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MODULE (ϕ,φ)-BIPROJECTIVITY AND MODULE (ϕ,φ)-BIFLATNESS OF BANACH ALGEBRAS

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Abstract. Let $\mathfrak{A}$ be a Banach algebra. In this paper for a Banach algebra A which is also an $\mathfrak{A}$-bimodule we introduce the notions of module (ϕ,φ)-biprojectivity and module (ϕ,φ)-biflatness of A, where $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and $\phi \in \Omega_A$, the space consisting of all linear maps $\phi : A \to \mathfrak{A}$ such that $\phi(ab) = \phi(a)\phi(b)$, $\phi(\alpha.a) = \varphi(\alpha)\phi(a)$ $(a,b \in A, \alpha \in \mathfrak{A})$. We investigate relations between module (ϕ,φ)-biprojectivity and $\varphi \circ \phi$-biprojectivity of A and we show that under some conditions A is module (ϕ,φ)-biflat if and only if A is module (ϕ,φ)-amenable. Finally, for an inverse semigroup S with the set of idempotents E, we show that the semigroup algebra $l^1(S)$, as an $l^1(E)$-module, is module (ϕ,φ)-biflat if and only if S is amenable.

Key words: Banach $\mathfrak{A}$-bimodule, module (ϕ,φ)-biprojectivity, module (ϕ,φ)-biflatness, module (ϕ,φ)-amenability.

1. INTRODUCTION AND PRELIMINARIES

The notion of Biprojective Banach algebras were introduced by A. Ya. Helemskii in [7]. Later he has studied biprojectivity and biflatness of the Banach algebras in more details in Chapters IV and VII of [8].

Let A be a Banach algebra and $\omega_A : A\widehat{\otimes}A \to A; a \otimes b \to ab$ be the canonical morphism. A is called biprojective if ω_A has a bounded right inverse which is an A-bimodule homomorphism. A Banach algebra A is said to be biflat if the adjoint $\omega_A^* : A^* \to (A\widehat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule homomorphism. The concepts of φ-biflatness and φ-biprojectivity for a Banach algebra A, where $\varphi \in \Delta(A)$, the character space of A, were introduced and studied in [15].

Let A be a Banach algebra and let $\varphi \in \Delta(A)$. Then A is called φ-biprojective if there exists a bounded A-bimodule homomorphism $\rho : A \to A\widehat{\otimes}A$ such that $\varphi \circ \omega_A \circ \rho(a) = \varphi(a)$ $(a \in A)$. A Banach algebra A is called φ-biflat if there exists a bounded A-bimodule homomorphism $\rho_A : A \to (A\widehat{\otimes}A)^{**}$ such that $\widehat{\varphi} \circ \omega_A^{**} \circ \rho(a) = \varphi(a)$ $(a \in A)$, where $\widehat{\varphi} : A^{**} \to \mathbb{C}$ denotes the extension of φ.

Let $\mathfrak{A}$ and A be Banach algebras such that A be a Banach $\mathfrak{A}$-bimodule with compatible actions $\alpha.(ab) = (\alpha.a)b$, $(ab).\alpha = a(b.\alpha)$ $(a,b \in A, \alpha \in \mathfrak{A})$. Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible left actions defined by

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x, \quad (\alpha.x).a = \alpha.(x.a) \quad (a \in A, \alpha \in \mathfrak{A}, x \in X), \tag{1.1}$$

and similar for the right or two-sided actions. Then we say that X is a Banach A-$\mathfrak{A}$-module. A Banach A-$\mathfrak{A}$-module X is called commutative A-$\mathfrak{A}$-module, if $\alpha.x = x.\alpha$ $(\alpha \in \mathfrak{A}, x \in X)$.

If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^*, whenever the actions of A and $\mathfrak{A}$ on X^* define by $\langle \alpha.f, x\rangle = \langle f, x.\alpha\rangle$, $\langle a.f, x\rangle = \langle f, x.a\rangle$ $(a\in A, \alpha\in\mathfrak{A}, x\in X, f\in X^*)$, and similarly for the right actions.

Let X and Y be two A-$\mathfrak{A}$-modules, then a bounded linear operator $h: X\to Y$ is called A-$\mathfrak{A}$-module homomorphism if $h(x\pm y)=h(x)\pm h(y)$ and

$$h(\alpha.x)=\alpha.h(X),\ \ h(x.\alpha)=h(x).\alpha,\ \ h(a.x)=a.h(x),\ \ h(x.a)=h(x).a,$$

for $x, y\in X, a\in A$ and $\alpha\in\mathfrak{A}$.

Let $A\widehat{\otimes}A$ be the projective tensor product of A and A which is a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule by the following actions: $\alpha.(a\otimes b)=(\alpha.a)\otimes b,\ c.(a\otimes b)=(ca)\otimes b\ (\alpha\in\mathfrak{A}, a,b,c\in A)$, similarly for the right actions. Let $I_{A\widehat{\otimes}A}$ be the closed ideal of $A\widehat{\otimes}A$ generated by elements of the form

$$\{a.\alpha\otimes b - a\otimes\alpha.b \mid \alpha\in\mathfrak{A}, a,b\in A\}. \tag{1.2}$$

Let J_A be the closed ideal of A generated by

$$\omega_A(I_{A\widehat{\otimes}A})=\{(a.\alpha)b-a(\alpha.b)\mid a,b\in A,\alpha\in\mathfrak{A}\}. \tag{1.3}$$

Then, the module projective tensor product $A\widehat{\otimes}_{\mathfrak{A}}A$, which is $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ by [14], and the quotient Banach algebra A/J_A are both Banach A-bimodules and Banach $\mathfrak{A}$-bimodules. Also, A/J_A is A-$\mathfrak{A}$-module with compatible actions when A acts on A/J_A canonically.

Define $\widetilde{\omega}_A\in\mathfrak{L}(A\widehat{\otimes}_{\mathfrak{A}}A, A/J_A)$ by $\widetilde{\omega}_A(a\otimes b+I_{A\widehat{\otimes}A})=ab+J_A$ and extend by linearity and continuity. Obviously, $\widetilde{\omega}_A$ is A-$\mathfrak{A}$-bimodule map. Moreover, $\widetilde{\omega}_A{}^*$, the first adjoints of $\widetilde{\omega}_A$ is also A-$\mathfrak{A}$-module homomorphism.

Let A be a Banach $\mathfrak{A}$-bimodule. A is called $\mathfrak{A}$-module biprojective if $\widetilde{\omega}_A$ has a bounded right inverse which is an A/J_A-$\mathfrak{A}$-module homomorphism, and A is called $\mathfrak{A}$-module biflat if $\widetilde{\omega}_A{}^*$ has a bounded left inverse which is an A/J_A-$\mathfrak{A}$-module homomorphism. Module biprojectivity and module biflatness of Banach algebras were introduced and investigated by Bodaghi and Amini in [4]. For every inverse semigroup S with subsemigroup E of idempotents, they showed that $l^1(S)$ is module biprojective, as an $l^1(E)$-module, if and only if an appropriate group homomorphic image G_S of S is finite. They also proved that module biflatness of $l^1(S)$ is equivalent to the amenability of the underlying semigroup S.

Let A be a Banach $\mathfrak{A}$-bimodule, $\varphi\in\Delta(\mathfrak{A})\cup\{0\}$ and $\phi\in\Omega_A$, the space consisting of all linear maps $\phi: A\to\mathfrak{A}$ such that $\phi(ab)=\phi(a)\phi(b),\ \phi(\alpha.a)=\varphi(\alpha)\phi(a)\ (a,b\in A,\alpha\in\mathfrak{A})$. Our aim in this paper is to introduce and study the notions of module (ϕ,φ)-biprojectivity and module (ϕ,φ)- biflatness of A. We briefly summarize the results in this paper.

In section 2 for a Banach $\mathfrak{A}$-bimodule A we investigate relation between module (ϕ,φ)- biprojectivity of A and $\varphi\circ\widetilde{\phi}$-biprojectivity of A/J_A. We also prove that if A/J_A has an identity, then $\varphi\circ\phi$-biprojectivity of A implies module (ϕ,φ)-biprojectivity of A.

In section 3 we investigate relation between module (ϕ,φ)-amenability of A and module (ϕ,φ)-biflatness of A. Indeed we show that if A has a bounded approximate identity and $\mathfrak{A}$ act on A trivially from the left, then A is module (ϕ,φ)-biflat if and only if A is module (ϕ,φ)-amenable. Finally, for an inverse semigroup S with the set of idempotents E, we give some conditions under which the semigroup algebra $l^1(S)$, as an $l^1(E)$-module, is module (ϕ,φ)-biflat if and only if S is amenable.

Note that, in this paper `Banach algebra' means complex associative Banach algebra, and in general Banach algebras are not assumed to have any unit element, unless they are otherwise specified explicitly.

2. MODULE (ϕ,φ)-BIPROJECTIVITY OF BANACH ALGEBRAS

We commence this section with the following definition:

Definition 2.1. We say the Banach algebra $\mathfrak{A}$ acts trivially on A from the left (right) if there is a multiplicative linear functional f on $\mathfrak{A}$ such that $\alpha.a = f(\alpha)a$ (resp. $a.\alpha = f(\alpha)a$) for all $\alpha \in \mathfrak{A}$ and $a \in A$.

Let $\phi \in \Omega_A$. Clearly $\phi((a.\alpha)b - a(\alpha.b)) = 0$ $(\alpha \in \mathfrak{A}, a,b \in A)$. so $\phi = 0$ on J_A and $\tilde{\phi} : A/J_A \to \mathfrak{A}$ given by $\tilde{\phi}(a + J_A) = \phi(a)$ is well defined. Hence $\tilde{\phi} \in \Omega_{A/J_A}$.

Definition 2.2. Let $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and $\phi \in \Omega_A$. A Banach $\mathfrak{A}$-bimodule A is called module (ϕ,φ)-biprojective if there exists A/J_A-$\mathfrak{A}$-module homomorphism $\widetilde{\rho} : A/J_A \to (A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ such that $\varphi \circ \tilde{\phi} \circ \widetilde{\omega}_A \circ \widetilde{\rho}(a + J_A) = \varphi \circ \tilde{\phi}(a + J_A)$ $(a \in A)$.

The proof of the following proposition is straightforward, so we omit its proof.

PROPOSITION 2.3. Let A be a Banach $\mathfrak{A}$-bimodule, $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and $\phi \in \Omega_A$. If A is $\mathfrak{A}$-module biprojective, then A is module (ϕ,φ)-biprojective.

For the proof of the following result we refer to Lemma 3.13 of [2].

LEMMA 2.4. *Let $\mathfrak{A}$ acts on A trivially from the left or right and A/J_A has a right bounded approximate identity, then for each $\alpha \in \mathfrak{A}$ and $a \in A$ we have $f(\alpha)a - a.\alpha \in J_A$.*

We recall the following remark from [4] for proof of the next results:

Remark 2.5. Let $I_{A\widehat{\otimes}A}$ and J_A be the closed ideals defined in (1.2) and (1.3), respectively. Suppose that A has a bounded approximate identity and $\mathfrak{A}$ acts on A trivially from the left. Then $(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ is an A/J_A-bimodule with the following actions given by

$$(a + J_A).(c \otimes b + I_{A\widehat{\otimes}A}) := a.(c \otimes b + I_{A\widehat{\otimes}A}) = ac \otimes b + I_{A\widehat{\otimes}A}, \tag{2.1}$$

and

$$(c \otimes b + I_{A\widehat{\otimes}A}).(a + J_A) := (c \otimes b + I_{A\widehat{\otimes}A}).a = c \otimes ba + I_{A\widehat{\otimes}A}, \tag{2.2}$$

for $a,b,c \in A$ and $\alpha \in \mathfrak{A}$.

PROPOSITION 2.6. *Let A be Banach algebra with a bounded approximate identity and $\mathfrak{A}$ acts on A trivially from the left. Let $\Phi_A : (A\widehat{\otimes}A)/I_{A\widehat{\otimes}A} \to A/J_A \widehat{\otimes} A/J_A$ be defined by*

$$\Phi_A\left((a_1 \otimes a_2) + I_{A\widehat{\otimes}A}\right) = (a_1 + J_A) \otimes (a_2 + J_A) \quad (a_1, a_2 \in A).$$

Then Φ_A is a bijective A/J_A-$\mathfrak{A}$-module homomorphism.

Proof. Let $\pi : A \to A/J_A$ is the projection map, then the map

$$F_1 : (A\widehat{\otimes}A)/\ker(\pi \otimes \pi) \to A/J_A \widehat{\otimes} A/J_A,\ a_1 \otimes a_2 + \ker(\pi \otimes \pi) \to (a_1 + J_A) \otimes (a_2 + J_A),$$

is well defined. By Lemma 2.4, for every $a_1, a_2 \in A$ and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}(\pi \otimes \pi)(a_1\alpha \otimes a_2 - a_1 \otimes \alpha.a_2) &= (a_1.\alpha + J_A) \otimes (a_2 + J_A) - (a_1 + J_A) \otimes (\alpha.a_2 + J_A) \\ &= (f(\alpha)a_1 + J_A) \otimes (a_2 + J_A) - (a_1 + J_A) \otimes (f(\alpha)a_2 + J_A) \\ &= f(\alpha)(a_1 + J_A) \otimes (a_2 + J_A) - f(\alpha)(a_1 + J_A) \otimes (a_2 + J_A) = 0.\end{aligned}$$

Thus $I_{A\widehat{\otimes}A} / \ker(\pi\otimes\pi)$. Hence the map

$$F_2 : (A\widehat{\otimes}A)/I_{A\widehat{\otimes}A} \to (A\widehat{\otimes}A)/\ker(\pi\otimes\pi),\ \ a_1\otimes a_2 + I_{A\widehat{\otimes}A} \mapsto a_1\otimes a_2 + \ker(\pi\otimes\pi),$$

is also well defined. So $\Phi_A = F_1\circ F_2$ is well defined. Since $\pi\otimes\pi$ is bounded, for every $a_1, a_2\in A,$ it follows that

$$\begin{aligned}\left\|F_1\left(a_1\otimes a_2+\ker(\pi\otimes\pi)\right)\right\| &= \|(a_1+J_A)\otimes(a_2+J_A)\| = \|\pi\otimes\pi(a_1\otimes a_2)\| \\ &= \inf_{x\in\ker(\pi\otimes\pi)}\|\pi\otimes\pi(a_1\otimes a_2)+\pi\otimes\pi(x)\| \le k'\|a_1\otimes a_2+\ker(\pi\otimes\pi)\|,\end{aligned}$$

where $k'>0$ is bound for $\pi\otimes\pi$. Thus F_1 is bounded. Also since $I_{A\widehat{\otimes}A}\subseteq\ker(\pi\otimes\pi)$, it follows that F_2 is bounded. So Φ_A is bounded. We show that Φ_A is a bijective map.

Clearly, Φ_A is surjective. Let (e_i) be a bounded approximate identity for A with bound $m>0$. By (2.1) and (2.2), for every $a_1, a_2\in A,$ we have

$$\begin{aligned}\left\|a_1\otimes a_2+I_{A\widehat{\otimes}A}\right\| &= \lim_i\left\|a_1e_i\otimes e_ia_2+I_{A\widehat{\otimes}A}\right\| = \lim_i\left\|(a_1+J_A).\left((e_i\otimes e_i+I_{A\widehat{\otimes}A}).(a_2+J_A)\right)\right\| \\ &\le k\lim_i\left\|e_i\otimes e_i+I_{A\widehat{\otimes}A}\right\|\|a_1+J_A\|\|a_2+J_A\| \le k\lim_i\|e_i\otimes e_i\|\|(a_1+J_A)\otimes(a_2+J_A)\| \\ &\le km^2\|(a_1+J_A)\otimes(a_2+J_A)\|.\end{aligned}$$

This shows that Φ_A is injective and so Φ_A is a bijective map. Obviously Φ_A is an $\mathfrak{A}$-bimodule homomorphism. Again by using (2.1) and (2.2), and the facts that $A/J_A\widehat{\otimes}A/J_A$ is A/J_A-homomorphism, it is easy to see that Φ_A is A/J_A-bimodule map. Therefore Φ_A is a bijective A/J_A-$\mathfrak{A}$-module homomorphism.

Let Φ_A be as in above Proposition. If we denote the inverse of Φ_A by Φ_A^{-1}, then it is easy to see that Φ_A^{-1} is a A/J_A-$\mathfrak{A}$-module homomorphism.

PROPOSITION 2.7. *Let A be a Banach $\mathfrak{A}$-bimodule with a bounded approximate identity, where $\mathfrak{A}$ act on A trivially from the left. Let $\varphi\in\Delta(\mathfrak{A})\cup\{o\}$ and $\phi\in\Omega_A$. If A is module (ϕ,φ)-biprojective, then A/J_A is $\varphi\circ\tilde{\phi}$-biprojective.*

Proof. Let A be module (ϕ,φ)-biprojective. Then there exists A/J_A-$\mathfrak{A}$-module homomorphism $\tilde{\rho}: A/J_A\to(A\widehat{\otimes}A)/I_{A\widehat{\otimes}A}$ such that $\varphi\circ\tilde{\phi}\circ\tilde{\omega}_A\circ\tilde{\rho}(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A)$. Let Φ_A be as in Proposition 2.6. A direct verication shows that the equalities $\omega_{A/J_A}\circ\Phi_A=\tilde{\omega}_A$ are valid. Define $\rho: A/J_A\to(A/J_A\widehat{\otimes}A/J_A)$ by $\rho\left(a+J_A\right)=\Phi_A\circ\tilde{\rho}(a+J_A)$ $(a\in A)$. Since $\mathfrak{A}$ act on A trivially from the left, we may take $\alpha_0\in\mathfrak{A}$ such that $f(\alpha_0)=1$. Hence for every $a\in A$ and $\lambda\in\mathbb{C},$ we have

$$\rho\left(\lambda(a+J_A)\right)=\rho\left(\lambda(\alpha_0.a+J_A)\right)=\lambda\alpha_0\rho(a+J_A)=\lambda\rho(\alpha_0.a+J_A)=\lambda\rho(a+J_A). \tag{2.3}$$

That is ρ is $\mathbb{C}$-linear. Then ρ is a A/J_A-bimodule homomorphism and for every $a\in A,$ we have

$$\varphi\circ\tilde{\phi}\circ\omega_{A/J_A}\circ\rho(a+J_A)=\varphi\circ\tilde{\phi}\circ\omega_{A/J_A}\circ\Phi_A\circ\tilde{\rho}\left(a+J_A\right)=\varphi\circ\tilde{\phi}\circ\tilde{\omega}_A\circ\tilde{\rho}(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A).$$

Consequently A/J_A is $\varphi\circ\tilde{\phi}$-biprojective.

PROPOSITION 2.8. *Let A be a Banach $\mathfrak{A}$-bimodule, where $\mathfrak{A}$ act on A trivially from the left and let A/J_A has an identity. Then the following statements are valid:*

(i) *If A/J_A is $\varphi\circ\tilde{\phi}$-biprojective, then A is module (ϕ,φ)-biprojective*;

(ii) *If A* is *$\varphi\circ\phi$-biprojective, then A is module (ϕ,φ)-biprojective.*

Proof. Let $e+J_A$ be the identity of A/J_A. (i) Suppose that A/J_A is $\varphi\circ\tilde{\phi}$-biprojective. Then there exists A/J_A-module homomorphism $\rho : A/J_A \to (A/J_A\hat{\otimes}A/J_A)$ such that

$$\varphi\circ\tilde{\phi}\circ\omega_{A/J_A}\circ\rho(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A) \quad (a\in A).$$

Define $\tilde{\rho}: A/J_A \to (A\hat{\otimes}A)/I_{A\hat{\otimes}A}$ by $\tilde{\rho}(a+J_A)=\Phi_A^{-1}\circ\rho(e+J_A).(a+J_A)$ $(a\in A).$ For every $\alpha\in\mathfrak{A}$ and $a\in A,$ we have

$$\tilde{\rho}\big(\alpha.(a+J_A)\big)=\Phi_A^{-1}\circ\rho(e+J_A).(\alpha.a+J_A)=f(\alpha)\Phi_A^{-1}\circ\rho(e+J_A).(a+J_A)=\alpha.\tilde{\rho}(a+J_A),$$

and similarly, $\tilde{\rho}\big((a+J_A).\alpha\big)=\tilde{\rho}(a+J_A).\alpha.$ Since Φ_A^{-1} and ρ are A/J_A-module map for every $a,a'\in A,$ we obtain that

$$\begin{aligned}\tilde{\rho}\big((a'+J_A).(a+J_A)\big)&=\Phi_A^{-1}\circ\rho(e+J_A).(a'a+J_A)=\big(\Phi_A^{-1}\circ\rho(e+J_A).(a'+J_A)\big).(a+J_A)\\&=\big((a'+J_A).\Phi_A^{-1}\circ\rho(e+J_A)\big).(a+J_A)=(a'+J_A).\tilde{\rho}(a+J_A),\end{aligned}$$

and similarly, $\tilde{\rho}\big((a+J_A).(a'+J_A)\big)=\tilde{\rho}(a+J_A).(a'+J_A).$ So $\tilde{\rho}$ is a A/J_A-$\mathfrak{A}$-module homomorphism. Now for every $a\in A,$ we have

$$\begin{aligned}\varphi\circ\tilde{\phi}\circ\tilde{\omega}_A\circ\tilde{\rho}(a+J_A)&=\varphi\circ\tilde{\phi}\circ\tilde{\omega}_A\big(\Phi_A^{-1}\circ\rho(e+J_A).(a+J_A)\big)=\varphi\circ\tilde{\phi}\big(\tilde{\omega}_A\big(\Phi_A^{-1}\circ\rho(e+J_A)\big)(a+J_A)\big)\\&=\varphi\circ\tilde{\phi}\circ(\tilde{\omega}_A\circ\Phi_A^{-1})\circ\rho(e+J_A)\varphi\circ\tilde{\phi}(a+J_A)=\varphi\circ\tilde{\phi}\circ\omega_{A/J_A}\circ\rho(e+J_A)\varphi\circ\tilde{\phi}(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A).\end{aligned}$$

Therefore A is module (ϕ,φ)-biprojective.

(ii) Suppose that A is $\varphi\circ\phi$-biprojective and $\rho : A\to(A\hat{\otimes}A)$ is a A-module homomorphism such that $\varphi\circ\phi\circ\omega_A\circ\rho(a)=\varphi\circ\phi(a)$ $(a\in A).$ Define $\tilde{\rho}: A/J_A\to(A\hat{\otimes}A)/I_{A\hat{\otimes}A}$ by $\tilde{\rho}(a+J_A)=(\rho(e)+I_{A\hat{\otimes}A}).(a+J_A)$ $(a\in A).$ A similar argument as in (i) shows that $\tilde{\rho}$ is a A/J_A-$\mathfrak{A}$-module homomorphism. Hence for every $a\in A,$ we have

$$\begin{aligned}\varphi\circ\tilde{\phi}\circ\tilde{\omega}_A\circ\tilde{\rho}(a+J_A)&=\varphi\circ\tilde{\rho}\circ\tilde{\omega}_A\big((\rho(e)+I_{A\hat{\otimes}A}).(a+J_A)\big)=\varphi\circ\tilde{\rho}\circ\tilde{\omega}_A\big(\rho(a)+I_{A\hat{\otimes}A}\big)\\&=\varphi\circ\tilde{\rho}\big(\omega_A(\rho(a))+J_A\big)=\varphi\circ\phi\circ\omega_A\circ\rho(a)=\varphi\circ\phi(a)=\varphi\circ\tilde{\rho}(a+J_A).\end{aligned}$$

This means that A is module (ϕ,φ)-biprojective.

3. MODULE (ϕ,φ)-AMENABILITY AND MODULE (ϕ,φ)-BIFLATNESS OF BANACH ALGEBRAS

Let $\varphi\in\Delta(A)$. Then φ has a unique extension $\hat{\varphi}\in\Delta(A^{**})$ which is denote by $\hat{\varphi}(F)=F(\varphi)$ for every $F\in A^{**}$.

Definition 3.1. Let $\varphi\in\Delta(A)\cup\{0\}$ and $\phi\in\Omega_A$. A Banach algebra A is called module (ϕ,φ)-biflat if there exists A/J_A-$\mathfrak{A}$-module homomorphism $\tilde{\rho}_A : A/J_A\to\big((A\hat{\otimes}A)/I_{A\hat{\otimes}A}\big)^{**}$ such that

$$\widehat{\varphi\circ\tilde{\phi}}\circ\tilde{\omega}_A^{**}\circ\tilde{\rho}_A(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A) \quad (a\in A).$$

We recall following definition from [5].

Definition 3.2. Let A be a Banach $\mathfrak{A}$-bimodule, $\varphi \in \Delta(A) \cup \{0\}$ and $\phi \in \Omega_A$. A bounded linear functional $m: A^* \to \mathbb{C}$ is called a module (ϕ,φ)-mean on A^* if $m(f.a) = \varphi \circ \phi(a) m(f)$, $m(f.\alpha) = \varphi(\alpha) m(f)$ and $m(\varphi \circ \phi) = 1$ for all $f \in A^*, a \in A$ and $\alpha \in \mathfrak{A}$. A is called module (ϕ,φ)-amenable if there exists a module (ϕ,φ) mean on A^*.

Remark 3.3. Let X be a Banach A-$\mathfrak{A}$-module. A bounded map $D: A \to X$ is called an $\mathfrak{A}$-module derivation if

$$D(a \pm b) = D(a) \pm D(b),\ \ D(ab) = D(a).b + a.D(b),\ D(\alpha.a) = \alpha.D(a),\ \ D(a.\alpha) = D(a).\alpha \tag{3.1}$$

for all $a, b \in A$ and $\alpha \in \mathfrak{A}$. Although D in general is not linear, but still its boundedness implies its norm continuity. A $\mathfrak{A}$-module derivation D is said to be inner if there exists $x \in X$ such that $D(a) = a.x - x.a$. $(a \in A)$. (see [1]).

PROPOSITION 3.4. *Let A be a Banach $\mathfrak{A}$-bimodule, and let $\varphi \in \Delta(A) \cup \{0\}$ and $\phi \in \Omega_A$. Then A is module (ϕ,φ)-amenable if and only if A/J_A is module $(\tilde{\phi},\varphi)$-amenable.*

Proof. Suppose that A/J_A is $(\tilde{\phi},\varphi)$-module amenable. Let X be a Banach A-$\mathfrak{A}$-module such that $a.x = \phi(a).x$ and $\alpha.x = x.\alpha = \varphi(\alpha)x$ for every $a \in A, x \in X$ and $\alpha \in \mathfrak{A}$. Let $D: A \to X^*$ be a bounded module derivation. Using (1.1) and commutativity of X, we have $J_A X = X J_A = 0$ and so X is a Banach A/J_A-$\mathfrak{A}$-module by following actions $(a + J_A).x = a.x,\ x.(a + J_A) = x.a\ (a \in A, x \in X)$. Also using (3.1) we see that D vanishes on J_A. Therefore, D induces a bounded module derivation $\widetilde{D}: A/J_A \to X^*$. Since X is a Banach A/J_A-$\mathfrak{A}$-module such that $(a + J_A).x = \tilde{\phi}(a + J_A).x\quad (a \in A, x \in X)$, $\quad \alpha.x = x.\alpha = \varphi(\alpha)x$ $(\alpha \in \mathfrak{A})$ and A/J_A is module $(\tilde{\phi},\varphi)$-amenable, by Theorem 2.1 of [5], we conclude that $\widetilde{D}$ is inner. Hence D is inner. Again Theorem 2.1 of [5], implies that A is module (ϕ,φ)-amenable. Similarly, we can proof the other direction.

PROPOSITION 3.5. *Let A be a Banach $\mathfrak{A}$-bimodule with a bounded approximate identity, and let $\varphi \in \Delta(A) \cup \{0\}$ and $\phi \in \Omega_A$. Let $\mathfrak{A}$ act on A trivially from the left. If A is module (ϕ,φ)-biflat, then A/J_A is $\varphi \circ \tilde{\phi}$-biflat.*

Proof. Assume that A is module (ϕ,φ)-biflat. Thus there exists a A/J_A-$\mathfrak{A}$-module homomorphism $\rho_A : A/J_A \to \left(A \widehat{\otimes} A / I_{A\widehat{\otimes}A}\right)^{**}$ such that $\widehat{\varphi \circ \tilde{\phi}} \circ \widetilde{\omega}_A^{**} \circ \widetilde{\rho}_A(a + J_A) = \varphi \circ \tilde{\phi}(a + J_A)\ \ (a \in A)$. Let Φ_A be as in Proposition 2.6. Define $\rho : A/J_A \to \left(A/J_A \widehat{\otimes} A/J_A\right)^{**}$ by $\rho = \Phi_A^{**} \circ \widetilde{\rho}_A$. By a similar argument as in (2.3), we may show that ρ is $\mathbb{C}$-linear. Let $G \in \left(A \widehat{\otimes} A / I_{A\widehat{\otimes}A}\right)^{**}$. Take the net $(x_\alpha) \subset \left(A \widehat{\otimes} A\right) / I_{A\widehat{\otimes}A}$ such that $\widehat{x_\alpha} \to G$ in w^*-topology. For every α let $x_\alpha = \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha + I_{A\widehat{\otimes}A}$, for some sequences $\left(a_i^\alpha\right)_i$ and $\left(b_i^\alpha\right)_i$ in A with $\sum_{i=1}^{\infty} \left\|a_i^\alpha\right\| \left\|b_i^\alpha\right\| < \infty$. Then for every $f \in \left(A/J_A\right)^*$, we have

$$\begin{aligned}
\left\langle f, \widetilde{\omega}_A^{**}(G)\right\rangle &= \left\langle \widetilde{\omega}_A^{*}(f), G\right\rangle = \lim_\alpha \left\langle \widetilde{\omega}_A^{*}(f), \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha + I_{A\widehat{\otimes}A}\right\rangle = \lim_\alpha \left\langle f, \sum_{i=1}^{\infty} a_i^\alpha b_i^\alpha + J_A\right\rangle \\
&= \lim_\alpha \left\langle \omega_{A/J_A}^{*}(f), \sum_{i=1}^{\infty} (a_i^\alpha + J_A) \otimes (b_i^\alpha + J_A)\right\rangle = \lim_\alpha \left\langle \Phi_A^* \circ \omega_{A/J_A}^{*}(f), \sum_{i=1}^{\infty} a_i^\alpha \otimes b_i^\alpha + I_{A\widehat{\otimes}A}\right\rangle \\
&= \left\langle \Phi_A^* \circ \omega_{A/J_A}^{*}(f), G\right\rangle = \left\langle f, \omega_{A/J_A}^{**} \circ \Phi_A^{**}(G)\right\rangle.
\end{aligned}$$

That is $\widetilde{\omega}_A^{**}(G)=\omega_{A/J_A}^{**}\circ\Phi_A^{**}(G)\quad (G\in(A\widehat{\otimes}A/I_{A\widehat{\otimes}A})^{**})$. So $\widetilde{\omega}_A^{**}=\omega_{A/J_A}^{**}\circ\Phi_A^{**}$ and

$$\widehat{\varphi\circ\tilde{\phi}}\circ\omega_{A/J_A}^{**}\circ\rho(a+J_A)=\widehat{\varphi\circ\tilde{\phi}}\circ\omega_{A/J_A}^{**}\circ\Phi_A^{**}\circ\tilde{\rho}_A(a+J_A)=\widehat{\varphi\circ\tilde{\phi}}\circ\widetilde{\omega}_A^{**}\circ\tilde{\rho}_A(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A),$$

for all $a\in A$. Consequently A/J_A is $\varphi\circ\tilde{\phi}$-biflat.

THEOREM 3.6. *Let A be a Banach $\mathfrak{A}$-bimodule with a bounded approximate identity, where $\mathfrak{A}$ act on A trivially from the left. Let $\varphi\in\Delta(A)\cup\{0\}$ and $\phi\in\Omega_A$. Then A is module (ϕ,φ)-biflat if and only if A is module (ϕ,φ)-amenable.*

Proof. Suppose that A is module (ϕ,φ)-biflat. By Proposition 3.5, A/J_A is $\varphi\circ\tilde{\phi}$-biflat. So Theorem 3.1 of [16], implies that A/J_A is $\varphi\circ\tilde{\phi}$-amenable. Let $D:A/J_A\to X^*$ be an $\mathfrak{A}$-module derivation for some A/J_A-$\mathfrak{A}$-bimodule X such that $(a+J_A).x=\tilde{\phi}(a+J_A)$ and $\alpha.x=x.\alpha=\varphi(\alpha)x$. We may assume X as a A/J_A-bimodule with the following actions

$$x\bullet(a+J_A)=x.(a+J_A),\ (a+J_A)\bullet x=\varphi\circ\tilde{\phi}(a+J_A)x\quad(a\in A,x\in X).$$

Since $\mathfrak{A}$ act on A trivially from the left, we may take $\alpha_0\in\mathfrak{A}$ such that $f(\alpha_0)=1$. Hence for every $a\in A$ and $\lambda\in\mathbb{C}$, we have $D(\lambda(a+J_A))=D(\lambda\alpha_0.a+J_A)=\lambda\alpha_0D(a+J_A)=\lambda D(\alpha_0.a+J_A)=\lambda D(a+J_A)$. Thus D is linear map. Now Theorem 1.1 of [9], yield that D is inner and so by Theorem 2.1 of [5], A/J_A is module $(\tilde{\phi},\varphi)$-amenable. Therefore A is module (ϕ,φ)-module amenable by Proposition 3.4.

Conversely, assume that A is module (ϕ,φ)-amenable. We consider the Banach A-bimodule $A\widehat{\otimes}A$ with module actions $(a\otimes b).a'=a'.(a\otimes b)=\varphi\circ\phi(a')a\otimes b\ (a',a,b\in A)$. A similar argument as in the proof of Theorem 2.10 of [5], shows that there exists a $\widetilde{M}\in((A\widehat{\otimes}A)/I_{A\widehat{\otimes}A})^{**}$ such that

$$a.\widetilde{M}=\widetilde{M}.a=(\varphi\circ\phi)(a)\widetilde{M},\quad \widetilde{\omega}^{**}(M)(\varphi\circ\tilde{\phi})=1\quad(a\in A).\tag{3.2}$$

Define $\tilde{\rho}_A:A/J_A\to((A\widehat{\otimes}A)/I_{A\widehat{\otimes}A})^{**}$ by $\tilde{\rho}_A(a+J_A)=\varphi\circ\tilde{\phi}(a+J_A)\widetilde{M}\ (a\in A)$. By (2.1), (2.2) and (3.2), one can easily show that $\tilde{\rho}$ is a A/J_A-$\mathfrak{A}$-module homomorphism. Thus for every $a\in A$, we have

$$\begin{aligned}\widehat{\varphi\circ\tilde{\phi}}\circ\widetilde{\omega}_A^{**}\circ\rho_A(a+J_A)&=\widehat{\varphi\circ\tilde{\phi}}\circ\widetilde{\omega}_A^{**}(\varphi\circ\tilde{\phi}(a+J_A)\widetilde{M})=\widetilde{\omega}_A^{**}(\varphi\circ\tilde{\phi}(a+J_A)\widetilde{M})(\varphi\circ\tilde{\phi})\\&=\varphi\circ\tilde{\phi}(a+J_A)\widetilde{\omega}_A^{**}(\widetilde{M})(\varphi\circ\tilde{\phi})=\varphi\circ\tilde{\phi}(a+J_A).\end{aligned}$$

Therefore A is module (ϕ,φ)-biflat.

Remark 3.7. A inverse semigroup is a discrete semigroup S such that for each $s\in S$, there is a unique element $s^*\in S$ with $ss^*s=s$ and $s^*ss^*=s^*$. An element $e\in S$ is called an idempotent if $e^2=e^*=e$. The set of idempotent elements of S is denoted by E.

Let S be an inverse semigroup with the set of idempotents E. We let $l^1(E)$ acts on $l^1(S)$ by multiplication from the right and trivially from the left, that is: $\delta_e.\delta_s=\delta_s$, $\delta_s.\delta_e=\delta_{se}=\delta_s*\delta_e\ (e\in E,s\in S)$. By these actions, $l^1(S)$ becomes a Banach $l^1(E)$-module. In this case, $J_{l^1(S)}=\{\delta_{set}-\delta_{st}\mid e\in E,s,t\in S\}$. We consider an equivalence relation on S as follows $s\approx t\Leftrightarrow\delta_s-\delta_t\in J_{l^1(S)}\quad(s,t\in S)$. For inverse semigroup S, the quotient semigroup $S/\approx$ is discrete group and so $l^1(S/\approx)$ has an identity (see [3] and [11]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S of S (see [10] and [12]). It is also shown in Theorem 3.3 of [13], that $l^1(s)/J_{l^1(s)}\cong l^1(S/\approx)=l^1(G_s)$, is a commutative $l^1(E)$

bimodule with the following actions: $\delta_e.\delta_{[s]} = \delta_{[s]}, \delta_{[s]}.\delta_e = \delta_{[se]}$ $(s \in S, e \in E)$. where $[s]$ denotes the equivalence class of s in G_S. Duncan and Namioka in Theorem 16 of [6], proved that for any inverse semigroup S, $l^1(S)$ has a bounded approximate identity if and only if E satifies condition D_k for some k (Let $k \in \mathbb{N}$. E satifies conditions D_k if for $f_1, f_2, \ldots, f_{k+1} \in E$ there exist $e \in E$ and i, j such that $1 \le i \le j \le k+1, f_i e = f_i, f_j e = f_j$.

THEOREM 3.8. *Let S be an inverse semigroup with the set of idempotents E. Consider $l^1(S)$ as a Banach module over $l^1(E)$ with the trivial left actions and natural right action. Let $\varphi \in \Delta(l^1(E)) \cup \{0\}$ and $\phi \in \Omega_{l^1(S)}$. Then the following statements are valid*:

(i) *If E satifies condition D_k for some k, then S is amenable if and only if $l^1(S)$ is module (ϕ, φ)-biflat*;

(ii) *S is amenable if and only if $l^1(G_S)$ is module $(\tilde{\phi}, \varphi)$-biflat*.

Proof. (i) Let E satifies condition D_k for some k. Since $l^1(S)$ has a bounded approximate identity by Theorem 16 of [6] and $l^1(E)$ act on $l^1(S)$ trivially from the left, result follows from Theorem 3.1 of [5] and Theorem 3.6.

(ii) By Theorem 3.6, $l^1(G_S)$ is module $(\tilde{\phi}, \varphi)$-biflat if and only if $l^1(G_S)$ is module $(\tilde{\phi}, \varphi)$-amenable. It follows from Theorem 3.1 of [5] that S is amenable if and only if $l^1(G_S)$ is module $(\tilde{\phi}, \varphi)$-biflat.

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