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SPIN COHERENT STATES DEFINED IN THE BARUT-GIRARDELLO MANNER

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Abstract. In this paper we construct, for the first time to the best of our knowledge, the Barut-Girardello coherent states for the spin systems, and examine some of their properties. We apply the previously deduced diagonal ordering operation technique (DOOT) to the spin coherent states. In this manner, implicitly we have showed that the DOOT can be applied not only to the coherent states of systems with infinite number of energy bound states but also for systems with finite bound states, among them being the spin systems.

Key words: spin systems, coherent states, operator ordering, thermal states.

1. INTRODUCTION

In condensed matter physics and not only, the more natural basis to study the magnetic problems is provided by the use of spin coherent states (CSs) $|z;S>$, where $z=|z|\exp(i\varphi)$ is the complex variable labeling these states and S is the spin quantum number. These states so far have been defined using the Klauder-Perelomov's procedure, i.e. by applying the spin displacement operator to the ground state $|0;S>$. In the present paper we will show that the spin CSs may also be defined in the Barut-Girardello manner, i.e. as the eigenvalues of the lowering operator S_-. Moreover, in different calculations involving CSs it is necessary to achieve commutation of operators or their ordering, according to certain rules. For the harmonic one-dimensional CSs it is used the Integration Within an Ordered Product of Operators (IWOP) technique formulated by Fan et al. (see, e.g. [1] and references therein). In a previous paper [2] we have proposed a new, more general, technique for ordering the product operators applicable only to the normally ordered product of a pair of *generalized raising and lowering operators* A_+ and A_-. These operators define the generalized hypergeometric coherent states (GH-CSs) of Barut-Girardello (BG) kind. As many kinds of Barut-Girardello coherent states (BG-CSs), e. g. the CSs of harmonic oscillator (HO-1D), the pseudoharmonic oscillator and the Morse oscillator, the CSs are also the particular cases of GH-BG-CSs. The new elaborated technique in [2] we have called the *diagonal ordering operation technique* (DOOT) and we have denoted the corresponding operation by the symbol # #. Because the CSs of HO-1D are a particular case of GH-BG-CSs, the IWOP technique [1] can be regarded also as a particular case of the DOOT [2].

2. SPIN OPERATORS AND DOOT

The Fock vectors $|n;S>$, $n=0,1,...,2S$, of a single quantum particle with total spin S form a complete orthonormal set, so that

$$\sum_{n=0}^{2S}|n;S><n;S|=I_{2S+1}, \qquad (1)$$

where I_{2S+1} is the identity operator in the Hilbert space of the single-mode system. The spin operators $S_{\pm} = S_x \pm \mathrm{i} S_y$, $S_3 = S_z$ act on a finite $(2S+1)$-dimensional Hilbert space spanned by Fock vectors $|n;S>$ as follows

$$S_+ | n;S >= \sqrt{(n+1)(2S-n)}\, | n+1;S >, \ \ S_- | n;S >= \sqrt{n \ \ (2S+1-n)}\, | n-1;S >, \tag{2}$$

$$S_z | n;S >= (-S+n) | n;S > . \tag{3}$$

Moreover, the practical use of spin operators frequently requires the normal ordering, i.e. in a product, the raising (creation) operator S_+ is found on the left and the lowering (annihilation) S_- one on the right, S_+S_-. The normal ordered product S_+S_- is a diagonal operator in the Fock vectors basis $|n;S>$:

$$S_+S_- | n;S >= n(2S+1-n) | n;S > \equiv L_S(n) | n;S > \tag{4}$$

or, using the completeness relation:

$$S_+S_- = \sum_{n=0}^{2S} \left[(2S+1)n - n^2 \right] | n;S >< n;S | \equiv \sum_{n=0}^{2S} L_S(n) | n;S >< n;S |, \tag{5}$$

where, for shortness, we have used the following notation $L_S(x) \equiv (2S+1)x - x^2$.

In short, the newly introduced DOOT for GH-BG-CSs is based on the following rules [2]: *a*) Inside the symbol # # the order of the operators A_- and A_+ (generating the GH-BG-CSs) can be permuted like commutable operators, but so that finally will result an operator function that depends only on the powers of normally ordered product A_+A_-, i.e. $\#(A_-)^n (A_+)^n \# = \# (A_+)^n (A_-)^n \# = (A_+A_-)^n$. *b*) A symbol # # inside another symbol # # can be deleted. *c*) If the integration is convergent, expressions containing the normally ordered product of operators can be integrated or differentiated, with respect to *c*-numbers, according to the usual rules. In addition, the *c*-numbers can be taken out from the symbol # #. *d*) The projector $| 0;\lambda >< 0;\lambda |$ of the normalized vacuum (or unperturbed ground) state $| 0;\lambda >$, in the frame of DOOT, has the following normal ordered form:

$$| 0;\lambda >< 0;\lambda | = \# \frac{1}{{}_pF_q(\{a_i\}_1^p ; \{b_j\}_1^q ; A_+A_-)} \#, \tag{6}$$

where λ is a real parameter whose physical meaning differs from one kind of quantum examined system to another. Generally, the projector $| 0;\lambda >< 0;\lambda |$ is just the inverse of the normalization function of the GH-BG-CSs, but having the product operator A_+A_- as variable.

Let us apply this general technique to the spin operators by the following identifications: $A_- \equiv S_-$ and $A_+ \equiv S_+$. In this case the real parameter λ is just the particle total spin S and the integer parameters of GH-BG-CSs are $p = 0$ and $q = 1$ as we will see below.

The ground or vacuum state $| 0;S >$ is defined as usual, i.e. $S_- | 0;S >= 0 | 0;S >$, while the repeatable action of the raising operator on the vacuum state is

$$(S_+)^n | 0;\lambda >= \sqrt{n! \frac{\Gamma(2S+1)}{\Gamma(2S+1-n)}}\, | n;S > \equiv \sqrt{(-1)^n n! (-2S)_n}\, | n;S >, \tag{7}$$

where $\Gamma(x)$ is Euler Gamma function and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, particularly $(-a)_n = (-1)^n \Gamma(a+1)/\Gamma(a+1-n)$. Next we will use the following short notation:

$$\rho(n;S) \equiv n! \frac{\Gamma(2S+1)}{\Gamma(2S+1-n)} \equiv (-1)^n n! (-2S)_n. \tag{8}$$

In this manner, the Fock vectors ket and bra can be expressed as

$$|\,n;S>=\frac{1}{\sqrt{\rho(n;S)}}\left(S_{+}\right)^{n}|\,0;S>\ ;\ <n;S\,|=\frac{1}{\sqrt{\rho(n;S)}}<0;S\,|\left(S_{-}\right)^{n}. \tag{9}$$

Using the DOOT rules, the completeness relation is successively written as

$$\sum_{n=0}^{2S}|\,n;S><n;S\,|=\#\sum_{n=0}^{2S}\frac{1}{\rho(n;S)}\left(S_{+}\right)^{n}|\,0;S>\ <0;S\,|\left(S_{-}\right)^{n}\#=$$
$$=\#|\,0;S><0;S\,|\ \sum_{n=0}^{2S}\frac{1}{(-2S)_{n}}\frac{\left(-S_{+}S_{-}\right)^{n}}{n!}\#=\#|\,0;S><0;S\,|\ \#\#_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#=1, \tag{10}$$

from which we obtain the expression of the vacuum projector:

$$|\,0;S><0;S\,|=\ \#\frac{1}{{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})}\#. \tag{11}$$

We see that the vacuum projector is in fact the inverse of the *truncated operator confluent hypergeometric function* ${}_{0}F_{1}(;\,-2S\,;-S_{+}S_{-})$ because the sum stops at $2S$, i.e. in fact this is the inverse of a *confluent hypergeometric operator polynomial* of degree $2S$, with operatorial variable $S_{+}S_{-}$ defined as

$$\#_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#=\ \sum_{n=0}^{2S}\frac{1}{(-2S)_{n}}\frac{\#\left(-S_{+}S_{-}\right)^{n}\#}{n!}\ =\ \sum_{n=0}^{2S}\frac{\Gamma(2S+1-n)}{\Gamma(2S+1)}\frac{\#\left(S_{+}S_{-}\right)^{n}\#}{n!}. \tag{12}$$

Consequently, the projector on the Fock state $|\,n;S>$ becomes

$$|\,n;S><n;S\,|=\ \#\frac{1}{{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})}\#\frac{1}{(-2S)_{n}}\frac{\#\left(-S_{+}S_{-}\right)^{n}\#}{n!}. \tag{13}$$

Substituting this relation in Eq. (5) and using the DOOT rules, we obtain

$$S_{+}S_{-}\ =\ \#\frac{1}{{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})}\ \left[(2S+1)S_{+}S_{-}\frac{\partial}{\partial S_{+}S_{-}}-\left(S_{+}S_{-}\frac{\partial}{\partial S_{+}S_{-}}\right)^{2}\right]{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#\equiv$$
$$\equiv\#\frac{1}{{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})}L_{S}\left(S_{+}S_{-}\frac{\partial}{\partial S_{+}S_{-}}\right){}_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#. \tag{14}$$

Now we can make a fundamental observation: by acting *only on the truncated confluent hypergeometric polynomial* ${}_{0}F_{1}(;\,-2S\,;-S_{+}S_{-})$, the normally ordered product operator $S_{+}S_{-}$ becomes equivalent to the operator in the square brackets, with respect to the DOOT rules, i.e.

$$S_{+}S_{-}\#_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#\equiv\ \#L_{S}\left(S_{+}S_{-}\frac{\partial}{\partial S_{+}S_{-}}\right){}_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#. \tag{15}$$

Reciprocally, it follows that whenever the operator $\#L_{S}\left(S_{+}S_{-}\,\partial/\partial\,S_{+}S_{-}\right)\#$ acts on the confluent hypergeometric polynomial ${}_{0}F_{1}(;\,-2S\,;-S_{+}S_{-})$ this can be substituted by the normally ordered product operator $S_{+}S_{-}$ acting on the ${}_{0}F_{1}(;\,-2S\,;-S_{+}S_{-})$.

Similarly, for an integer power m of the ordered product $S_{+}S_{-}$ we obtain

$$\#\left(S_{+}S_{-}\right)^{m}\#\ =\#\frac{1}{{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})}\left[L_{S}\left(S_{+}S_{-}\frac{\partial}{\partial S_{+}S_{-}}\right)\right]^{m}{}_{0}F_{1}(\,;-2S;-S_{+}S_{-})\#. \tag{16}$$

Consequently, for a function of the ordered product operator S_+S_-, after power series expansion, $\# f(S_+S_-)\# = \sum_{m=0}^{\infty} c_m \#(S_+S_-)^m \#$, and using the DOOT rules, we obtain

$$\#f\left(S_+S_-\right)\# = \#\frac{1}{{}_0F_1(\,;-2S;-S_+S_-)}\, f\left[L_S\left(S_+S_-\frac{\partial}{\partial S_+S_-}\right)\right]{}_0F_1(\,;-2S;-S_+S_-)\#. \tag{17}$$

Particularly, any integer power $m=1,\ 2,\ \ldots$ of the particle number operator $\boldsymbol{N}$, can be written as

$$\boldsymbol{N}^m = \sum_{n=0}^{2S} n^m \,|\, n;S><n;S\,| = \#\frac{1}{{}_0F_1(;-2S;-S_+S_-)}\left(S_+S_-\frac{\partial}{\partial S_+S_-}\right)^m {}_0F_1(;-2S;-S_+S_-)\# \tag{18}$$

This expression will be useful in the next Section in order to calculate the Mandel parameter.

3. SPIN COHERENT STATES

Let us consider a spin operator $\vec{S}$ with the magnitude S. The spin coherent states (SCSs) so far have been defined in the Klauder-Perelomov (KP) manner, i.e. by applying the spin displacement operator to the ground state $|\,0;S>$ [3, 4]. Now, we can define the SCSs also in the Barut-Girardello (BG) manner [5], as eigenvalues of the lowering spin operator S_- :

$$S_-\,|\,z\,;S>= z\,|\,z\,;S>. \tag{19}$$

So, the Barut - Girardello spin coherent states (BG-SCSs) $|\,z\,;S>$ are defined on the entire complex plane of variable $z = |\,z\,|\exp(\mathrm{i}\varphi)$, $0<|\,z\,|\le\infty$, $0\le\varphi\le 2\pi$, with real parameter $\lambda\equiv S$.

Using the normalization condition $<z\,;S\,|\,z\,;S>=1$, the expansion of BG-SCSs in Fock basis is

$$|\,z;S>=\frac{1}{\sqrt{{}_0F_1(\,;-2S;-|\,z\,|^2)}}\sum_{n=0}^{2S}\frac{z^n}{\sqrt{\rho(n\,;S)}}\,|\,n\,;S>, \tag{20}$$

where ${}_0F_1(;-2S\,;-|\,z\,|^2)$ is the truncated confluent hypergeometric polynomial in variable $|\,z\,|^2$.

The BG-SCSs defined above accomplish all conditions required for a coherent state, as stated in [6]:

1) *Continuity in the complex label*, i.e. if $z'\to z$, then $|\,z';S>\to|\,z\,;S>$:

$$\lim_{z'\to z}\left\|\,|z';S>-|z;S>\right\|^2 = \left[2-\lim_{z'\to z}\left(<z;S\,|\,z';S>+<z';S\,|\,z;S>\right)\right]=0\,. \tag{21}$$

2) The BG-SCSs fulfill *the resolution of unity operator* (or satisfy the completeness relation):

$$\int \mathrm{d}\mu_{0\,,\,1}\left(z\ ;\ S\right)\ |\,z\ ;\ S><z\ ;\ S\,|=1, \tag{22}$$

where the positive defined integration measure is structured as

$$\mathrm{d}\mu_{0\,;\,1}\left(z;S\right)=\frac{\mathrm{d}^2z}{\pi}h_{0\,;\,1}\left(|\,z\,|^2;S\right)=\frac{\mathrm{d}\varphi}{2\pi}\,\mathrm{d}\left(|\,z\,|^2\right)h_{0\,;\,1}\left(|\,z\,|^2;S\right), \tag{23}$$

and where the positive weight function $h_{0;1}\left(|\,z\,|^2;S\right)$ must be determined. Here we have used the indexes 0 and 1 in order to emphasize that the BG-SCSs are in fact *one of particular cases* of the Barut-Girardello generalized hypergeometric coherent states (GH-BG-CSs) with the indexes $p=0$ and $q=1$ [2].

After performing the angular integration we must solve the following relation

$$\sum_{n=0}^{2S}\frac{|\,n;S><n;S\,|}{n!\ \dfrac{\Gamma(2S+1)}{\Gamma(2S+1-n)}}\int_0^\infty \mathrm{d}(|\,z\,|^2)\ (|\,z\,|^2)^n\,h_{0\,;\,1}\left(|\,z\,|^2;S\right)\frac{1}{{}_0F_1(\,;-2S;-|\,z\,|^2)}=1. \tag{24}$$

Changing the exponent $n = s-1$ and the function $\widetilde{h}_{0;1}\left(|z|^2;S\right) \equiv h_{0;1}\left(|z|^2;S\right)\left[{}_0F_1(;-2S;-|z|^2)\right]^{-1}$ leads to the Stieltjes moment problem [7]:

$$\int_0^\infty \mathrm{d}(|z|^2)\,(|z|^2)^n\,\widetilde{h}_{0\,;\,1}\left(|z|^2;S\right) = \Gamma(2S+1)\frac{\Gamma(n+1)}{\Gamma(2S+1-n)}. \tag{25}$$

The solution of this problem i.e. $\widetilde{h}_{0;1}\left(|z|^2;S\right)$ is expressed through the Bessel function of the first kind $J_{2S+1}(2|z|)$ [7] and finally, the integration measure becomes:

$$\mathrm{d}\mu_{0\,;\,1}\left(z\ ;\ S\right) = \Gamma(2S+1)\frac{\mathrm{d}\varphi}{2\pi}\ \mathrm{d}\left(|z|^2\right)\ \frac{1}{|z|^{2S+1}}\ {}_0F_1(\ ;\ -2S\ ;\ -|z|^2)\ J_{2S+1}(2|z|). \tag{26}$$

Generally, the convergence radius $\widetilde{R}$ of the radial integral is determined by calculating the limit $\widetilde{R} = \lim_{n\to\infty}\sqrt[n]{\rho(n;S)}$ [8]. Because $n \le 2S$, if $n \to \infty$, it follows also that $2S \to \infty$, and for the BG-SCSs the convergence radius is infinite [9] (Eq. 8.328.2). So, the BG-SCSs are defined on the whole complex *z*-plane.

3) The BG-SCSs are *normalized but not orthogonal* and this is evident from the overlap, i.e.

$$< z;S\,|\,z';S >= \frac{{}_0F_1(;-2S;-z^*z')}{\sqrt{{}_0F_1(;-2S;-|z|^2)}\sqrt{{}_0F_1(;-2S;-|z'|^2)}}. \tag{27}$$

On the other hand, using the DOOT rules, the BG-SCSs can be written in an operatorial manner as

$$|\,z;S >= \frac{1}{\sqrt{{}_0F_1(;-2S;-|z|^2)}}\,{}_0F_1(;-2S;-z\,S_+)\,|\,0;S >, \tag{28}$$

and similarly for their dual state $< z;S\,|$. Consequently, the BG-SCSs projector is

$$|\,z;S >< z;S\,| = \frac{1}{{}_0F_1(;-2S;-|z|^2)}\#\frac{{}_0F_1(;-2S;-z\,S_+)\ {}_0F_1(;-2S;-z^*\,S_-)}{{}_0F_1(;-2S;-S_+S_-)}\#\ . \tag{29}$$

For $z = 0$ we obtain the projector on the vacuum state in accordance with the rule *d*) of DOOT.

The mean value of an operator $\boldsymbol{A}$ in the BG-SCSs representation, i.e. $< \boldsymbol{A} >_{z;S}$ is

$$< \boldsymbol{A} >_{z;S} \equiv < z;S\,|\,\boldsymbol{A}\,|\,z;S >= \frac{1}{{}_0F_1(;-2S;-|z|^2)}\sum_{n=0}^{2S}\frac{\left(z^*\right)^{n'} z^n}{\sqrt{\rho(n';S)\,\rho(n;S)}} < n';S\,|\,\boldsymbol{A}\,|\,n;S >. \tag{30}$$

In the present paper we are interested *only in diagonal operators* in the Fock vectors basis, with the eigenequation $\boldsymbol{A}\,|\,n;S >= A(n)\,|\,n;S >$ so this relation becomes

$$< \boldsymbol{A} >_{z;S} = \frac{1}{{}_0F_1(;-2S;-|z|^2)}\sum_{n=0}^{2S}\frac{\left(|z|^2\right)^n}{\rho(n;S)}A(n) = \frac{1}{{}_0F_1(;-2S;-|z|^2)}A\left(|z|^2\frac{\partial}{\partial|z|^2}\right){}_0F_1(;-2S;-|z|^2) \tag{31}$$

So, the expectation value of the product operator $\boldsymbol{A}$ in the BG-SCSs representation $< \boldsymbol{A} >_{z;S}$ can be regarded as the eigenvalue of the operator $A\left(|z|^2\,\partial/\partial|z|^2\right)$ and with the corresponding eigenfunction ${}_0F_1(;-2S;-|z|^2)$.

Particularly, for $\boldsymbol{A} = S_+S_-$ the result is

$$< S_+S_- >_{z;S} = < z;S\,|\,L_S\left(S_+S_-\frac{\partial}{\partial S_+S_-}\right)|\,z;S >= \frac{1}{{}_0F_1(;-2S;-|z|^2)}L_S\left(|z|^2\frac{\partial}{\partial|z|^2}\right){}_0F_1(;-2S;-|z|^2) \tag{32}$$

Moreover, for a DOOT ordered function like $\boldsymbol{A} = \# f(S_+S_-)\# = \sum_{m=0}^{\infty} c_m \#(S_+S_-)^m\#$, we obtain also

$$<\# f(S_+S_-)\#>_{z;S} = \frac{1}{{}_0F_1(;\,-2S;-|z|^2)} f\left[L_S\left(|z|^2 \frac{\partial}{\partial |z|^2}\right)\right]{}_0F_1(;\,-2S;-|z|^2)\,. \tag{33}$$

By comparing Eqs. (17) and (33) we can see that it exists a correspondence between the normally ordered product operator S_+S_- and the square of complex variable $|z|^2$, i.e. $S_+S_- \to |z|^2$, respectively ${}_0F_1(;\,-2S;-S_+S_-) \to {}_0F_1(;\,-2S;-|z|^2)$. This means that if we have to compute, in the BG-SCSs representation, the average of some functions that depend only on the ordered product S_+S_-, it is sufficient to simply replace this product of operators with $|z|^2$ and perform the corresponding algebraic operations.

On the other hand, the average of the integer power $m = 1,\ 2,\ ...$ of the particle number operator, or the average values of the moments of weighting distribution probability in the BG-SCSs representation is

$$<\boldsymbol{N}^m>_{z;S} = \frac{1}{{}_0F_1(;-2S;-|z|^2)} \sum_{n=0}^{2S} \frac{\left(|z|^2\right)^n}{\rho(n;S)} n^m = \frac{1}{{}_0F_1(;-2S;-|z|^2)} \left(|z|^2 \frac{\partial}{\partial |z|^2}\right)^m {}_0F_1(;-2S;-|z|^2) \tag{34}$$

This result can be verified by using the direct formula for average, i.e. Eq. (31).

Using this formula we can examine the nature of weighting distribution of BG-SCSs by computing the Mandel parameter [10], [11], which is used as a convenient measure of the statistics of coherent states:

$$Q_{|z|^2} = \frac{<\boldsymbol{N}^2>_{z;S} - \left(<\boldsymbol{N}>_{z;S}\right)^2}{<\boldsymbol{N}>_{z;S}} - 1\,. \tag{35}$$

For BG-SCSs this expression finally can be written like in the manner presented in [8]:

$$Q_{|z|^2} = x\left[\frac{{}_0F_1^{(2)}(;\,-2S;-x)}{{}_0F_1^{(1)}(;\,-2S;-x)} - \frac{{}_0F_1^{(1)}(;\,-2S;-x)}{{}_0F_1(;\,-2S;-x)}\right] = x\frac{d}{dx}\left[\log\left(\frac{d}{dx}\log {}_0F_1(;\,-2S;-x)\right)\right], \tag{36}$$

where $x \equiv |z|^2$ and where we have used the notation ${}_0F_1^{(m)}(;\,-2S;-x) \equiv \left(\frac{d}{dx}\right)^m {}_0F_1(;\,-2S;-x)$.

The behavior of the BG-SCSs, is sub-Poissonian (if $Q_{|z|^2} < 0$), Poissonian (if $Q_{|z|^2} = 0$) or super-Poissonian (if $Q_{|z|^2} > 0$), and this can be examined by calculating the Mandel parameter $Q_{|z|^2}$ with respect to the variable $x = |z|^2$. Thus, the statistical properties of the BG-SCSs are dependent on the analytical properties of the expressions involving function ${}_0F_1(;\,-2S;-x)$ and their derivatives. In other words, because the expectation value $<\boldsymbol{N}>_{z;S}$ is less, equal respectively higher than the variance $(\Delta\boldsymbol{N})^2_{z;S} = <\boldsymbol{N}^2>_{z;S} - \left(<\boldsymbol{N}>_{z;S}\right)^2$, then the statistics are sub-Poissonian, Poissonian, or super-Poissonian.

The probability to occupy the n-th Fock state in the BG-SCSs $|z;S>$ is

$$\mathrm{P}_n(z;S) = |<n;S\,|\,z;S>|^2 = \frac{1}{{}_0F_1(;\,-2S;-|z|^2)} \frac{\left(|z|^2\right)^n}{n!\dfrac{\Gamma(2S+1)}{\Gamma(2S+1-n)}}\,. \tag{37}$$

It is obvious that for (*unphysical*) limit $2S \to \infty$ we recover the standard Poisson distribution $\mathrm{P}_n^{(P)}(z;S)$ with the shape parameter $|z|^2\,(2S)^{-1}$:

$$\lim_{2S\to\infty} \mathrm{P}_n(z;S) = \mathrm{P}_n^{(P)}(z;S) = \exp\left(-\frac{1}{2S}|z|^2\right)\frac{\left(\frac{1}{2S}|z|^2\right)^n}{n!}. \quad (38)$$

In order to calculate this relation we have used the limit [9] (Eq. 8.328.2): $\lim_{|x|\to\infty} \frac{\Gamma(x+a)}{\Gamma(x)} \mathrm{e}^{-a \log x} = 1$.

For finite (physical) values of $2S$, the probability $\mathrm{P}_n(z;S)$ can be either narrower or broader than Poisson distribution and this situation depends on the values chosen for $2S$ and for $|z|^2$.

At the end, we will say that this construction of the BG-SCSs by means of DOOT can be used also in the case of mixed (thermal) states. We present here, only some considerations. As an example, we consider a quantum system of N_{tot} non-interacting particles with spin S placed in an external constant magnetic field $\vec{B}$ in the z-direction, described by the Zeeman Hamiltonian $H_{\text{int}} = -\gamma \vec{B}\vec{S} = \gamma B S_z$, and energy eigenvalues $E_n = \gamma B(-S+n)$, where γ is the gyromagnetic ratio. If the spin system is in thermal contact with a reservoir (or "thermal bath"), the corresponding state is mixed, described by the canonical equilibrium normalized density operator

$$\rho_S = \frac{1}{Z_S}\exp(-\beta H_{\text{int}}) = \frac{1}{Z_S}\exp(-\Theta S_z), \quad (39)$$

where Z_S is the partition function of one spin, $\Theta = \beta\gamma B$, and $\beta = 1/k_B T$, as usual.

We indicate here just that in the Fock-vector basis, by using the DOOT rules, the density operator is expressed as

$$\rho_S = \frac{1}{Z_S} e^{\Theta S} \sum_{n=0}^{2S} \left(e^{-\Theta}\right)^n |n;S><n;S| = \frac{1}{Z_S} e^{\Theta S} \# \frac{{}_0F_1(;-2S;-S_+S_-e^{-\Theta})}{{}_0F_1(;-2S;-S_+S_-)} \#. \quad (40)$$

The exhaustive examination of the mixed (thermal) states of spin systems, using the DOOT, will be the subject matter of a forthcoming paper.

4. CONCLUDING REMARKS

In this paper we have constructed, for the first time to the best of our knowledge, the coherent states for the spin systems using the Barut-Girardello manner (BG-CSs), and have examined some of their properties, by applying the previously deduced diagonal ordering operation technique (DOOT). We showed that the newly constructed coherent states in the Barut-Girardello manner for spin systems satisfy all Klauder's minimal prescriptions for a coherent state: continuity in the complex label, normalization, non orthogonality, and the unity operator resolution, with positive weight function of the integration measure. Using the DOOT we have found an interesting situation: by acting only on the confluent hypergeometric polynomial ${}_0F_1(;-2S;-S_+S_-)$, the normally ordered product operator S_+S_- is equivalent to the operator $\# L_S\left(S_+S_- \partial/\partial S_+S_-\right)\#$, with respect to the DOOT rules. At the same time, the expectation value of the product operator S_+S_- in the BG-CSs representation $<S_+S_->_{z;S}$ can be regarded as the eigenvalue of the operator $L_S\left(|z|^2 \partial/\partial |z|^2\right)$ acting on the corresponding eigenfunction ${}_0F_1(;-2S;-|z|^2)$. This means that if we have to compute, in the BG-CSs representation, the average of some functions that depend only on the ordered product S_+S_-, it is sufficient to simply replace this product of operators with $|z|^2$ and perform the corresponding algebraic operations. This way of approaching (i.e. the DOOT) is not surprising since many kind of coherent states are actually particular cases of more general coherent states – the GH-BG-CSs [10, 12, 13]. For pure states, the DOOT calculations are the same even if the system has finite or infinite number of bound energy states and may be successfully used to avoid many relatively complicated algebraic calculations that appear when using the coherent state formalism.

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