2006.
2. A.H. BHRAWY, M.A. ZAKY, *A method based on the Jacobi tau approximation for solving multi-term time-space fractional partial differential equations*, J. Comput. Phys., **281**, pp. 876–895, 2015.
3. A.H. BHRAWY, *An efficient Jacobi pseudospectral approximation for nonlinear complex generalized Zakharov system*, Appl. Math. Comput., **247**, pp. 30–46, 2014.
4. E.H. DOHA *et al.*, *An accurate Legendre collocation scheme for coupled hyperbolic equations with variable coefficients.* Rom. J. Phys. **59**, pp. 408–420, 2014.
5. H. SCHAMEL, K. ELSÄSSER, *The application of the spectral method to nonlinear wave propagation*, J. Comput. Phys., **22**, pp. 501–516, 1976.
6. E.H. DOHA *et al.*, *Numerical treatment of coupled nonlinear hyperbolic Klein-Gordon equations*, Rom. J. Phys., **59**, pp. 247–264, 2014.
7. J. MA, B-W. LI, J.R. HOWELL, *Thermal radiation heat transfer in one- and two-dimensional enclosures using the spectral collocation method with full spectrum k-distribution model*, Int. J. Heat Mass Transfer, **71**, pp. 35–43, 2014.
8. X. MA, C. HUANG, *Spectral collocation method for linear fractional integro-differential equations*, Appl. Math. Modell., **38**, pp. 1434–1448, 2014.
9. E.H. DOHA, A.H. BHRAWY, R.M. HAFEZ, *On shifted Jacobi spectral method for high-order multi-point boundary value problems*, Commun. Nonlinear Sci. Numer. Simul., **17**, pp. 3802–3810, 2012.
10. S.R. LAU, H. PRICE, *Sparse spectral-tau method for the three-dimensional helically reduced wave equation on two-center domains*, J. Comput. Phys., **231**, pp. 7695–7714, 2012.
11. M.A. EL-KHATEB, M.E. AL-HOHALY, H.S. HUSSIEN, *Spectral Galerkin method for optimal control problems governed by integral and integro- differential equations*, Mathematical Sciences Letters, **1**, pp. 33–42, 2012.
12. Y.YANG, *Jacobi spectral Galerkin methods for fractional integro-differential equations*, Calcolo, DOI 10.1007/s10092-014-0128-6, 2014.
13. M. ZAYERNOURI, M. AINSWORTH, G.E. KARNIADAKIS, *A unified Petrov-Galerkin spectral method for fractional PDEs,* Computer Methods in Applied Mechanics and Engineering, doi:10.1016/j.cma.2014.10.051, 2014.
14. G.W. WANG, T.Z. XU, *The improved fractional sub-equation method and its applications to nonlinear fractional partial differential equations*, Rom. Rep. Phys., **66**, pp. 595–602, 2014.
15. M.A. AKINLAR *et al.*, *Numerical Solution of Fractional Benney Equation*, Appl Math. Infor. Sci., **8**, pp. 1633–1637, 2014.
16. S.J. SADATI *et al.*, *Some fractional comparison results and stability theorem for fractional time delay systems*, Rom. Rep. Phys., **65**, pp. 94–102, 2013.
17. A.M.O. ANWAR *et al.*, *Fractional Caputo heat equation within the double Laplace transform*, Rom. J. Phys., **58**, pp. 15–22, 2013.
18. A. BORHANIFAR, KH. SADRI, *A new operational approach for numerical solution of generalized functional integro-differential equations*, J. Comput. Appl. Math., **279**, pp. 80–96, 2015.
19. M.H. HEYDARI *et al.*, *Wavelets method for the time fractional diffusion-wave equation*, Phys. Lett. A, **379**, pp. 71–76, 2015.
20. H.A.A. EL-SAKA, *The Fractional-order SIR and SIRS Epidemic Models with Variable Population Size*, Math. Sci. Lett., **2**, pp. 195–200, 2013.
21. G.A. NEMNES, D.V. ANGHEL, *Fractional exclusion statistics in non-homogeneous interacting particle systems*, Rom. Rep. Phys., **66**, pp. 336–358, 2014.
22. A.A.M. ARAFA, S.Z. RIDA, M. KHALIL, *Solutions of Fractional Order Model of Childhood Diseases with Constant Vaccination Strategy*, Mathematical Sciences Letters, **1**, pp. 17–23, 2012.
23. D. CAFAGNA, *Fractional calculus: a mathematical tool from the past for the present engineer*, IEEE Transactions on Industrial Electronics, pp. 35–40, 2007.
24. J. MA, J. LIU, Z. ZHOU, *Convergence analysis of moving finite element methods for space fractional differential equations*, J. Comput. Appl. Math., **255**, pp. 661–670, 2014.
25. B. JIN, R. LAZAROV, Y. LIU, Z. ZHOU, *The Galerkin finite element method for a multi-term time-fractional diffusion equation*, J. Comput. Phys., **281**, pp. 825–843, 2015.
26. L. LI, D. XU, M. LUO, *Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation*, J. Comput. Phys., **255**, pp. 471–485, 2013.
27. M.M. MEERSCHAERT, C. TADJERAN, *Finite difference approximations for two-sided space-fractional partial differential equations*, Appl. Numer. Math., **56**, pp. 80–90, 2006.
28. Z. DING, A. XIAO, M. LI, *Weighted finite difference methods for a class of space fractional partial differential equations with variable coefficients*, J. Comput. Appl. Math., **233**, pp. 1905–1914, 2010.

29. H. WANG, N. DU, *Fast alternating-direction finite difference methods for three-dimensional space-fractional diffusion equations*, J. Comput. Phys., **258**, pp. 305–318, 2014.
30. A.H. BHRAWY, A.A. AL-ZAHRANI, Y.A. ALHAMED, D. BALEANU, *A new generalized Laguerre-Gauss collocation scheme for numerical solution of generalized fractional pantograph equations*, Rom. J. Phys., **59**, pp. 646–657, 2014.
31. P. MOKHTARY, *Reconstruction of exponentially rate of convergence to Legendre collocation solution of a class of fractional integro-differential equations*, J. Comput. Appl. Math., **279**, pp. 145–158, 2015.
32. E.H. DOHA, D. BALEANU, A.H. BHRAWY, R.M. HAFEZ, *A Jacobi collocation method for Troesch's problem in plasma physics*, Proc. Romanian Acad., Series A, **15**, pp. 130–138, 2014.
33. A. KAYEDI-BARDEH, M. ESLAHCHI, M. DEHGHAN, *A method for obtaining the operational matrix of fractional Jacobi functions and applications*, Journal of Vibration and Control, **20**, pp. 736–748, 2014.
34. S. IRANDOUST-PAKCHIN *et al*., *Numerical solution for a class of fractional convection–diffusion equations using the flatlet oblique multiwavelet*s, Journal of Vibration and Control, **20**, pp. 913–924, 2014.
35. K. MILLER, B. ROSS, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons Inc., New York, 1993.
36. G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, XXIII, American Mathematical Society, 1939.
37. Y. LUKE, *The Special Functions and Their Approximations*, Vol. 2, Academic Press, New York, 1969.
38. A.H. BHRAWY, M.A. ZAKY, D. BALEANU, *New numerical approximations for space-time fractional Burgers' equations via a Legendre spectral-collocation method*, Rom. Rep. Phys., **67**, 2015.
39. E.H. DOHA, A.H. BHRAWY, R.M. HAFEZ, M.A. ABDELKAWY, *A Chebyshev-Gauss-Radau scheme for nonlinear hyperbolic system of first order*, Appl. Math. Infor. Sci., **8**, pp. 535–544, 2014.
40. F. LIU, C. YANG, K. BURRAGE, *Numerical method and analytical technique of the modified anomalous subdiffusion equation with a nonlinear source term*, J. Comput. Appl. Math., **231**, pp. 160–176, 2009.
41. Q. LIU, F. LIU, I. TURNER, V. ANH, *Finite element approximation for a modified anomalous subdiffusion equation*, Appl. Math. Model., **35**, pp. 4103–4116, 2011.
42. A. MOHEBBI, M. ABBASZADEH, M. DEHGHAN, *A high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term*, J. Comput. Phys., **240**, pp. 36–48, 2013.
43. C. LI, Z. ZHAO, Y.Q. CHEN, *Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion*, Comput. Math. Appl., **62**, pp. 855–875, 2011.
44. J. REN, Z-Z. SUN, X. ZHAO, *Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions*, J. Comput. Phys., **232**, pp. 456–467, 2013.

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