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# A LOGISTIC LAW OF GROWTH AS A BASE FOR METHODS OF COMPANY'S LIFE CYCLE PHASES FORECASTING

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**Abstract.** A logistic law of growth can be easily identified in natural sciences, social sciences, physics or technology. In economy, the logistic law of growth, expressed by a logistic law function (S-curve) can be employed to describe the processes of economic growth or a company's cycle of life. However, according to the fact, that in almost all economic phenomena values cannot be limited, the logistic function often doesn't fully work, mainly due to the problem of "limited growth" (expressed by an asymptote). This inconvenience can be avoided by employing the modified S-curve (loglogistic function). In this paper we present the application of the S-curve, as well as the modified S-curve in company's life cycle forecasting. We have also proposed our own method of iterative estimation, which can be used to estimate all functions' parameters, especially those, which cannot be converted to linear form.

***Keywords:*** Constructal law, Law of growth, Forecasting, Business cycle, Time series analysis, S-curve, Modified S-curve.

## 1. INTRODUCTION

The every system, including economic ones, can be described as a flow system enable to generate and evolve structures, which can increase flow access [1, 2]. It also emphasise the time direction of the analysed phenomenon, which can take several forms – one of them is manifested through a logistic law of growth. The logistic law of growth, initially proposed by Verhulst, from the beginning, is employed to analyse natural processes in biology, demography or physics [3–8], where the phenomena are described using statistical and econometric models – basing on an assumption, that some phenomena have a very similar tenor. The logistic law of growth might be applied also in economy. There are, generally, two important factors which influence economic activity – there are technological and sociological progress. Both those factors are caused by human needs, activity, creativity, which, in turn, are the result of political, economic and technical decisions. This activity submits to the logistic growth law, which can take a form of varied economic laws: the law of diminishing returns from the land, or the law of relatively decreasing efficiency of inputs. These laws base not only on experience, but also on empirical research; we can observe, that, in the case of almost every economic process, after a beginning stage, characterized by slow growth, further increase of inputs initiate the dynamic growth of effects, up to certain maximum level. From that moment, the growth of effects is getting smaller, until it reach some stable level, or even, in some cases, a dramatic reduction of results.

The S-curve is a mathematic expression of logistic growth law, and becomes a popular tool for forecasting and analysing of economic processes, which follow the rule of the logistic growth, like GDP growth [8–14] or company's valuation [17]. Despite undeniable advantages, S-curve is characterized by several disadvantages, from which the most important is a problem of limited growth – S-curve has an asymptote, what, in practice, means, that after reaching some level, further growth cannot be achieved. We should be aware, that in many economy, and finance, this limited growth cannot be accepted because those data, cannot be limited in values. This important disadvantage can be avoided by employing the modified S-curve, which is both elastic, and offers the unlimited growth.

The S-curve, as well as the modified S-curve, can be also employed primarily in the analyses of company's cycle of life [16]. It is widely confirmed, that enterprise's development has a cyclical form. Companies, depending on the industry or the environment, are characterized by different development trajectories. The analysis of financial data (the value of the assets, revenues, goodwill), allow the construction of a variety of development trajectories. The shape of growth trajectory depends not only on a financial situation; different shapes characterise also young and mature enterprises [17]. We can generalise, that a healthy company should have the trajectory similar to the logistic function, or the modified logistic function, which describes the classic phases of growth: an initial phase, a phase of intensive growth and a phase of stable development. This observation opens a new field of applications both for the S-curve and the modified S-curve.

The aim of this research is to apply the S-curve, as well as the modified S-curve, in company's life cycle forecasting, what would allow to predict a further trajectory of enterprise's development and company's valuation.

This paper is organised as follow: first we present the formal form of the S-curve and the modified S-curve, as well as, their most important characteristics a simple method of. We also propose our own method of iterative estimation, which can be employ to estimate all functions' parameters, especially those, which cannot be converted to linear form (section 2). Then, we show the application of the S-curve, and the modified S-curve in the company's valuation, based on the analysis of discounted cash flows (DCF) (section 3). All results are concluded in the section 4.

## 2. LOGISTIC GROWTH LAW MATHEMATICAL MODELS

The S-curve is a mathematic expression of logistic growth law, presented, for the first time, by P.F. Verhulst [1]. The S-curve is the only solution of a differential equation, called the Robertson's, Prescott's, Kuznets' law [18–20] (compare: [16, 21]):

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{K}{r} y(K - N). \tag{1}$$

Solution of this equation:

$$N(t) = \frac{K}{1 + e^{b-rt}}, \tag{2}$$

where: $K > 0$, $b > 0$, $r > 0$.

It can be observed, that a rate of change of flows (which can be of a different nature) is directly proportional to the product of $N$ ($a$–$N$), where $y$ is the momentum factor and ($K$–$N$) is an inhibiting factor [22, 23]. When $t$ tends to infinity – the function tends to a maximum value of the ratio (saturation level). This function has two asymptotes $N = 0$ and $N = K$, constituting an interval of variability of a given process. The upper point determines the saturation level. The function has one inflexion point, separating the phase of accelerated growth from the phase of decreasing growth rate.

The literature review suggests a lot of function describing the logistic law, based on the generalized logistic growth function. The most known are: exponential growth, Verhulst logistic equation, Generic Growth Function, Blumberg's equation, Von Bertalanffy's growth equation, Richards growth equation, Richards growth equation, Gompertz growth function, Hyper-Gompertz growth function [8, 12]. Unfortunately, in finance and economy (or even demography) those functions usually do not work well, mainly due to the problem of "the unlimited growth" – in finance variables such as GDP, stock market indexes, salaries, sales or company value cannot be limited. That implies the necessity to employ the modified S-curve, which assumes the unlimited growth, by introducing the ln($t$) factor. The modified function is called the log-logistic function (logarithmic-logistic) and is proposed by Hellwig and Siedlecki [24]. The modification of the S-curve, which takes into account the character of financial data, assumes that:

$$N(t) = \frac{K}{1 + e^{b-rt}} \varphi(t). \tag{3}$$

The best function has a first derivative of the following form:

$$\varphi©(t) = \alpha t^{p}, \text{ for } -1 \le p < 0, \tag{4}$$

Assuming, that $p = -1$, and omitting $\alpha$, we get:

$$\varphi(t) = \ln(t). \tag{5}$$

This modified, log logistic function (modified S-curve) is expressed by the following formula [24-25]:

$$N(t) = \frac{K\ln(t)}{1+e^{b-rt}}, \tag{6}$$

where $K > 0$, $b > 0$, $c > 0$.

First derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} = K\frac{1+e^{b-rt}+rte^{b-rt}\ln t}{t\left(1+e^{b-rt}\right)^2}. \tag{7}$$

Second derivative:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = K\frac{r^2t^2e^{b-rt}\left(e^{b-rt}-1\right)\ln t - 2rte^{b-rt}\left(1+e^{b-rt}\right) - \left(1+e^{b-rt}\right)^2}{t^2\left(1+e^{b-rt}\right)^3}. \tag{8}$$

When examining the function variability graph, its basic properties can be analysed: it is monotonic (for $t_1 < t_2$, $N(t_2) > N(t_1)$) and characterised by the unlimited growth – it doesn't have the extreme points and its first derivative is always positive.

When forecasting and determining the phases of economic cycles using the modified S-curve, the special attention should be paid to the determination of characteristic points – points of inflection and points indicating changes: from the early stage into the intensive growth stage, and from the intensive growth stage into the stable stage. The modified S-curve phase has usually two points of inflection, where the first point is of less importance from the point of view of the growth cycle. The second point of inflection, which usually indicates a centre of the intensive growth phase. Points of inflection, for both functions, indicate the change of function convexity (from convex into concave) what signalises the change of a growth rate.

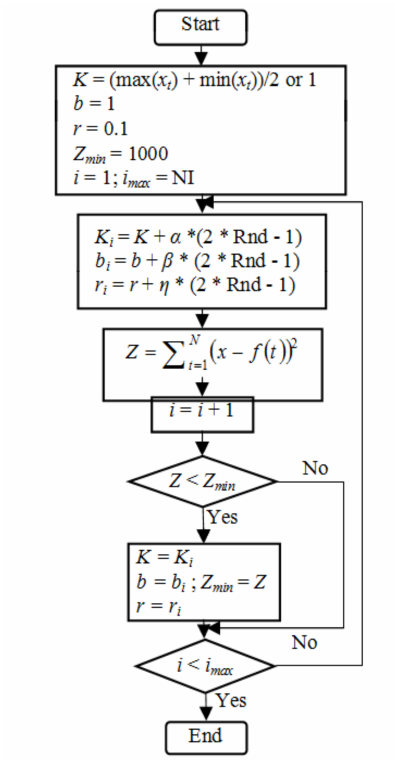


Fig. 1 – Flowchart of numerical estimation of function parameters, where $\alpha > \beta > \eta$ are parameters of algorithm, $N$ is number of observations, $NI$ is number of iterations and Rnd is random number (source: own study).

It can be concluded, that both the first and the second derivative of the modified S-curve are very complex, and it is impossible to estimate its parameters using analitical methods. It is also very difficult to convert this function into a linear form. According to that, we propose the method of parameters' estimation for functions with three or more parameters, *e.g.*,

$$N(t) = \frac{KN_0\ln(t)}{(K-N_0)e^{-rt}+N_0},$$

where $N_0 = \dfrac{K}{1+e^b}$, $N(t) = \ln(t)\left(\dfrac{K}{t}+\dfrac{b}{t^2}+\dfrac{r}{t^3}\right)$.

This is relatively simple, iterative, numerical method (see [14]). The procedure is presented on Fig. 1.

Our proposal is based on a random searching of “the parameters space”. In each step, the search is narrowed due to the use of $\alpha$, $\beta$ and $\eta$. Values of parameters $\alpha$, $\beta$ and $\eta$ are chosen carefully, after several trials, using an experimental method and an empirical constant. This method can be successfully employed not only in the case of the modified S-curve, but also for other functions which cannot be presented in a linear form. The proposed algorithm is not only simple, but, in the same time, efficient and reliable, and can be implemented using popular software tools, like Excel, Matlab, Gretl, etc. This algorithm can be also extended to include an intercept ($d$). For example, as a starting value of $d$ we can choose $x_1$.

## 3. FORECASTING OF A COMPANY LIFE CYCLE' PHASE. A VALUATION PROBLEM

The S-curve, as well as the modified S-curve, allows the extrapolation of long time series, among all the determining of the phases of company's development, revenues and future cash flows, by estimating a moment of transition from one phase of development to another, and, finally, to the saturation level or the slow growth rate. The identification of "turning points" in economic cycle's phases, can be achieved using the analytical or expert methods. Usually, in order to determine those points of transition, historical data, as well as the analysis of a sector, must be employed. The moments of transition into the phase of intensive growth, or into the stagnation period, is detected based on the analysis of I and II derivatives [12]. The moment of transition into the phase of intensive growth, or the moment of transition into the phase of stagnation, is determined by the first derivative, while the second derivative determines the change of a function's bulge. After solving the equation of II derivative, we can observe the following characteristic: the function is convex for $0 \le t < b/r$ and concave for $t > b/r$ (Fig. 2).

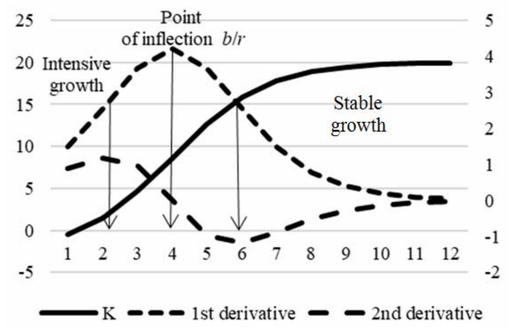


Fig. 2 – Phases of company life cycle based on the S-curve (source: [14]).

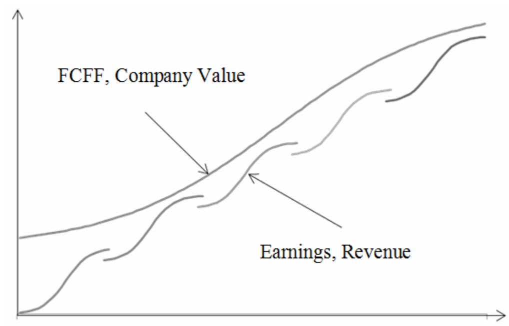


Fig. 3 – Revenue and earnings analysis.

Using financial data (EAT, EBIT), we can conclude, that the S-curve shows a successful start, then grows rapidly, and peaks while transforming to the maturity and decreasing phase. After this moment, we can usually observe "a jump" to the new S-curve – again and again [26]. This "jump" can be achieved by investments in fixed and current assets, tax management or a human capital development, according to the fact, the fact, companies must maximizing owners' wealth (see Fig. 3).

The most popular, both in practice and theory, method of the company's valuation, is discounted cash flow (DCF) method, where the enterprise's value is equal to the sum of all implemented investment projects represented by the sum of the discounted cash flows. The effectiveness of DCF depends on the possibility of identifying value drivers, and thus, is determined by the phases of the company's development (see [15, 27]). In the early stages of development, namely the creation of intense growth, when there is too little information and the valuation is mainly based on the financial planning and analysis, the use of this method is not easy - in the initial period, it is very difficult to determine the company's condition, because companies often have relatively low sales revenues or losses – due to a large scale of investments or a small market share. When a company grows dynamically, different financial variables, such as revenues, profits, assets or working capital, allow the easier analysis and forecasting of cash flows and a cost of capital.

In order to valuate or forecast a cash flows growth, S-curve and modified S-curve can be successfully employed. In this approach, it can be assumed, that the differential equation [18, 23] can be used to describe the formation of a company's cash flow (FCFF):

$$\frac{\mathrm{dFCFF}}{\mathrm{d}t} = \frac{r}{\mathrm{FCFF}_{\max}} \mathrm{FCFF}\left(\mathrm{FCFF}_{\max} - \mathrm{FCFF}\right), \tag{9}$$

for S-curve, but it is not known for modified S-curve, where: $\mathrm{FCFF}_{\max}$ – free cash flow during saturation period (maximum level); $t$ – time

Cash flows can be described as the S-curve, or the modified S-curve:

$$\mathrm{FCFF}(t)=\frac{\mathrm{FCFF}_{\max}}{1+e^{b-rt}},\ \text{or}\ \mathrm{FCFF}(t)=\frac{\ln(t)A}{1+e^{b-rt}}, \tag{10}$$

where $\mathrm{FCFF}_{\max} > 0$, $b > 0$, $r > 0$, $A > 0$.

Assuming that the initial cash flows are negative, we need to modify this function by the inclusion of the intercept. In this case, the intercept ($d$) represents the value of a minimum flow (negative) and the saturation level reach a level $\mathrm{FCFF}_{\max} = a + d$. Accordingly to those assumptions, the goodwill can be described as:

$$V=\int_t^{\infty}\mathrm{FCFF}(t)\cdot e^{-\mathrm{WACC}\cdot t}\mathrm{d}t. \tag{11}$$

The main problem of this method is a little undervaluation, due to the unlimited growth's assumption. Using data for CCC company, which is one of the biggest footwear distributors and one of the biggest shoe manufacturers in Poland, we can present the application of the presented method (ex post valuation). Three years cash flows, in the beginning of the company's activity, are: 27.075 mln PLN (in 2001), 3.903 mln PLN (in 2002), 10.065 mln PLN (in 2003) and 13.834 mln PLN (in 2004).

Assuming that cash flow growth is shaped by the logistic law of growth, with the assumption of a level market saturation at the level of 227.075 million PLN, the transition from the phase of rapid growth takes place after four years, and after seven years – into the stabilization phase (the moment of change of the convexity of the function – five years). Results of the company's valuation, at the beginning of 2005, is presented in Table 1.

*Table 1*

Result of valuation

| **WACC** | DCF basic models | DCF based on S-curve | DCF based on modified S-curve |
|---|---|---|---|
| **Firm value** (in millions PLN) | | | |
| **10%** | 9057.660151 | 1218.03091 | ? |
| **12%** | 2759.104054 | 921.2325028 | ? |
| **Stock price** | | | |
| **10%** | 235.16 | 31.62 | ? |
| **12%** | 75.58 | 23.85 | ? |

Big differences can be observed in then company's value using both models (we compare the DCF model and the S-curve model). DCF model significantly overvalues company's valuation. On the other hand, DCF based on S-curve also bring some undervaluation, because the stock price was, in 2005, between 39-50 PLN. Unfortunately, it is impossible to estimate analysed values based on modified S-curve, because we can't calculate the integral's value.

## 4. CONCLUSION

In this paper we, successfully, show the procedure, which allows to use the modified S-curve, based on the logistic law, in order to determine the shape of company life cycle. This procedure can be also employed in the company's valuation. Based on presented empirical examples, we can form some concluding remarks:

- the modified S-curve is a very good tool, which allow smoothing of time series, because it is monotonic and flexible,
- the modified S-curve allows far extrapolation of economic and financial time series,
- it can be employed in forecasting, or modelling, of economic processes, which follow the logistic growth law.

The paper presents also the concept of valuation methods, using logistic law of growth. That tool can be presented in the functional form, can be effective, especially form the point of view of investors and managers, who analyse company's condition in order to determine a market price or a moment of the exit from the investment, but also for owners, who want to determine a value of its operations for informational purposes, planning the development or transformation into a commercial company.

We still lack:

- a differential equation which help us to describe unlimited growth law,
- possibilities to use the modified S-curve for the company's valuation,
- a calculation of significance of estimated parameters of the S-curve and the modified S-curve,
- a building of interval forecasts based on the modified S-curve (we use RMSE but we think it is not sufficient).

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