THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,*
*Volume 19, Number 2/2018, pp. 321–328*

# COMPARING THE EXPECTED SYSTEM LIFETIMES OF K-OUT-OF-M SYSTEMS USING TRANSMUTED-G DISTRIBUTIONS

Irina BĂNCESCU

University of Bucharest, School of Mathematics
E-mail: irina_adrianna@yahoo.com

**Abstract.** An open problem in reliability is comparison of expected system lifetimes. In this paper, using transmuted-G distributions (Nofaly et al. [5]), we can represent large k-out-of-m and series-parallel/parallel-series systems. We study the asymptotic behavior of the order statistics of these type of systems, along with stochastic ordering, namely likelihood ratio order. An example is provided.

***Key words**:* transmuted-G distributions of order *n*, order statistics, stochastic ordering, series-parallel/parallel-series systems.

## 1. INTRODUCTION

Coherent systems are characterized by parallel-series (a parallel system with series subsystems as components) or series-parallel systems (a series system with parallel subsystems as components). In this work, using transmuted-G distributions introduced in 2016 by Nofaly *et al*. [5], we construct new classes of distributions. These new classes of distributions are obtained by the multiple application of the method mentioned. The advantage of the new distributions is that they can represent the lifetime distribution function of large k-out-of-m and series-parallel/parallel-series systems.

An open problem in reliability and stochastic processes [10] is comparison of systems. This can be done in many ways and considering different types of comparisons. The simpler and most commonly used metric for system performance is the expected system lifetime [7]. Boland and Samaniego [2] discussed this problem, but they only provided conditions for ordering the expected system lifetimes for a particular group of small coherent systems. This paper discusses the stochastic ordering of some particular types of systems which are used in characterizing coherent systems [7]. We also provide some examples.

Finding the asymptotic distribution of a series-parallel/parallel-series system can be quite difficult. In this work, we study the asymptotic behavior of the order statistics of these type of systems, and we show that they are not dependent on all parameters.

This paper is organized as follows. In Section 2, we introduce the class of transmuted-G distributions of order n, along with motivation and interpretation. Section 3 deals with the asymptotic behavior of the order statistics, while Section 4 discusses the stochastic ordering.

## 2. METHOD OF CONSTRUCTION

Let $F$ be an arbitrary continuous cumulative distribution function (CDF) with corresponding density function (PDF) $f$.

Let, for all $n > 1$

$$GT.T_1(F, a_0, b_0, \lambda_0) : F(x)^{a_0}\left[1 + \lambda_0 - \lambda_0 F(x)^{b_0}\right]$$

and

$$GT.T_{n-1}(F, a_0, b_0, \lambda_0, \ldots, a_{n-1}, b_{n-1}, \lambda_{n-1}) : F_n(x) = F_{n-1}(x)^{a_{n-1}}\left[1 + \lambda_{n-1} - \lambda_{n-1} F_{n-1}(x)^{b_{n-1}}\right]. \quad (1)$$

The corresponding PDFs of $F_1, F_2, \cdots, F_n$ are defined recursively as follows

$$f_1(x) = f(x)F(x)^{a_0 - 1}\left[a_0(1+\lambda_0) - \lambda_0(a_0 + b_0)F(x)^{b_0}\right] \quad \text{and}$$

$$f_n(x) = f_{n-1}(x)F_{n-1}(x)^{a_{n-1}-1}\left[a_{n-1}(1+\lambda_{n-1}) - \lambda_{n-1}(a_{n-1} + b_{n-1})F_{n-1}(x)^{b_{n-1}}\right]. \tag{2}$$

where $a_i > 0$, $b_i > 0$ for $-1 \le \lambda_i \le 0$, while for $0 \le \lambda_i \le 1$ we have $a_i + b_i > 0$, $a_i \ge b_i$.

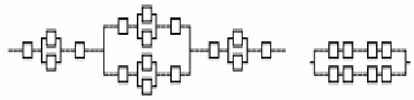

Fig. 1 – $GT.T_2(F, 3, 1, 1, 3, 1, 1)$ and $GT.T_1(F, 4, 4, 1)$.

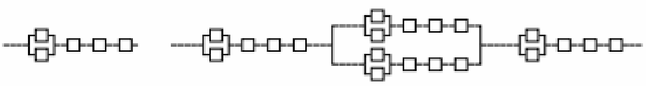

Fig. 2 – $GT.T_1(F, 4, 1, 1)$ and $GT.T_2(F, 4, 1, 1, 3, 1, 1)$.

We denote by $h_i(x) = \dfrac{f_i(x)}{\overline{F_i}(x)}$ the corresponding hazard rate functions, where $\overline{F_i}(x) = 1 - F_i(x)$ are the survival functions, for all $i = \overline{1,n}$. Also, we denote by $GT.T_1(F, a_0, b_0, \lambda_0)$ the class of distributions defined by $F_1$, by $GT.T_2(F, a_0, b_0, \lambda_0, a_1, b_1, \lambda_1)$ the class of distributions defined by $F_2$, and so no, denoting by $GT.T_n(F, a_0, b_0, \lambda_0, \ldots, a_{n-1}, b_{n-1}, \lambda_{n-1})$ the class of distributions defined by $F_n$.

### 2.1. Motivation and interpretation

The $GT.T_n$ classes of distributions can represent k-out-of-m and series-parallel/parallel-series systems. For $b_0 = 1$, $\lambda_0 = 1$ and $a_0$ integer, $F_1$ represents the lifetime function of a complex system with $a_0$ components linked in series: $a_0 - 1$ components have $F$ as their lifetime distribution function, while the last component is a parallel system with two components, each component having $F$ as their lifetime distribution function. For $\lambda_0 = -1$ and $a_0$, $b_0$ integers, $F_1$ represents the lifetime distribution function of a series system with $a_0 + b_0$ components. For $a_0 = b_0$, $a_0$, $b_0$, integers and $\lambda_0 = -1$, $F_1$ represents the lifetime distribution function of a system with $2a_0$ components linked in series. For $a_0 = b_0$, $a_0$, $b_0$ integers and $\lambda_0 = 1$ the distribution function $F_1$ represents the lifetime function of a system with two series subsystems with $a_0$ components, linked in parallel. For $a_0 = b_0 = 1$ and $\lambda_0 = 1$, $F_1$ represents the lifetime distribution function of a parallel system with two components. In Figs. 1 and 2, we have displayed some possible types of systems that can be represented by the $GT.T_n$ models. The empty square represents a system's component that has $F$ as its lifetime distribution function. For $a_i = b_i = 1$, $i = \overline{0, n-1}$, we get the $T_n$ models [9].

Other classes of distributions that model series-parallel/parallel-series systems are the generalized exponentiated distributions (2013) [3] and the generalized exponentiated transmuted distributions (2016) [6]. Many other distribution families are used in reliability and insurance models, [12] introduced the generalized exponential-Poisson distribution with increasing or decreasing failure rate function. [13] studied several optimization problems, under risk measure constraints and applied the results for insurance models, involving the generalized Pareto distribution.

## 3. CHARACTERISTICS OF THE TRANSMUTED-G OF ORDER *N* MODELS

In this section, we discuss the asymptotic behavior of order statistics of the transmuted-G of order n classes of distributions. We prove that the asymptotic distributions of order statistics are not dependent on all parameters. Let

$$C_1=\left\{F\,|\lim_{t\to 0}\frac{F(tx)}{F(t)}<\infty,\forall x\right\},\ C_2=\left\{F\,|\lim_{t\to\infty}\frac{\overline{F}(t+x)}{\overline{F}(t)}<\infty,\forall x\right\},\ C_3=\left\{F\,|\lim_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{h(x)}\right)=0\right\}. \quad (3)$$

THEOREM 1. If $F\in C_1$, then it follows that also $F_1\in C_1$.

*Proof.* We have

$$\lim_{t\to 0}\frac{F_1(tx)}{F_1(t)}=\lim_{t\to 0}\frac{F(tx)^{a_0}}{F(t)^{a_0}}\lim_{t\to 0}\frac{1+\lambda_0-\lambda_0F(tx)^{b_0}}{1+\lambda_0-\lambda_0F(t)^{b_0}}$$

For $b_0>0$, we get $\lim_{t\to 0}\frac{F_1(tx)}{F_1(t)}=l(x)^{a_0}<\infty$, where $l(x)=\lim_{t\to\infty}\frac{F(tx)}{F(t)}<\infty,\forall x$.

*Remark* 1. Theorem 1 shows that the limit of $F_1$ from $C_1$ is dependent only of $a_0$, so invariant to $\lambda_0$ and $b_0$.

THEOREM 2. If $F\in C_2$, then it follows that also $F_1\in C_2$.

*Proof.* Using l'Hospital rule, we obtain

$$\lim_{t\to\infty}\frac{1-F_1(t+x)}{1-F_1(t)}=\lim_{t\to\infty}\frac{f(t+x)F(t+x)^{a_0-1}\left[a_0(1+\lambda_0)-\lambda_0(a_0+b_0)F(t+x)^{b_0}\right]}{f(t)F(t)^{a_0-1}\left[a_0(1+\lambda_0)-\lambda_0(a_0+b_0)F(t)^{b_0}\right]}.$$

As $l(x)=\lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)}=\lim_{t\to\infty}\frac{f(t+x)}{f(t)}<\infty$, we get $\lim_{t\to\infty}\frac{1-F_1(t+x)}{1-F_1(t)}=l(x)$, for all real parameters $b_0>0$.

*Remark* 2. Theorem 2 shows that the limit of $F_1$ from $C_2$ is invariant with respect to $a_0$, $b_0$ and $\lambda_0$.

THEOREM 3. *If* $F\in C_3$, *then it follows that also* $F_1\in C_3$.

*Proof.* We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{h_1(x)}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-F(x)^{a_0}\left[1+\lambda_0-\lambda_0F(x)^{b_0}\right]}{f(x)F(x)^{a_0-1}\left[a_0(1+\lambda_0)-\lambda_0(a_0+b_0)F(x)^{b_0}\right]}\right).$$

Rewriting $1-F(x)^{a_0}\left[1+\lambda_0-\lambda_0F(x)^{b_0}\right]=1-\left(1-\overline{F}(x)\right)^{a_0}\left[1+\lambda_0-\lambda_0\left(1-\overline{F}(x)\right)^{b_0}\right]$ and using the power series expansion $(1-z)^{\rho-1}=\sum_{i\geq 0}\beta_iz^i$, $|z|<1$, $\rho$ real non-integer, $\beta_i=\sum_{i\geq 0}\frac{(-1)^i\Gamma(\rho)}{\Gamma(\rho-i)i!}$ and $\Gamma(x)$ is the $\Gamma$ function, we obtain

$$1-F_1(x)=1-\left(1+\sum_{i\geq 1}\gamma_i\overline{F}(x)^i\right)\left[1+\lambda_0-\lambda_0\left(1+\sum_{i\geq 1}\gamma_i'\overline{F}(x)^i\right)\right]=1-\left(1+\sum_{i\geq 1}\gamma_i\overline{F}(x)^i\right)\left(1-\lambda_0\sum_{i\geq 1}\gamma_i'\overline{F}(x)^i\right)$$

$$=\overline{F}(x)\left[-\sum_{i\geq 1}\gamma_i\overline{F}(x)^{i-1}+\lambda_0\sum_{i\geq 1}\gamma_i'\overline{F}(x)^{i-1}\left(1+\sum_{i\geq 1}\gamma_i\overline{F}(x)^i\right)\right].$$

where $\gamma_i = \dfrac{(-1)^i \Gamma(a_0+1)}{\Gamma(a_0+1-i)i!}$ and $\gamma'_i = \dfrac{(-1)^i \Gamma(b_0+1)}{\Gamma(b_0+1-i)i!}$ . Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{h_1(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\overline{F}(x)\left[-\sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1} + \lambda_0 \sum_{i\ge1}\gamma'_i \overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma_i \overline{F}(x)^i\right)\right]}{f(x)F(x)^{a_0-1}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right]}\right) = T_1(x) + T_2(x),$$

where

$$T_1(x) = \frac{-\sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1} + \lambda_0 \sum_{i\ge1}\gamma'_i \overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma_i \overline{F}(x)^i\right)}{F(x)^{a_0-1}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right]} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\overline{F}(x)}{f(x)}\right),$$

and

$$T_2(x) = \frac{\overline{F}(x)}{f(x)} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-\sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1} + \lambda_0 \sum_{i\ge1}\gamma'_i \overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma_i \overline{F}(x)^i\right)}{F(x)^{a_0-1}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right]}\right).$$

Because

$$\lim_{x\to\infty}\left[-\sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1} + \lambda_0 \sum_{i\ge1}\gamma'_i \overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma_i \overline{F}(x)^i\right)\right] =$$
$$= \lim_{x\to\infty}\left[-\gamma_1 - \sum_{i\ge2}\gamma_i \overline{F}(x)^{i-1} + \lambda_0\left(\gamma'_1 + \sum_{i\ge2}\gamma'_i \overline{F}(x)^{i-1}\right)\left(1+\sum_{i\ge1}\gamma_i \overline{F}(x)^i\right)\right] = -\gamma_1 + \lambda_0\gamma'_1,$$

and $\lim_{x\to\infty} F(x)^{a_0-1}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right] = a_0(1+\lambda_0) - \lambda_0(a_0+b_0)$ equals $a_0 - \lambda_0 b_0$ for $a_0 \ne \lambda_0 b_0$ and 0, otherwise, and because $F \in C_3$, the first term of the equation, $T_1(x)$ is 0 for $x \to \infty$. For the second term, $T_2(x)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[-\sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1} + \lambda_0 \sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma'_i \overline{F}(x)^i\right)\right] = -f(x)\left[\sum_{i\ge1}(i-1)\gamma_i \overline{F}(x)^{i-2} +\right.$$
$$\left. +\lambda_0 \sum_{i\ge1}(i-1)\gamma_i \overline{F}(x)^{i-2}\left(1+\sum_{i\ge1}\gamma'_i \overline{F}(x)^i\right) + \lambda_0 \sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1}\sum_{i\ge1} i\gamma'_i \overline{F}(x)^{i-1}\right] = -f(x)A(x),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{F(x)^{a_0-1}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right]\right\} = -f(x)\times$$
$$\times\left\{(a_0-1)F(x)^{a_0-2}\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right] - \lambda_0 b_0(a_0+b_0)F(x)^{a_0+b_0-2}\right\} = -f(x)B(x),$$

where

$$A(x) = \sum_{i\ge1}(i-1)\gamma_i \overline{F}(x)^{i-2} + \lambda_0 \sum_{i\ge1}(i-1)\gamma_i \overline{F}(x)^{i-2}\left(1+\sum_{i\ge1}\gamma'_i \overline{F}(x)^i\right) + \lambda_0 \sum_{i\ge1}\gamma_i \overline{F}(x)^{i-1}\sum_{i\ge1} i\gamma'_i \overline{F}(x)^{i-1},$$

and

$$B(x)=(a_0-1)F(x)^{a_0-2}\left[a_0(1+\lambda_0)-\lambda_0(a_0+b_0)F(x)^{b_0}\right]-\lambda_0 b_0(a_0+b_0)F(x)^{a_0+b_0-2}.$$

So, the second term, $T_2(x)$ becomes

$$T_2(x)=\overline{F}(x)\{F(x)^{a_0-1}[a_0(1+\lambda_0)-\lambda_0(a_0+b_0)F(x)^{b_0}]\}^{-2}\left\{A(x)F(x)^{a_0-1}[a_0(1+\lambda_0)-\right.$$

$$\left.-\lambda_0(a_0+b_0)F(x)^{b_0}\right]-B(x)\left[-\sum_{i\ge1}\gamma_i\overline{F}(x)^{i-1}+\lambda_0\sum_{i\ge1}\gamma'_i\overline{F}(x)^{i-1}\left(1+\sum_{i\ge1}\gamma_i\overline{F}(x)^i\right)\right]\Bigg\}.$$

Hence, for $x\to\infty$, $T_2(x)$ is 0. Thus, for $a_0\neq\lambda_0 b_0$, we have $\lim\limits_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{h_1(x)}\right)=0.$

Similar results regarding the asymptotic behavior of the order statistics of the $GT.T_n$ models we obtain using mathematical induction.

THEOREM 4. *If* $F\in C_1$, *then also* $F_n\in C_1$.

*Remark* 3. We have $\lim\limits_{t\to0}\frac{F_n(tx)}{F_n(t)}=l(x)^{a_0a_1\cdots a_{n-1}}$, where $l(x)=\lim\limits_{t\to0}\frac{F(tx)}{F(t)}$. As you can see, the limit is invariant of parameters $b_i$ and $\lambda_i$, $i=\overline{0,n-1}$. It only depends on $a_i$.

THEOREM 5. If $F\in C_2$, then also $F_n\in C_2$.

*Remark* 4. We have $\lim\limits_{t\to\infty}\frac{1-F_n(t+x)}{1-F_n(t)}=\lim\limits_{t\to\infty}\frac{1-F(t+x)}{1-F(t)}=l(x),\forall x$. The limit is invariant of all parameters.

THEOREM 6. *If* $F\in C_3$, *then also* $F_n\in C_3$.

### 3.1. Extreme order statistics

In this subsection, we give the asymptotic distributions of the extreme order statistics based on characteristics of the $GT.T_n$ classes of distributions discussed in the previous section.

THEOREM 7. *Let* $X_{1:m}$ and $X_{m:m}$ *be the minimum and the maximum of a random sample* $X_1,X_2,\ldots,X_m$ *with* $F_n$ *as their common* CDF, *defined for* $x>0$.

1. *If* $F\in C_1$ *and* $\lim\limits_{t\to0}\frac{F(tx)}{F(t)}=x^{\theta_1}$, *for each* $x>0$, $\theta_1>0$, *then*

$$\lim_{m\to\infty}P\left(\frac{X_{1:m}-A_m}{B_m}\le x\right)=1-\exp\left(-x^{\theta_1a_0a_1\cdots a_{n-1}}\right)\ \textit{for each}\ b_j>0, j=\overline{0,n-1}$$

2. *If* $F\in C_2$ *and* $\lim\limits_{t\to\infty}\frac{1-F(t+x)}{1-F(t)}=\exp(-\theta_2x)$, *for each* $x>0$, $\theta_2>0$, then

$$\lim_{m\to\infty}P\left(A'_m\left(X_{m:m}-B'_m\right)\le x\right)=\exp(-\exp(-\theta_2x))$$

3. *If* $F\in C_3$, *then*

$$\lim_{m\to\infty}P\left(\frac{X_{m:m}-C_m}{D_m}\le x\right)=\exp(-\exp(-x))$$

*where* $A_m$, $B_m$, $A'_m$, $B'_m$, $C_m$, $D_m$ *are normalizing constants* [1].

*Proof*. For (1) we apply Theorem 8.3.3 from [1] and Theorem 4, and for (2), Theorem 1.6.2 from [4] and Theorem 5. The last part, (3), follows from Theorem 8.3.3 [1] and Theorem 6.

The form of the normalizing constants can be determined following Corollary 1.6.3 from [4] and the results from [1].

THEOREM 8. *Let* $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ *be the order statistics of a random sample* $X_1, X_2, \ldots, X_m$ *with* $F_n$ *as their common* CDF *defined for* $x > 0$. *We have the following*

1. *If* $F \in C_1$ *and* $\lim\limits_{t\to 0}\dfrac{F(tx)}{F(t)} = x^{\theta_1}$ *, for each* $x > 0$, $\theta_1 > 0$, *then for each* $i = \overline{1, m}$

$$\lim_{m\to\infty} P\left(\frac{X_{i:m} - A_m}{B_m} \le x\right) = 1 - \sum_{k=0}^{i-1} \frac{x^{k\theta_1 a_0 a_1 \cdots a_{n-1}}}{k!}\exp\left(-x^{\theta_1 a_0 a_1 \cdots a_{n-1}}\right)$$

2. *If* $F \in C_3$, *then for each* $i = \overline{1, \mathrm{m}}$

$$\lim_{m\to\infty} P\left(\frac{X_{m-i+1:m} - C_m}{D_m} \le x\right) = \sum_{r=0}^{i-1} \exp(-\exp(-x))\frac{\exp(-rx)}{r!}$$

*where* $A_m$, $B_m$, $C_m$, $D_m$ *are normalizing constants* [1].

*Proof*. The conclusion follows from Eqs. (8.4.2) and (8.4.3) of [1], Theorem 4 and Theorem 6.

## 4. STOCHASTIC ORDERING

Using stochastic ordering, we can compare the expected system lifetimes of different systems.

*Definition* 1 [8]. Let $X_1$ and $X_2$ be two random variables with probability density functions $f_1$ and $f_2$, respectively. Then:

a) $X_1$ is said to be smaller than $X_2$ in the likelihood ratio order (denoted by $X_1 \le_{LR} X_2$), if $\dfrac{f_2(x)}{f_1(x)}$ is non-decreasing over the union of the supports of $X_1$ and $X_2$.

b) $X_1$ is said to be stochastically smaller than $X_2$ (denoted by $X_1 \le_{ST} X_2$), if $F_1(x) \ge F_2(x)$ for all $x$, where $F_1$ and $F_2$ are the CDFs of $X_1$ and $X_2$, respectively.

c) $X_1$ is said to be smaller than $X_2$ in the hazard rate order (denoted by $X_1 \le_{HR} X_2$), if $h_1(x) \le h_2(x)$ for all $x$, where $h_1$ and $h_2$ are the hazard rate functions of $X_1$ and $X_2$, respectively.

*Remark* 5 [8]. It is well-known that the likelihood ratio order is stronger than the hazard rate order and the stochastic order, $X_1 \le_{LR} X_2 \Rightarrow X_1 \le_{HR} X_2 \Rightarrow X_1 \le_{ST} X_{2.}$. Also, we have that $X_1 \le_{ST} X_2$ implies $E(X_1) \le E(X_2)$.

Let $F$, $G$ be two arbitrary continuous CDFs. Also, let for $n \ge 1$

$$G_1(x) = G(x)^{a'_0}\left[1 + \lambda'_0 - \lambda'_0\, G(x)^{b'_0}\right]$$

$$G_n(x) = G_{n-1}(x)^{a'_{n-1}}\left[1 + \lambda'_{n-1} - \lambda'_{n-1}\, G_{n-1}(x)^{b'_{n-1}}\right] \tag{4}$$

where $a'_i > 0$, $b'_i > 0$ for $-1 \le \lambda'_i \le 0$, while for $0 \le \lambda'_i \le 1$ we have $a'_i + b'_i > 0$, $a'_i \ge b'_i$.

THEOREM 9. *Let* $X$ *and* $Y$ *be two random variables with* $F$ *and* $G$, *as* CDFs, *respectively. Let* $X_1$ *and* $Y_1$ *be two random variables with* CDFs *of the form* $GT.T_1$, $F_1$ *and* $G_1$, *respectively. If* $0 < a_0 \le 1$,

$a'_0 \geq 1$, $b_0 > 0$, $b'_0 > 0$, $0 \leq \lambda_0 \leq 1$, $-1 \leq \lambda'_0 \leq 0$ *and* $X \leq_{LR} Y$, *then* $X_1 \leq_{LR} Y_1$.

*Proof*. We have

$$\log\frac{g_1(x)}{f_1(x)} = \log\frac{g(x)}{f(x)} + (a'_0 - 1)\ \log(G(x)) - (a_0 - 1)\log(F(x)) +$$
$$+\log\left(a'_0(1+\lambda'_0) - \lambda'_0(a'_0 + b'_0)G(x)^{b'_0}\right) - \log\left(a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\log\frac{g_1(x)}{f_1(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{f(x)}\right) + \frac{g(x)\left[(a'_0-1)a'_0(1+\lambda'_0) - (a'_0+b'_0)\lambda'_0 G(x)^{b'_0}\right]}{G(x)\left[a'_0(1+\lambda'_0) - \lambda'_0(a'_0+b'_0)G(x)^{b'_0}\right]} +$$
$$+\frac{f(x)\left[-(a_0-1)a_0(1+\lambda_0) + (a_0+b_0)\lambda_0 F(x)^{b_0}\right]}{F(x)\left[a_0(1+\lambda_0) - \lambda_0(a_0+b_0)F(x)^{b_0}\right]}.$$

It is easy to see that $0 < a_0 \leq 1$, $a'_0 \geq 1$, $b_0 > 0$, $b'_0 > 0$, $0 \leq \lambda_0 \leq 1$, $-1 \leq \lambda'_0 \leq 0$ and $X \leq_{LR} Y$ imply $\frac{\mathrm{d}}{\mathrm{d}x}\left(\log\frac{g_1(x)}{f_1(x)}\right) \geq 0$ for all $x$, and the result holds.

COROLLARY 1. *Let* $X_1$ *and* $Y_1$ *be two random variables with densities of the form* $GT.T_1$, $f_1$ *and* $g_1$, *respectively*, *having a common baseline* CDF, $F$. *If* $a_0 = a'_0$, $b_0, b'_0 > 0$, $0 \leq \lambda_0 \leq 1$ *and* $-1 \leq \lambda'_0 \leq 0$, *then* $X_1 \leq_{LR} Y_1$.

A more general result of Theorem 9 is the next theorem.

THEOREM 10. *Let* $X, X_1, X_2, \cdots, X_n$ *be random variables with* $F, F_1, F_2, \cdots, F_n$ *as* CDFs *of forms* (1). *Let* $Y, Y_1, Y_2, \cdots, Y_n$ *be random variables with* $G, G_1, \cdots, G_n$ *as* CDFs *of forms* (4). *If* $0 < a_i \leq 1$, $a'_i \geq 1$, $b_i > 0$, $b'_i > 0$, $0 \leq \lambda_i \leq 1$, $-1 \leq \lambda'_i \leq 0$ *and* $X \leq_{LR} Y$, *then* $X_i \leq_{LR} Y_i$ *for all* $i = \overline{1,n}$.

*Example.* Let $F$ be a Weibull CDF of parameters $\mu > 0$ and $\rho > 0$, $F(x) = 1 - \exp(-(x/\rho)^{\mu})$ [11]. Let $X$ and $Y$ be random variables having $F_3$ and $G_3$ as their CDFs obtained using the method described in Section 1 as follows: $F_3$ is the CDF of the $GT.T_3(F,3,1,1,1,1,1,1,1,1)$ model, while $G_3$ is the CDF of the $GT.T_3(F,3,1,1,1,1,1,1,2,-1)$ model. In Fig. 3, we have displayed the series-parallel/parallel-series systems that these random variables $X$ and $Y$ represent. Using Theorems 9 and 10 and Remark 5, we have that $X \leq_{LR} Y$, and therefore the expected system lifetime of $X$ is smaller than the expected system lifetime of $Y$. Generating values from these two random variables, we have calculated the expected system lifetimes of them. These values are displayed in Table 1.

*Table 1*

The expected system lifetimes of $X$ and $Y$

| Parameters (μ, ρ) | *E*(*X*) | *E*(*Y*) |
|---|---|---|
| (10,20) | 18.94729 | 20.70291 |
| (9,10) | 18.94575 | 20.81047 |
| (8,32) | 29.87808 | 33.44599 |
| (4,32) | 32.89666 | 39.62359 |
| (2,32) | 28.71142 | 43.55383 |

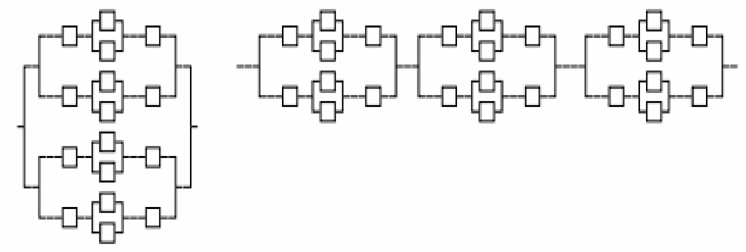

Fig. 3 – The systems represented by $X$ and $Y$.

## REFERENCES


1. B.C. ARNOLD, N. BALAKRISHNAN, H.N. NAGARAJA, *A first course in Order Statistics,* John Wiley & Sons, New York, 1992.
2. P.J. BOLAND, F.J. SAMANIEGO, *The Signature of a Coherent System and its Applications in Reliability*, in: R. SOYER, T. MAZZUCHI, N. SINGPURWALLA (ed.), *Mathematical Reliability*: *An Expository Perspective*, Kluwer Academic Publishers, Boston 129, 2004.
3. G.M. CORDEIRO, E.M.M. ORTEGA, C.C. Da CUNHA DANIEL, *The exponentiated Generalized Class of Distributions,* Journal of Data Science, **11**, pp. 1–27, 2013.
4. M.R. LEADBETTER, G. LINDGREN, H. ROOTZ'EN, *Extremes and Related Properties of Random Sequences and Processes,* Springer Verlag, New York, 1987.
5. Z.M. NOFALY, A.Z. AFIFY, H.M. YOUSOF, G.M. CORDEIRO, *The generalized transmuted-G family of distributions,* Communication in Statistics-Theory and Methods, **46**, *8*, pp. 4119–4136, 2017.
6. V. PREDA, I. BĂNCESCU, *A new family of distributions with a general generic distribution for reliability studies. Log-concavity and Applications*, International Journal of Risk Theory, Alexandru Myller Publishing Iasi, **1**, *6*, pp. 13–38, 2016.
7. F.J. SAMANIEGO, *System signatures and their applications in Engineering Reliability*, Springer, 2007.
8. M. SHAKED, J.G. SHANTHIKUMAR, *Stochastic Orders,* Springer Series in Statistics, Springer, 2007.
9. I. BĂNCESCU, *Some classes of statistical distributions. Properties and Applications*, Analele Ştiinţifice ale Universităţii Ovidius Constanţa, accepted paper.
10. M. IOSIFESCU, L. LIMNIOS, G. OPRISAN*, Introduction to Stochastic Models*, John Wiley and Sons, 2013.
11. M. DUMITRESCU, *Curs de statistică matematică*, Tipografia Universităţii Bucureşti, 1989.
12. V. PREDA, E. PANAITESCU, R. CIUMARA, *The modified Exponential-Poisson distribution*, Proceedings of the Romanian Academy, **12**, *1*, pp. 21–29, 2011.
13. S. DEDU, R. CIUMARA*, Restricted optimal retention in Stop-Loss reinsurance under VaR and CTE risk measures*, Proceedings of the Romanian Academy, **11**, *3*, pp. 213–217, 2010.


*Received November 2, 2017*