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THERMAL STATES IN THE *k*-GENERALIZED HYPERGEOMETRIC COHERENT STATES REPRESENTATION

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Abstract. In our previous paper (Proc. Romanian Acad. A **18**, 323, 2017) we have built and examined some properties of the *k*-coherent states, expressed through the *k*-hypergeometric functions. As a continuation of that paper where we have focussed only on the pure states, in the present paper we deal with mixed (thermal) states for the systems with linear, as well as quadratic energy spectra.

***Key words*:** coherent states, *k*-hypergeometric function, thermal state, density operator.

1. INTRODUCTION

The classical generalized hypergeometric (GHG) functions have a series of applications in different branches of physics [1]. Among them, they are used for defining the most general class of coherent states (CSs) – generalized hypergeometric coherent states (GHG-CSs) introduced in [2] and applied to thermal states of the pseudoharmonic oscillator in [3]. Even if the classical GHG functions turned out to be very useful in terms of applications, in the last decades were defined the so called k – generalized hypergeometric (k-GHG) functions ${}_pF_{q,k}\left(\left\{(a_i)_{n,k}\right\}_{i=1}^{p};\left\{(b_j)_{n,k}\right\}_{j=1}^{q};x\right)$, as a formal power series [4, 5]

$$ {}_pF_{q,k}\left(\left\{(a_i)_{n,k}\right\}_{i=1}^{p};\left\{(b_j)_{n,k}\right\}_{j=1}^{q};x\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{j=1}^{q}(b_j)_{n,k}}\frac{x^n}{n!}\equiv\sum_{n=0}^{\infty}\frac{1}{\rho_k^{(p,q)}(n)}x^n \tag{1.1}$$

where $x\in\mathrm{C}$, $k\in\mathrm{R}$ and $n\in\mathrm{N}^+$. Particularly, for $k=1$ the k-GHG lead to the usual GHG functions.

In a previous paper [6] we have built and examined some properties of the k-GHG-CSs, by using the diagonal ordering operation technique (DOOT). We have limited only to the k-CSs of the Barut-Girardello kind (k-GHG-BG-CSs), although this procedure can be used also in the case of Klauder-Perelomov or Gazeau-Klauder CSs. In order to avoid rehearsals, we appeal and use here the notations and formulae of paper [6].

We consider two Hermitic conjugate operators that act on the Fock vectors $|\,n>$ as

$$ A_-\,|\,n>=\sqrt{e_{n,k}^{(p,q)}}\,|\,n-1>,\qquad A_+\,|\,n>=\sqrt{e_{n+1,k}^{(p,q)}}\,|\,n+1>. \tag{1.2}$$

They are connected with the dimensionless Hamilton operator $H\equiv A_+A_-$ as follows:

$$ H\,|\,n>=A_+A_-\,|\,n>=e_{n,k}^{(p,q)}\,|\,n>. \tag{1.3}$$

The dimensionless energy eigenvalues $e_{m,k}^{(p,q)}$, $m=1,2,\ldots$ and the structure constants $\rho_k^{(p,q)}(n)$ are

$$e_{m,k}^{(p,q)} = m\frac{\prod_{j=1}^{q}\left(b_j+(m-1)k\right)}{\prod_{i=1}^{p}\left(a_j+(m-1)k\right)} \quad - \quad \rho_k^{(p,\,q)}(n) \equiv \prod_{m=1}^{n} e_{m,\,k}^{(p,\,q)} = n\ ! \ \frac{\prod_{j=1}^{q}\left(b_j\right)_{n,\,k}}{\prod_{i=1}^{p}\left(a_i\right)_{n,\,k}}, \tag{1.4}$$

which, as we will see below, appear in the expression of k-GHG-CSs. The following equality holds

$$|\, n> = \frac{1}{\sqrt{\rho_k^{(p,q)}(n)}}\left(A_+\right)^n |\, 0> \tag{1.5}$$

The vacuum state projector can be obtained through the standard procedure, using the DOOT rules [6]:

$$|\, 0 >< 0\,| = \#\frac{1}{{}_pF_{q,k}\left(\left\{(a_i)_{n,k}\right\}_{i=1}^{p};\left\{(b_j)_{n,k}\right\}_{j=1}^{q};\ A_+A_-\right)}\#. \tag{1.6}$$

The k-generalized hypergeometric Barut-Girardello coherent states (k-GHG-BG-CSs) $|\, z>_k$ are defined as usual, *i.e.* as the eigenstates of the lowering operator A_- [7]: $A_- |\, z>_k = z|\, z>_k$ where the complex number $z = |\, z\,|\exp(\mathrm{i}\,\varphi)$, with $|\, z\,| \le \infty$ and $0 \le \varphi \le 2\pi$, labels the k-GHG-BG-CSs.

Their expansion into the Fock vector basis $|\, n>$ is

$$|\, z>_k = \frac{1}{\sqrt{{}_pF_{q,k}\left(\left\{(a_i)_{n,k}\right\}_{i=1}^{p};\left\{(b_j)_{n,k}\right\}_{j=1}^{q};|\, z\,|^2\right)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_k^{(p,q)}(n)}}|\, n> \ . \tag{1.7}$$

In [6] we have showed that the k-GHG-BG-CSs satisfy all Klauder's requirements imposed to any CS: *a*) continuity in complex label, *b*) normalization, *c*) non orthogonality, *d*) unity operator resolution with unique positive weight function of the integration measure, *e*) temporal stability, and *f*) action identity [8].

Generally, the integration measure is expressed using the Meijer's G-function:

$$\mathrm{d}\mu_k^{(p,q)}(z) \ = \ \frac{\prod_{i=1}^{p}\Gamma\left(\frac{a_i}{k}\right)}{\prod_{j=1}^{q}\Gamma\left(\frac{b_j}{k}\right)}\ \frac{1}{k^{q-p}}\ \frac{\mathrm{d}\varphi}{2\pi}\ \mathrm{d}\left(|\, z\,|^2\right) \ \times$$

$$\times \quad {}_pF_{q,\,k}\left(\left\{(a_i)_{n,\,k}\right\}_{i=1}^{p}\ ;\ \left\{(b_j)_{n,\,k}\right\}_{j=1}^{q};\ |\, z\ |^2\right)\ G_{p,\,q+1}^{q+1,\,0}\left(\frac{|\, z\,|^2}{k^{q-p}}\ \middle|\ \begin{matrix} / \ ; & \left\{\frac{a_i}{k}-1\right\}_{i=1}^{p} \\ 0\ ,\ \left\{\frac{b_j}{k}-1\right\}_{j=1}^{q} & ; \ \ / \end{matrix}\right) \tag{1.8}$$

2. MIXED STATES

Let us we examine the thermal states described by the density operator ρ_k

$$\rho_k = \frac{1}{Z_k}\mathrm{e}^{-\beta\,\hbar\omega\,H} = \frac{1}{Z_k}\mathrm{e}^{-\tilde{\beta}\,A_+A_-} = \frac{1}{Z_k}\sum_{n=0}^{\infty}\mathrm{e}^{-\tilde{\beta}\,e_{n,\,k}^{(p,\,q)}}\ |\, n >< n\,| \tag{2.1}$$

where $\widetilde{\beta} = \beta\hbar\,\omega$ is a dimensionless thermal parameter and Z_k is the partition function:

$$Z_k = \mathrm{Tr}\rho_k = \sum_{n=0}^{\infty} < n \,|\, \mathrm{e}^{-\tilde{\beta}\, A_+ A_-} \,|\, n > = \sum_{n=0}^{\infty} \mathrm{e}^{-\tilde{\beta}\; e_{n,k}^{(p,q)}} . \tag{2.2}$$

By using the DOOT, Eq. (1.5) and their conjugate counterpart, the density operator becomes:

$$\rho_k = \frac{1}{Z_k} \sum_{n=0}^{\infty} \mathrm{e}^{-\tilde{\beta}\; e_{n,k}^{(p,q)}} \frac{\#\ \left(A_+ A_-\right)^n\ \#}{\rho_k^{(p,q)}(n)} . \tag{2.3}$$

The Husimi's Q-distribution function defined in the k-GHG-BG-CSs representation is [9]

$$Q_k\left(|\, z\,|^2\right) = {}_k< z \,|\, \rho_k \,|\, z >_k = \frac{1}{Z_k} \frac{1}{{}_pF_{q,k}\left(\left\{(a_i)_{n,k}\right\}_{i=1}^{p}; \left\{(b_j)_{n,k}\right\}_{j=1}^{q}; |\, z\,|^2\right)} \sum_{n=0}^{\infty} e^{-\tilde{\beta}\; e_{n,k}^{(p,q)}} \frac{\left(|\, z\,|^2\right)^n}{\rho_k^{(p,q)}(n)} \tag{2.4}$$

and it is normalized to unity: $\int \mathrm{d}\mu_k(z)\ Q_k\left(|\, z\,|^2\right) = 1$.

The density operator ρ_k can be expanded as a superposition of CSs projectors (see, *e.g.* [9]):

$$\rho_k = \int \mathrm{d}\mu_k(z)\ P_k\left(|\, z\,|^2\right)\ |\, z >_k \ {}_k< z\,| . \tag{2.5}$$

To determine the expression of the quasi-distribution function $P_k\left(|\, z\,|^2\right)$ we follow the same procedure as there for determining the integration measure, with the function change and the resolution of suitable Stieltjes moment problem (see, [6]). This function is also normalized to unity: $\int \mathrm{d}_k(z) P_k\left(|\, z\,|^2\right) = 1$.

The expectation value of a physical observable $\mathcal{A}$ in a mixed (thermal) state is calculated as

$$< \#\ \mathcal{A}\ \# >_k = \mathrm{Tr}(\#\rho_k \mathcal{A}\ \#) = \int \mathrm{d}\mu_k(z)\ P_k\left(|\, z\,|^2\right)\ < \#\mathcal{A}\ \# >_{z,\,k} , \tag{2.6}$$

where the notation $\#\ldots\ \#$ signifies the diagonal ordering (DOOT) of product operators $\#\left(A_+ A_-\right)^s \#$ [3].

In analogy with the Mandel parameter calculated in the k – GHG-BG-CSs, we have defined the Mandel parameter for the thermal states, called the thermal counterpart of the Mandel parameter $Q_k(T)$ [10]:

$$Q_k(T) = \frac{< N^2 >_k - \left(< N >_k\right)^2}{< N >_k} - 1 . \tag{2.7}$$

The sign (positive, zero or negative) of $Q_k(T)$ as function of equilibrium temperature T show whose kind of statistics (super-Poissonian, Poissonian or sub-Poissonian) obey the thermal states.

3. SOME SIMPLE PARTICULAR CASES

We will illustrate the above ideas in two cases of dimensionless energy eigenvalues: *a*) the *linear spectra* $e_{n,k}^{(p,q)} = e_{0,k}^{(p,q)} + k\,n$, and *b*) the *quadratic spectra* $e_{n,k}^{(p,q)} = e_{0,k}^{(p,q)} + n - k\,B n^2$, with respect to the energy quantum number n. Here B is a positive constant and $e_{0,k}^{(p,q)} \equiv e_0$ is the dimensionless zero energy term. We will particularize, without customizing it again, the main results obtained above.

***a*) The linear spectra** is characteristic for some quantum oscillators, *e.g.* the one-dimensional quantum oscillator (HO-1D), for which $e_0 = 1/2$ [9], or the pseudoharmonic oscillator (PHO), for which $e_0 = 1/2 + \alpha/2 + m\omega^2 r_0^2 / 4\hbar$ [10]. The dimensionless energy eigenvalues $e_{m,\,k}^{(p,\,q)}$, $m = 1,\ 2,\ ...$ are

$$e_{m,k}^{(p,q)} = e_{0,k}^{(p,q)} + k\,m = m\frac{e_0 + k + (m-1)k}{k + (m-1)k} \equiv e_{m,k}^{(1,1)} \ . \tag{3.1}$$

Consequently, $p = 1$ and $q = 1$, so that the structure constants become

$$\rho_k^{(p,q)}(n) \equiv \prod_{m=1}^{n} e_{m,k}^{(p,q)} = \prod_{m=1}^{n}\left(e_0 + k\,m\right) = \left(e_0 + k\right)_{n,k} = n!\frac{\left(e_0 + k\right)_{n,k}}{\left(k\right)_{n,k}} k^n \equiv \rho_k^{(1,1)}(n) \tag{3.2}$$

and the k – GHG function in this case is the Kummer confluent k-hypergeometric function:

$$ {}_1F_{1,k}\left(\left(k\right)_{n,k}; \left(e_0 + k\right)_{n,k}; \frac{|\,z\,|^2}{k}\right) = \sum_{n=0}^{\infty} \frac{\left(k\right)_{n,k}}{\left(e_0 + k\right)_{n,k}} \frac{1}{n!}\left(\frac{|\,z\,|^2}{k}\right)^n \equiv \sum_{n=0}^{\infty} \frac{1}{\rho_k^{(1,1)}(n)}\left(|\,z\,|^2\right)^n \ . \tag{3.3}$$

By particularizing $p = 1$ and $q = 1$, we easily get the vacuum projector

$$|\,0 \rangle\langle 0\,| = \# \frac{1}{{}_1F_{1,k}\left(\left(k\right)_{n,k}; \left(e_0 + k\right)_{n,k}; \frac{A_+ A_-}{k}\right)} \# \ . \tag{3.4}$$

Then, the k –GHG-BG-CSs for the case of linear spectra becomes

$$|\,z\,\rangle_k = \frac{1}{\sqrt{{}_1F_{1,k}\left(\left(k\right)_{n,k}; \left(e_0 + k\right)_{n,k}; \frac{|\,z\,|^2}{k}\right)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho_k^{(1,1)}(n)}}\,|\,n\,\rangle \tag{3.5}$$

where $a_1 = k$ and $b_1 = e_0 + k$, while the integration measure that assure the unity operator decomposition is

$$\mathrm{d}\mu_k^{(1,1)}(z) \ = \ \frac{1}{\Gamma\left(\frac{e_0}{k} + 1\right)} \frac{\mathrm{d}\varphi}{2\pi} \mathrm{d}\left(\frac{|\,z\,|^2}{k}\right) e^{-\frac{|z|^2}{k}} \left(\frac{|\,z\,|^2}{k}\right)^{\frac{e_0}{k}} {}_1F_{1,\,k}\left(\left(k\right)_{n,\,k};\ \left(e_0 + k\right)_{n,\,k}; \frac{|\,z\,|^2}{k}\right). \tag{3.6}$$

For the particular case of HO-1D, $k = 1$ and $e_0 = 0$ (*i.e.* if we neglect the insignificant term of zero energy), we recover the integration measure $\mathrm{d}\mu_{HO}^{(1,1)}(z) \ = \ \frac{\mathrm{d}\varphi}{2\pi}\mathrm{d}\left(|\,z\,|^2\right) = \ \frac{\mathrm{d}^2 z}{\pi}$.

The density operator for the case of linear spectra is independent on the constant e_0

$$\rho_k = \frac{1}{Z_k}\mathrm{e}^{-\beta\,\hbar\omega\,H} = \frac{1}{Z_k}\mathrm{e}^{-\tilde{\beta}\,A_+ A_-} = \frac{1}{\overline{n}_k + 1}\sum_{n=0}^{\infty}\left(\frac{\overline{n}_k}{\overline{n}_k + 1}\right)^n |\,n\,\rangle\langle\,n\,| \tag{3.7}$$

where $\widetilde{\beta} = \beta\hbar\,\omega$ is a dimensionless thermal parameter, and $\overline{n}_k = \left(\mathrm{e}^{\tilde{\beta}\,k} - 1\right)^{-1}$ is the Bose-Einstein distribution function or the average of the photon number in the thermal state. Consequently, by normalizing the density operator to unity, we obtain the partition function: $Z_k = \sum_{n=0}^{\infty}\mathrm{e}^{-\tilde{\beta}\,(e_0 + n\,k)} = \mathrm{e}^{-\tilde{\beta}\,e_0}\left(\overline{n}_k + 1\right)$.

With the help of the DOOT rules and the vacuum projector, the density operator becomes:

$$\rho_k = \frac{1}{\overline{n}_k + 1} \# \left[{}_1F_{1,k}\left((k)_{n,k} ; (e_0 + k)_{n,k} ; \frac{A_+ A_-}{k} \right) \right]^{-1} {}_1F_{1,k}\left((k)_{n,k} ; (e_0 + k)_{n,k} ; \frac{\overline{n}_k}{\overline{n}_k + 1} \frac{A_+ A_-}{k} \right) \# . \tag{3.8}$$

The Husimi's *Q*- function is obtained by replacing the operator product $A_+ A_-$ with the $|z|^2$ [6]

$$Q_k\left(|z|^2\right) = \frac{1}{\overline{n}_k + 1} \left[{}_1F_{1,k}\left((k)_{n,k} ; (e_0 + k)_{n,k} ; \frac{|z|^2}{k} \right) \right]^{-1} {}_1F_{1,k}\left((k)_{n,k} ; (e_0 + k)_{n,k} ; \frac{\overline{n}_k}{\overline{n}_k + 1} \frac{|z|^2}{k} \right) \tag{3.9}$$

This function is normalized to unity *i.e.* $\int \mathrm{d}\mu_k^{(1,1)}(z)\ Q_k\left(|z|^2\right) = 1$ and this can be demonstrated by using an integral of kind 7.621.4 (see [11, p. 822]).

The quasi-distribution function $P_k\left(|z|^2\right)$ from the spectral decomposition of the density operator

$$\rho_k = \int \mathrm{d}\mu_k^{(1,1)}(z) P_k\left(|z|^2\right) |z>_k\ {}_k<z| \tag{3.10}$$

is determined by solving a suitable Stieltjes moment problem and the final (normalizable to unity) result is

$$P_k\left(|z|^2\right) = \frac{1}{\overline{n}_k} \left(\frac{\overline{n}_k + 1}{\overline{n}_k} \right)^{\frac{e_0}{k}} \mathrm{e}^{-\frac{1}{\overline{n}_k}\frac{|z|^2}{k}} . \tag{3.11}$$

The thermal expectation value of integer powers $s = 0,1,2,...$ of number particle operator is

$$<N^s>_k = \frac{1}{\overline{n}_k + 1} \sum_{n=0}^{\infty} \left(\frac{\overline{n}_k}{\overline{n}_k + 1} \right)^n n^s = \frac{1}{\overline{n}_k + 1} \left(X \frac{\mathrm{d}}{\mathrm{d}X} \right)^s \left(\frac{1}{1 - X} \right) \tag{3.12}$$

with $X = \exp(-\widetilde{\beta}\, k)$. With the above formula, we obtain $<N>_k = \overline{n}_k$, and $<N^2>_k = \overline{n}_k + 2\overline{n}_k^{\ 2}$, so that the thermal counterpart of the Mandel parameter is $Q_k(\overline{n}_k) = \overline{n}_k = \left(e^{\widetilde{\beta}\, k} - 1\right)^{-1}$. As a function of equilibrium absolute temperature $T = \hbar\,\omega / k_B\, \widetilde{\beta}$, the thermal Mandel parameter is a positive function for all finite temperatures. So, the corresponding thermal states obey super Poissonian statistics.

***b*) The quadratic spectra**, whose dimensionless energy eigenvalues are $e_{n,k}^{(p,q)} = e_0 + n - k\,B\,n^2$, is characteristic for some anharmonic oscillators, e.g. the Morse oscillator (MO) [12]. Here, the constant B is strictly positive and $B \notin \mathrm{N}^+$. Without affecting the generality of the problem we can consider $e_0 = 0$, so the energy eigenvalues are $e_{n,k}^{(p,q)} = n(1 - k\,B\,n) \equiv e_{n,k}^{(0,1)}$. Consequently, the structure constants becomes

$$\rho_k^{(p,q)}(n) \equiv \prod_{m=1}^{n} e_{m,k}^{(p,q)} = \prod_{m=1}^{n} m(1 - k\,B\,m) = (-B)^n\, n! \left(\frac{1}{-B} + k \right)_{n,k} \equiv \rho_k^{(0,1)}(n) \tag{3.13}$$

and the indexes of normalization function are $p = 0$ and $q = 1$.

The k – GHG function associated with this case is

$${}_0F_{1,k}\left(; \left(\frac{1}{-B} + k \right)_{n,k} ; \frac{|z|^2}{-k\,B} \right) = \sum_{n=0}^{n_{\max}} \frac{1}{\left(\frac{1}{-B} + k \right)_{n,k}} \frac{1}{n!} \left(\frac{|z|^2}{-k\,B} \right)^n \equiv \sum_{n=0}^{n_{\max}} \frac{1}{\rho_k^{(0,1)}(n)} \left(|z|^2\right)^n . \tag{3.14}$$

This function can be expressed also in an equivalent manner

$$ {}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{|z|^2}{-k\,B}\right)={}_0F_1\left(\ ;\ \frac{1}{-kB}+1;\ \frac{|z|^2}{-k^2\,B}\right)=\Gamma\left(\frac{1}{-kB}+1\right)\left(\frac{|z|}{k\sqrt{B}}\right)^{\frac{1}{kB}}J_{\frac{1}{-kB}}\left(\frac{|z|}{k\sqrt{B}}\right) \tag{3.15} $$

where we used the expression of the hypergeometric function through Bessel function (in fact, the Bessel polynomial of the degree the integer part of $n_{\max}=\left[(2kB)^{-1}\right]$) of the first kind $J_\nu(x)$ [13].

The vacuum projector is

$$ |\,0><0\,|=\#\frac{1}{{}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{A_+A_-}{-k\,B}\right)}\#=\frac{1}{\Gamma\left(\frac{1}{-kB}+1\right)}\#\frac{1}{\left(\frac{A_+A_-}{k\sqrt{B}}\right)^{\frac{1}{kB}}J_{\frac{1}{-kB}}\left(\frac{A_+A_-}{k\sqrt{B}}\right)}\#\ . \tag{3.16} $$

The k –GHG- BG-CSs for the case of quadratic spectra is

$$ |\,z>_k=\frac{1}{\sqrt{{}_0F_{1,k}\left(\ ;\left(-\frac{1}{B}+k\right)_{n,k};-\frac{|z|^2}{k\,B}\right)}}\sum_{n=0}^{n_{\max}}\frac{z^n}{\sqrt{\rho_k^{(0,1)}(n)}}\,|\,n> \tag{3.17} $$

where $b_1=\frac{1}{-B}+k$, while the k – HG-BG-CSs projector, by respecting the DOOT rules is

$$ |\,z>_k\,{}_k<z\,|=\#\frac{{}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{z\,A_+}{\sqrt{-k\,B}}\right){}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{z^*A_-}{\sqrt{-k\,B}}\right)}{{}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{A_+A_-}{-k\,B}\right){}_0F_{1,k}\left(\ ;\left(\frac{1}{-B}+k\right)_{n,k};\frac{|z|^2}{-k\,B}\right)}\# \tag{3.18} $$

The integration measure that assures the unity operator decomposition is

$$ \mathrm{d}\mu_k^{(0,1)}(z)\ =\ 2\ \ \frac{d\varphi}{2\pi}\mathrm{d}\left(\frac{|z|^2}{-k\ \ B}\right)J_{\frac{1}{-kB}}\left(\frac{|z|}{k\sqrt{B}}\right)K_{\frac{1}{kB}}\left(2\ \ \frac{|z|}{\sqrt{-kB}}\right). \tag{3.19} $$

Using the following integral 6.561.16, p. 676 of [11] it can be proved that this integration measure assure the resolution of the unity operator.

In order to write the normalized canonical density operator we will reorganize the energy exponential according to our previous introduced *ansatz* [12]:

$$ \mathrm{e}^{-\tilde{\beta}\,e_{n,k}^{(0,1)}}=\mathrm{e}^{-\tilde{\beta}\,n\,(1-k\,B\,n)}=\mathrm{e}^{-\tilde{\beta}\,n}\sum_{j=0}^{\infty}\frac{\left(\tilde{\beta}kB\right)^j}{j!}n^{2j}=\exp\left[\tilde{\beta}kB\left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\left(\mathrm{e}^{-\tilde{\beta}}\right)^n \tag{3.20} $$

where in the front part appears the exponential operator acting on the variable $\tilde{\beta}$.

Then, the density operator for the case of quadratic spectra is

$$ \rho_k=\frac{1}{Z_k}\exp\left[\tilde{\beta}\ \ k\ \ B\ \left(\frac{\partial}{\partial\tilde{\beta}}\right)^2\right]\sum_{n=0}^{n_{\max}}\left(\mathrm{e}^{-\tilde{\beta}}\right)^n|\,n><n\,| \tag{3.21} $$

where $n_{\max}=\left[(2k\,B)^{-1}\right]$ (the integer part) represents the number of bound states of the examined Morse system (*e.g*. the diatomic molecule). The partition function Z_k is calculated as

$$Z_k = \sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}} = \exp\left[\tilde{\beta} k B \left(\frac{\partial}{\partial \tilde{\beta}}\right)^2\right]\left[\frac{1-\mathrm{e}^{-\tilde{\beta}\,(n_{\max}+1)}}{1-\mathrm{e}^{-\tilde{\beta}}}\right]. \tag{3.22}$$

With the DOOT rules, as well as the expression of the vacuum projector, we can write the density operator in the following manner:

$$\rho_k = \frac{1}{Z_k} \# \frac{1}{{}_0F_{1,k}\left(\; ;\; \left(\frac{1}{-B}+k\right)_{n,k} \; ; \; \frac{A_+ A_-}{-k\;\; B}\right)} \exp\left[\tilde{\beta} k B \left(\frac{\partial}{\partial \tilde{\beta}}\right)^2\right] {}_0F_{1,k}\left(\; ;\; \left(\frac{1}{-B}+k\right)_{n,k} \; ; \mathrm{e}^{-\tilde{\beta}} \; \frac{A_+ A_-}{-k\;\; B}\right) \# . \tag{3.23}$$

We observe that here the k – GHG function ${}_0F_{1,k}(\; ; ...;...)$ reduces to a *polynomial* of degree $n_{\max}$. By replacing the product $A_+ A_-$ with $|z|^2$ we obtain the Husimi's Q-distribution function

$$Q_k\left(|z|^2\right) = \frac{1}{Z_k} \frac{\exp\left[\tilde{\beta} k B \left(\frac{\partial}{\partial \tilde{\beta}}\right)^2\right] {}_0F_{1,k}\left(\; ;\; \left(\frac{1}{-B}+k\right)_{n,k} \; ; \mathrm{e}^{-\tilde{\beta}} \; \frac{|z|^2}{-k\;\; B}\right)}{{}_0F_{1,k}\left(\; ;\; \left(\frac{1}{-B}+k\right)_{n,k} \; ; \frac{|z|^2}{-k\;\; B}\right)}. \tag{3.24}$$

The density operator can be represented by the following spectral decomposition

$$\rho_k = \int \mathrm{d}\mu_k^{(0,1)}(z) P_k\left(|z|^2\right) |z>_k \; {}_k<z| \tag{3.25}$$

and we will choose the quasi-distribution function $P_k\left(|z|^2\right)$ of the form

$$P_k\left(|z|^2\right) = \frac{1}{Z_k} \exp\left[\tilde{\beta}\, k\, B \left(\frac{\partial}{\partial \tilde{\beta}}\right)^2\right] P_{j,k}\left(|z|^2; \tilde{\beta}\right). \tag{3.26}$$

After a function change and the resolution of a suitable Stieltjes moment problem, the final expression (normalizable to unity) becomes

$$P_k\left(|z|^2\right) = \frac{1}{Z_k}\left[K_{\frac{1}{kB}}\left(2\frac{|z|}{\sqrt{-kB}}\right)\right]^{-1} \exp\left[\tilde{\beta}\, k\, B \left(\frac{\partial}{\partial \tilde{\beta}}\right)^2\right] \frac{\overline{n}_k+1}{\overline{n}_k}\left(\sqrt{\frac{\overline{n}_k+1}{\overline{n}_k}}\right)^{-\frac{1}{kB}} K_{\frac{1}{kB}}\left(2\sqrt{\frac{\overline{n}_k+1}{\overline{n}_k}}\frac{|z|}{\sqrt{-kB}}\right). \tag{3.27}$$

By integrating this expression on the variable $|z|^2$ and using the integral 6.561.16, p. 676 of [11], we lead to the definition of the partition function, which means that $P_k\left(|z|^2\right)$ is normalized to unity.

The thermal expectations of the integer powers of number particle operator is

$$< N^s >_k = \frac{1}{Z_k} \sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}} n^s \tag{3.28}$$

as well as the corresponding thermal Mandel parameter

$$Q_k\left(\tilde{\beta}\right) = \frac{< N^2 >_k}{< N >_k} - < N >_k - 1 = \frac{\sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}} n^2}{\sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}} n} - \frac{\sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}} n}{\sum_{n=0}^{n_{\max}} \mathrm{e}^{-\tilde{\beta}\, e_{n,k}^{(0,1)}}} - 1. \tag{3.29}$$

The character of distribution that governs these thermal states must be examined numerically.

4. CONCLUDING REMARKS

In this paper we have examined the statistical (thermal) properties of the k – generalized hypergeometric Barut-Girardello coherent states (k – GHG-BG-CSs), by using the rules of the diagonal ordering operation technique (DOOT) [3]. We dealt with two kinds of systems: linear (harmonic, pseudoharmonic oscillators) and quadratic (Morse oscillator) energy spectrum for which we calculated both distribution functions: $Q_k\left(|z|^2\right)$ and $P_k\left(|z|^2\right)$. Statistical characteristics of the k – GHG-BG-CSs for thermal states were examined by calculating the thermal Mandel parameter. Their values (negative, zero or positive) show what kind of statistics is associated to examined states (sub Poissonian, Poissonian or super Poissonian) as function of the equilibrium temperature T. At the limit $k \to 1$ all formulae and expressions for the k – GHG-BG-CSs tend to the corresponding formulae and expressions for the usual (classical) generalized hypergeometric coherent states. For the *harmonic limit* $k \to 1$, $e_0 \to 0$ and $B \to 0$ we recover formulae and expressions for the canonical CSs of the one-dimensional harmonic oscillator.

By concluding, the present paper is a step forward regarding the applications of the k – GHG functions, concretely, in defining the k – GHG-BG-CSs, and examining their thermal properties. Through particularization of the indices and parameters of k – GHG-BG-CSs, it can be obtained the all known CSs with physical significance. In this context the paper enriches the literature referring to the CSs.

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