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A NONSTANDARD APPROACH OF HELLY' SELECTION PRINCIPLE IN COMPLETE METRIC SPACES

Grigore CIUREA

Bucharest University of Economic Studies, Department of Applied Mathematics
Piața Romană 6, Bucharest 010374, Romania
E-mail: grigore.ciurea@gmail.com

Abstract. The classical Helly' selection theorem asserts that any infinite set of real functions of one variable $\{f(x): x\in[a, b]\}$, satisfying the condition $|f(a)|$ + Var $(f : [a, b]) \leq C$, contains a pointwise convergent subsequence to a function of bounded variation on $[a, b]$. We generalize this principle to functions with values in complete metric spaces and obtain a similar result to Arzela-Ascoli theorem. Then, we introduce an equivalence relation on the space of bounded functions and equip the quotient space with a metric, which turns it out in a complete metric space. We establish a sufficient condition such that a subset of this complete metric space is compact. To prove these results, we use methods and techniques of nonstandard analysis.

Key words: Helly's theorem, bounded variation.

1. INTRODUCTION

It is well known that the classical Helly's theorem [16] can fail if we drop the uniformly bounded variation condition. The role played by this condition in the proof is similar to the equicontinuous role in Arzela-Ascoli theorem. The proof is based on Jordan's decomposition theorem and the fact that [10] *any bounded sequence of monotone real functions contains a pointwise convergent subsequence*.

Fuchino and Plewik [6] extended this theorem in terms of monotone functions on linearly ordered sets. Using the splitting number, they obtain a positive answer to a problem of S. Saks [19]: *for an arbitrary sequence* $(f_n)_{n\in\mathbf{N}}$ *of real functions, do there exist an infinite* $I \subseteq \mathbf{N}$ *and an uncountable* $Y \subset \mathbf{R}$ *such that, for each* $x \in Y$ *, the sequence of real numbers* $(f_n(x))_{n\in I}$ *has a finite or infinite limit?*

Generally, under the Continuum Hypothesis, a negative answer is given to this question in [19]. If the splitting number is greater than the first uncountable cardinal, a positive answer is given in [6].

For functions of several variables, there are many approaches to the notion of variation. Several authors, including Adams [1], Hahn [8], Hobson [11], Hardy [9], Vitushkin [21], suggested different definitions to the concept of variation for functions of several variables. Namely, they characterized these functions by a series of variations of different dimensions. Even if such an approach is a fruitful one in some constructions (see [13] or [21]), there are significant properties in the one dimensional case that can not be transferred automatically to the multidimensional one. In particular, this is the true for the well-known Helly' selection principle. However, there are some statements essentially close to Helly's principle (see [2] or [3]), but these statements refer to "essential convergence" instead of the pointwise convergence.

To remove this lack, Leonov [14] has introduced a notion of *the total variation* for functions of several variables, which allows to state a multidimensional analog for this principle and some important applications (see [15]).

In the present paper, we give a generalized version for functions of bounded variation from a closed interval I of $\mathbf{R}$ to a complete metric space (X,ρ). To do that, we start with a list of definitions and elementary facts needed for further developments. We also consider the variation for functions from an arbitrary subset A of $\mathbf{R}$ to a metric space X, which is useful in our construction.

Section 2 is devoted to the study of Helly's general theorem. Our core result which enable us to extend this theorem will be proved by nonstandard methods. Thereafter, we can easily find an extended version of Helly' selection theorem with the same conclusion, provided by a similar assumption. It can be seen that this result is fairly close to Arzela-Ascoli's theorem.

The last section focuses on the applications of this principle. We introduce an equivalence relation with respect to a filter F, on the space of all bounded $X-valued$ functions defined on I, denoted by $l_\infty(I,X)$. The quotient space $l_\infty(I,X)_F$ will be equipped with a complete metric. The result of this section, which is a consequence of Helly's general theorem, yields information on compactness in $l_\infty(I,X)_F$.

In a forthcoming paper, using results from [4], we intend to give other applications of this principle in measure theory.

2. PRELIMINARIES AND NOTATIONS

Let $J \subseteq \mathbf{N}$ be an infinite subset. For a sequence $(x_n)_{n\in J}$ its convergence to a point x is denoted by $\lim\limits_{n\in J} x_n = x$. A sequence $(f_n)_{n\in J}$ of functions from a closed interval I of $\mathbf{R}$ to a metric space X, *converges pointwise* to $f : I \to X$, if $\lim\limits_{n\in J} f_n(x) = f(x)$ holds for every $x \in I$. We say that $(f_n)_{n\in J}$ is *pointwise convergent* if there is some function f to which the sequence converges pointwise. If $(x_n)_{n\in J}$ is a sequence of real numbers, we denote by $\limsup\limits_{n\in J} x_n = \inf\limits_k \sup\limits_{n\in J_k} x_n$, respectively $\liminf\limits_{n\in J} x_n = \sup\limits_k \inf\limits_{n\in J_k} x_n$, where $J_k = \{n \in J : n \geq k\}$.

We shall assume throughout this paper that (X,ρ) is a complete metric space, I a compact interval of $\mathbf{R}$ and Δ a dense countable subset in I.

Now, we present a more general formalism for the concept of the total variation of a function.

Definition 1. Let A be an arbitrary subset of I. By a partition of A, denoted by $\pi = \{x_0 < \ldots < x_n\}$, is meant a finite ordered subset of A. For $f : A \to X$, the variation of f over π is

$$\mathrm{var}(f:\pi) = \sum_{i=0}^{n-1} \rho\big(f(x_{i+1}), f(x_i)\big).$$

The total variation of f on A is the number

$$\mathrm{Var}(f:A) = \sup \mathrm{var}(f:\pi),$$

where the supremum is taken over all possible partitions, π*, of* A.

Functions that satisfy

$$\mathrm{Var}(f:A) < \infty$$

are called of *bounded variation* on A and the class of such functions is denoted by $\mathrm{BV}(A,X)$.

An infinite family of functions of $\mathrm{BV}(I,X)$ has uniformly bounded variation, if there exists a positive constant *M* such that $\mathrm{Var}(f:I) \leq M$ for all functions f from this family. The below properties of bounded variation are straightforward or well known, so we omit their proofs [16].

LEMMA 1. *A function f of bounded variation on the set Δ has, at each interior point x of the interval* I*, a right-hand limit* $f(x+0)$ *and a left-hand limit* $f(x-0)$*. Moreover,* $f(a+0)$ *and* $f(b-0)$ *exist, where* a *and* b *are the ends of* I.

LEMMA 2. *If* $f(x) = \lim\limits_{n\in J} f_n(x)$ *in* A*, then* $\mathrm{Var}(f:A) \leq \liminf\limits_{n\in J} \mathrm{Var}(f_n:A)$.

LEMMA 3. *Let* $(f_n)_{n\in\mathbf{N}}$ *be a sequence of functions which has uniformly bounded variation on I, and for all* $x\in\Delta$, *the set* $F_x = \{ f_n(x) : n \in \mathbf{N}\}$ *is compact. Then there exists an infinite subset* $I \subseteq \mathbf{N}$ *such that the sequence* $(f_n)_{n\in I}$ *converges pointwise in* Δ *to a function from* $\mathrm{BV}(\Delta, X)$.

Further, we assume that the reader is familiar with the elements of nonstandard analysis as treated, for example, in [5, 12 or 20]. For an axiomatic viewpoint see [17]. We take a structure including X and the set $\mathbf{R}$ of the reals, and one considers **N**-*saturated* models for this structure [12].

Let ${}^*X$ be the nonstandard extension of X in an **N**-*saturated* model. If $\mathbf{T}$ is the system of open sets, for any $x \in X$ we call

$$m(x) = \bigcap\{{}^*T : x \in T,\, T \in \mathbf{T}\}$$

the *monad* of x. If $y\in {}^*X$ and $y\in m(x)$ for some $x \in X$, we write ${}^\circ y = x$ and say that x is the *standard part* of y. As X is a metric space, the standard part ${}^\circ$ is uniquely determined, if it exists. Any element having a standard part is called *nearstandard.*

By $\mathbf{N}_\infty$ we denote the set ${}^*\mathbf{N} \setminus \mathbf{N}$, where ${}^*\mathbf{N}$ is the extension of $\mathbf{N}$ in our model. The next two results can be found in [20].

LEMMA 4. *Let* $A \subseteq X$. *A point* $x \in X$ *belongs to the closure of* A *if and only if there is some* $y\in {}^*A$ *with* $y \in m(x)$.

LEMMA 5. *For any bounded real sequence* $(x_n)_{n\in\mathbf{N}}$ *we have*

$$\limsup_{n\in\mathbf{N}} x_n = \sup\{{}^\circ x_h : h \in \mathbf{N}_\infty\}\,.$$

3. GENERALIZED HELLY'S THEOREM

At this point we can formulate and prove the Helly' selection principle for functions with values in complete metric spaces. It is interesting to compare the following proof with the classical ones [16], [18]. The next lemma is the core of the generalization of Helly's theorem.

LEMMA 6. *Let* $(f_n)_{n\in\mathbf{N}}$ *be a sequence of functions which has uniformly bounded variation on* I *and assume that for all* $x \in \Delta$, *the set* $F_x = \{f_n(x) : n \in \mathbf{N}\}$ *is compact. Then, there exists a function* f *from* I *to* X *with the following property: for all positive numbers* ε *there exists an infinite subset* $J_\varepsilon \subseteq \mathbf{N}$ *and an at most finite subset* $H_\varepsilon \subseteq I$ *such that* $x \in I \setminus H_\varepsilon$ *and* $n \in J_\varepsilon$ *implies* $\rho\big(f_n(x), f(x)\big) < \varepsilon$.

Proof. Let M be a positive constant such that $\mathrm{Var}(f_n : I) \le M$ for all $n \in \mathbf{N}$. By applying Lemma 3 we may assume that $(f_n)_{n\in\mathbf{N}}$ converges pointwise in Δ to a function f_Δ from $\mathrm{BV}(\Delta, X)$. We first notice from Lemma 4 that for each x in I we can choose y in ${}^*\Delta$ with $y \in m(x)$. Based on Lemma 1 applied to the function f_Δ, there exists $f_{\overset{\circ}{\Delta}}(y)$ in X, and denote this point by $f(x)$. Then the formula

$$f(x) = f_{\overset{\circ}{\Delta}}(y) \tag{1}$$

defines a map from I to X. We check at once that $f(x) = f_\Delta(x)$ for every x in Δ. Given a positive real number ε and p in $\mathbf{N}_\infty$ let

$$A_p = \left\{ x \in I : \rho\big(f_p(x), f(x)\big) \ge \frac{\varepsilon}{2} \right\}.$$

If A_p is empty for all p in $\mathbf{N}_\infty$, by Lemma 5, we have

$$\limsup_{n\in\mathbf{N}} \rho\big(f_n(x), f(x)\big) < \varepsilon$$

for all x in I, and it is easily seen that this is our assertion. Otherwise, there would be some p in $\mathbf{N}_\infty$ such that A_p is nonempty. Since our model is $\mathbf{N}$ - *saturated*, it is sufficient to show that A_p is finite.

Suppose, contrary to our claim, that A_p is infinite and let $m \in \mathbf{N}$ such that $m > \dfrac{4M}{\varepsilon}$. Since A_p is infinite, it follows that there are $x_1,\dots,x_m$ in A_p such that

$$\rho\big(f_p(x_k), f(x_k)\big) \geq \frac{\varepsilon}{2} \tag{2}$$

for $k = 1,\dots, m$. Using the fact that $A_p \cap \Delta = \varnothing$ we deduce that $x_k \notin \Delta$ for $k = 1,\dots, m$. Since Δ is a dense subset in I, we have that there exists y_k in Δ^* such that $y_k \in m(x_k)$ and $f(x_k) = f_{\overset{\circ}{\Delta}}(y_k)$ for $k = 1,\dots, m$. The definition of f introduced in formula (1) implies that for any $y_k > x_k$ there is $\overset{\circ}{x}_k \in \Delta$ such that $\overset{\circ}{x}_k > x_k$ and

$$\rho\left(f(x_k), f(\overset{\circ}{x}_k) \right) < \frac{\varepsilon}{8}. \tag{3}$$

The same conclusion can be drawn for $y_k < x_k$. Without loss of generality, we may assume that the intervals with the ends $x_k, \overset{\circ}{x}_k$ are disjoint. Moreover, the sequence $\big(f_n(x)\big)_{n\in\mathbf{N}}$ converges to $f_\Delta(x)$ provided that $x \in \Delta$, which gives

$$\rho\left(f_p(\overset{\circ}{x}_k), f(\overset{\circ}{x}_k) \right) < \frac{\varepsilon}{8} \tag{4}$$

for $k = 1,\dots, m$. As a consequence of the triangle inequality, we have from (2), (3) and (4) the following estimate:

$$\rho\left(f_p(x_k), f_p(\overset{\circ}{x}_k) \right) > \frac{\varepsilon}{4}. \tag{5}$$

It then follows from (5) that

$$\sum_{k=1}^{m} \rho\left(f_p(x_k), f_p(\overset{\circ}{x}_k) \right) > \frac{m\varepsilon}{4} > M \tag{6}$$

and we have $\mathrm{Var}(f_p : I) > M$. This contradicts the fact that $\mathrm{Var}(f_p : I) \leq M$, which is obtained by the transfer principle, and A_p is finite as required.

Using the previous lemma, we establish the following result which in our view is the Helly' selection theorem for functions with values in complete metric spaces.

THEOREM 1. *Let X be a metric space as above. Let F be an infinite family of $X-$valued functions defined on I, which satisfies the following conditions:*

i) *F has uniformly bounded variation;*

ii) *The closure of the set $F_x = \{f(x) : f \in F\}$ is compact in X for each $x \in I$.*

Then every sequence $(f_n)_{n\in\mathbf{N}} \subseteq F$ contains a pointwise convergent subsequence on I to a function of bounded variation.

Proof. By Lemma 6, we are able to obtain an infinite family $(J_k)_{k\in\mathbf{N}}$ of subsets of $\mathbf{N}$, a function f from I to X, and at most infinite subsets $(H_k)_{k\in\mathbf{N}}$ of I with the following property:

$$\limsup_{n\in J_k} \rho\big(f_n(x), f(x)\big) < \frac{1}{k} \text{ on } I \setminus H_k .$$

Moreover, we may assume that $J_1 \supseteq ... \supseteq J_k \supseteq ...$. Choosing $n_k \in J_k$ such that $n_k < n_{k+1}$ and setting $J = \{n_k : k \in \mathbf{N}\}$, the sequence $(f_p)_{p\in J}$ satisfies

$$\limsup_{p\in J} \rho\big(f_p(x), f(x)\big) < \frac{1}{k} \text{ on } I \setminus H_k ,$$

where $H = \bigcup_{k=1}^{\infty} H_k$ .Therefore, the sequence $(f_p)_{p\in J}$ is uniformly convergent on $I \setminus H$ to f.

By Lemma 3, there exists an infinite subset $L \subseteq J$ such that the sequence $(f_p)_{p\in L}$ converges pointwise on H to f. But $(f_p)_{p\in L}$, as a subsequence of $(f_p)_{p\in J}$, is pointwise convergent on $I \setminus H$ to f, so it is convergent to f on the whole I. Lemma 2 leads now to the fact that f is a function of bounded variation, which was our claim.

4. COMPACTNESS IN $l_\infty(I,X)_F$

In this section we introduce the set-theoretic framework, some definitions and elementary facts which we use to define a compacity criteria in metric spaces of functions. We start with the space $l_\infty(I,X)$ of all functions which satisfy

$$\delta\big(f(I)\big) = \sup_{x,y\in X} \rho\big(f(x), f(y)\big) < \infty .$$

Clearly, $l_\infty(I,X)$ endowed with the uniform metric

$$d\big(f;g\big) = \sup_{x\in X} \rho\big(f(x), g(x)\big)$$

is a complete metric space.

Next, let F be the family of all $A \subseteq I$ such that A^{C} (the complement of A in I) is at most countable. Since F is a filter we can introduce the following notion (see also [7]).

Definition 2. The functions f, g *of* $l_\infty(I,X)$ are called equivalent with respect to the filter F if

$$\{x \in I : f(x) = g(x)\} \in F .$$

Thus, two functions f, g are equivalent if they coincide except on a countable set of points. It is easily seen that this induces an equivalence relation on $l_\infty(I,X)$.

Definition 3. The set of all equivalence classes of $l_\infty(I,X)$ with respect to the filter F is denoted by $l_\infty(I,X)_F$. For an element f in $l_\infty(I,X)$, the corresponding equivalence class in the quotient space $l_\infty(I,X)_F$ is denoted by $\widehat{f}$.

Having introduced the quotient space $l_\infty(I,X)_F$, we shall now pass to another important notion: the distance from our classes. In order to do this, we note that the uniform metric can be weakened, by means of the metric ρ and the filter F, in the following notion.

Definition 4. The following formula defines a pseudometric on $l_\infty(I,X)$:

$$\mu(f,g)=\inf_{A\in F}\sup_{x\in A}\rho\big(f(x),g(x)\big).$$

Now, we can equip the space $l_\infty(I,X)_F$ with a distance.

Definition 5. If $\widehat{f},\widehat{g}$ are two elements of $l_\infty(I,X)_F$, we define the metric

$$d\big(\widehat{f},\widehat{g}\big)=\mu(f,g).$$

Remark 1. This definition verifies the metric axioms and is independent of the special choice of representatives. Furthermore, it can be shown that $l_\infty(I,X)_F$ endowed with this metric is complete. Moreover, we have a canonical isometric embedding of $l_\infty(I,X)$ into its quotient space $l_\infty(I,X)_F$, defined by $\theta(f)=\widehat{f}$.

The best way to understand the significance of these definitions is to see how Lemma 6 works to prove the main theorem. It should now be clear how to obtain the filter F and the complete metric d_F. This metric induces a topology O on $l_\infty(I,X)_F$ in a canonical way. Our main interest in the reminder of this section is devoted to finding out the sufficient conditions under which a subset K of $l_\infty(I,X)_F$ is $O-$*compact.* Helly's general theorem enables us to do that.

THEOREM 2. *Let* $l_\infty(I,X)_F$ *be endowed with the O-topology. If a subset* K *of* $l_\infty(I,X)_F$ *has the following properties:*

i) *the closure of the set* $\{f(x): f\in\theta^{-1}(K)\}$ *is compact in* X*, for each* $x\in I$*;*

ii) *the set* $\theta^{-1}(K)$ *has uniformly bounded variation on* I*, then* K *is O-compact.*

Proof. Let $(\widehat{f}_n)_{n\in\mathbf{N}}$ be a sequence of elements from K. It follows from the proof of Theorem 1 applied to the sequence $(f_n)_{n\in\mathbf{N}}$ that there exists an infinite subset J of $\mathbf{N}$ and a function f from I to X such that $\lim\limits_{n\in J}\mu\big(f_n,f\big)=0$. Again by Theorem 1 and Lemma 2 we give that f is of bounded variation and $\widehat{f}\in K$. Now it is clear that $\lim\limits_{n\in J}d\big(\widehat{f}_n,\widehat{f}\big)=0$. Thus we have shown that K is $O-$*compact* and the proof is complete.

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