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OPTIMAL AUXILIARY FUNCTIONS METHOD FOR NONLINEAR THIN FILM FLOW OF A THIRD GRADE FLUID ON A MOVING BELT

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Abstract. The aim of this paper is to solve approximately the thin film flow of a third grade fluid on a moving belt problem by means of a novel approach: Optimal Auxiliary Functions Method (AOFM). The corresponding nonlinear differential equation is reduced to the linear differential equations which contain some unknown parameters. These parameters are optimally determined by means of the collocation method. Our approach is completely different in structure comparing to other procedures, is very efficient and effective.

Key words: thin film flow of a third grade fluid, optimal auxiliary functions method, optimal parameters, nonlinear problem.

1. INTRODUCTION

In the last time, the subject of non-Newtonian fluid mechanics is very popular due to technological and industrial points of view, where the nonlinear fluid has practical applications. A third grade fluid is one of the most acceptable fluid in this subclass of non-Newtonian fluids. Examples of non-Newtonian fluids include mustard, mayonnaise, toothpaste, asphalt, lava, mud slides, wire and fiber coating, paper production, oil wills, fluid cells, etc. Third grade fluid is one of the important fluid in this category and considerable efforts have been made to study non-Newtonian fluids for various geometrical configurations via analytical procedures. Some developments in this direction are presented in the literature. Wang et al. [1] examined the steady plane flows of an incompressible fluid between two plates in a porous medium in the presence of magnetic field. Electro-osmotic flow is considered by Akgül and Pakdemirli [2] and approximate solutions are obtained by perturbation techniques. Shah et al. [3] obtained an approximate analytical solutions for fluid velocity and temperature distribution for the heat transfer flow of a third grade fluid for the post treatment of wire coating by means of optimal homotopy asymptotic method. Adomian decomposition method is employed by Mahmood and Khan [4] to find the velocity field for three fundamental flow problems that frequently arise in this field, namely plane Couette flow, plane Poiseuille flow and generalized Couette flow. Zaidi et al. [5] applied variation of parameters method to solve the nonlinear differential equations of the thin film flow of a third grade fluid down an inclined plane. Siddiqui et al. [6] used variational iteration method and Adomian decomposition method to obtain analytic approximations of a nonlinear problem that arises in the thin film flow of a third grade fluid on a moving belt. Optimal homotopy asymptotic method and Adomian decomposition method are considered by Gul et al. [7] to solve the problem of heat transfer in electrically conducting thin film flow with slip boundary conditions. Many others researchers as Nayak et al. [8], Gul et al. [9], Siddiqui et al. [10], Shah et al. [11], B. Marinca and V. Marinca [12] contributed to this field of study.

The objective of the present work is to apply OAFM to solve a boundary value problem for nonlinear differential equation of thin film flow of a third grade fluid on a moving belt.

2. THE GOVERNING EQUATION

The basic equations of the thin film flow of third grade fluid on a moving belt are:

$$\nabla \cdot V = 0 \tag{1}$$

$$\rho \frac{\mathrm{D}V}{\mathrm{D}t} = -\nabla p + \rho \mathrm{g} + \mathrm{div}\tau, \tag{2}$$

where V, ρ, p, τ denote velocity vector, constant density, pressure and stress tensor respectively, and $\frac{\mathrm{D}}{\mathrm{D}t}$ is the material derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + V \cdot \nabla. \tag{3}$$

The stress tensor defining a third grade fluid is given by

$$\tau = S_1 + S_2 + S_3, \tag{4}$$

where

$$S_1 = \mu A_1$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2$$

$$S_3 = \beta_1 A_3 + \beta_2 \left(A_1 A_2 + A_2 A_1 \right) + \beta_3 \left(\mathrm{tr} A_2 \right) A_1.$$

Here, μ is the coefficient of viscosity and α_1, α_2, β_1, β_2 and β_3 are material constants. The Rivlin-Ericksen tensor A_n are defined by $A_0 = \mathrm{I}$ (the identity tensor) and

$$A_n = \frac{\mathrm{D}A_{n-1}}{\mathrm{Dt}} + A_{n-1}\left(\nabla V\right) + \left(\nabla V\right)^T A_{n-1}, \; n \geq 1$$

The ambient air is considered to be stationary so that the flow is due to gravity alone. Also, the surface tension can be assumed to be negligible and film is of uniform thickness δ. If the pressure gradient is absent and thermal effects are negligible, the velocity fields is:

$$V = \left(u\left(y\right), 0, 0\right) \tag{5}$$

and U_0 is the speed of the belt.

Boundary conditions are

$$u\left(0\right) = U_0, \; u'\left(\delta\right) = 0, \tag{6}$$

where prime denotes derivative with respect to y.

Inserting the velocity field (5) and the stress tensor (4) into Eqs. (1) and (2), we obtain the momentum equation as:

$$\mu u'' + 6\left(\beta_2 + \beta_3\right)\left(u'\right)^2 u'' - \rho g = 0. \tag{7}$$

The dimensionless variables are defined as:

$$\overline{y} = \frac{y}{\delta}, \quad \overline{u} = \frac{u}{U_0}, \quad m = \frac{\rho g \delta^2}{\mu U_0}, \quad \beta = \beta_2 + \beta_3 \tag{8}$$

and dropping the bar, we obtain dimensionless from of the momentum equation (7) with the boundary conditions (6) in the forms:

$$\overline{u} + 6\beta\left(u'\right)^2 u'' - m = 0 \tag{9}$$

$$u\left(0\right) = 1, \; u'\left(1\right) = 0. \tag{10}$$

3. BASIC IDEAS OF OAFM

In a more general form, Eq (9) with boundary conditions (10) can be written as:

$$\mathrm{L}\left[u(\mathrm{y})\right]+\mathrm{N}\left[u(y)\right]+\mathrm{g}(y)=0, \tag{11}$$

where L, N and g are linear operator, nonlinear operator and a known function, respectively subject to the boundary condition

$$B\left(u(y),\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right)=0. \tag{12}$$

The approximate solution of Eqs. (11) and (12) can be expressed in the form

$$\bar{u}(y,C_i)=u_0(\mathrm{y})+u_1(y,C_i),\quad i=1,2,3,\ldots,r \tag{13}$$

where the initial and the first approximation are determined as follows.

$$\mathrm{L}\left[u_0(y)\right]+\mathrm{g}(y)=0,\ B\left(u_0,\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)=0 \tag{14}$$

$$\mathrm{L}\left[u_1(y,C_i)\right]+\mathrm{N}\left[u_0(y)+u_1(y,C_i)\right]=0,\ B\left(u_1,\frac{\mathrm{d}u_1}{\mathrm{d}y}\right)=0. \tag{15}$$

The nonlinear term in the last equation can be expanded in the form:

$$\mathrm{N}\left[u_0+u_1\right]=\mathrm{N}(u_0)+\sum_{k=1}^{\infty}\frac{u_1^k}{k!}\mathrm{N}^{(k)}(u_0)=\sum_{i=1}^{s}l_i f_\mathrm{i}(y)+\sum_{k=1}^{\infty}\frac{u_1^k}{k!}\mathrm{N}^{(k)}(u_0), \tag{16}$$

where l_i we known parameters, $f_i(y)$ are the functions which appear in the developing of the operator $\mathrm{N}(u_0)$, namely "fundamental functions", *s* being the number of the fundamental functions. To avoid the difficulties that appear in solving of nonlinear differential equation (15) and to accelerate the rapid convergence of the first approximation $u_1(y,C_i)$ and implicit of the solution $\bar{u}(y)$ instead of the last term into Eq. (15), we propose an another expression, such that Eq. (15) can be rewritten as

$$\mathrm{L}\left[u_1(y,C_i)\right]+\sum_{j=1}^{p}A_j(y,C_i)f_j(y)=0,\quad p\le s,\quad \mathrm{B}\left[u_1,\frac{\mathrm{d}u_1}{\mathrm{d}y}\right]=0 \tag{17}$$

where $A_\mathrm{j}(y_1,C_i)$ are auxiliary arbitrary functions depending of variable y and of some unknown parameters C_i, and $f_j(y),\ \ j=1,2,...,p,\ \ p\le s$ are above fundamental functions. The unknown parameters $C_i,\ \ i=1,2,...,r$ can be optimally identified via different methods as Ritz, the least square, Galerkin, collocation method, and so on [12–14]. We will see that this approach is a very powerful tool for solving nonlinear differential problems, without depending on small or large parameters. Our procedure contains auxiliary functions $A_\mathrm{j}(y,C_i)$ which provide with a simple way to adjust and control convergence of the approximate solution after only one iteration.

4. APROVIMATE SOLUTIONS FOR EQS (9) AND (10)

The linear, nonlinear and the function g are defined in the following form:

$$\mathrm{L}\left[u(y)\right]=u'',\ \mathrm{N}\left[u(y)\right]=6\beta(u')^2u'',g(y)=-m. \tag{18}$$

The approximate solution of Eqs (9) and (10) is

$$\bar{u}\left(y,C_i\right)=u_0\left(y\right)+u_1\left(y,C_i\right),\quad i=1,2,...,r \tag{19}$$

where the initial approximation $u_0\left(y\right)$ is determined from Eq. (14), which means that

$$u_0''(y)-m=0,\quad u_0\left(0\right)=1,\quad u'_0\left(1\right)=0 \tag{20}$$

with the solution given by

$$u_0\left(y\right)=1+\frac{1}{2}m\left[\left(1-y\right)^2-1\right]. \tag{21}$$

Inserting Eq. (21) into Eq. (18), the nonlinear operator will then be

$$\mathrm{N}\left[u_0\left(y\right)\right]=-6\beta m^3\left(1-y\right)^2 \tag{22}$$

such that the fundamental functions field and auxiliary functions are respectively:

$$f_j\left(y\right)=\left(1-y\right)^j,\quad A_j=(j+1)(j+2)C_{j+1},\quad j=0,1,2,...,p-1.$$

The first approximation $\mathrm{u}_1\left(\mathrm{y},\mathrm{C}_1\right)$ is obtained from eq. (17), which reduces to

$$u_1''\left(\left(y,C_i\right)\right)+2C_1+6C_2\left(1-y\right)+12C_3\left(1-y\right)^2+...+p\left(p+1\right)C_p\left(1-y\right)^{p-1}=0,\quad u_1\left(0\right)=u_1'\left(1\right)=0 \tag{23}$$

where $C_i,\quad i=1,2,...,r=p$ are unknown parameters at this moment and p is a fixed integer number. From Eq. (23), we obtain the following expression

$$u_i\left(\left(y,C_i\right)\right)=C_1\left[\left(1-y\right)^2-1\right]+C_2\left[\left(1-y\right)^3-1\right]+...+C_p\left[\left(1-y\right)^p-1\right]. \tag{24}$$

From Eqs. (21), (24) and (19), we occur approximate solution of Eqs (9) and (10), through OAFM in the form:

$$\bar{u}\left(\left(y,C_i\right)\right)=1+\left(\frac{1}{2}m+C_1\right)\left[\left(1-y\right)^2-1\right]+C_2\left[\left(1-y\right)^3-1\right]+C_3\left[\left(1-y\right)^4-1\right]+...+C_p\left[\left(1-y\right)^{p+1}-1\right] \tag{25}$$

5. NUMERICAL RESULTS BY OAFM

The accuracy of this method is illustrated for different values of the parameters β, m and p. The results obtained using OAFM are compared with numerical integration results and the parameters C_i are determined by collocation method.

5.1. First, we consider $\beta=1$, $m=4$ and $p=r=10$. The parameters are:
$C_1=-4.222602063672$; $C_2=4.935152677731$; $C_3=9.391271827942$;
$C_4=10.669550079814$; $C_5=-1.090752834895$; $C_6=-18.035066773528$;
$C_7=30.846526371527$; $C_8=-25.547677957819$;
$C_9=11.871489087301$; $C_{10}=-2.0177919572711$.
In this subcase, the approximate solution using OAFM becomes

$$\begin{aligned}\bar{u}\left(y\right)=&1.017551031407+2.222602063672\left(1-y\right)^2-4.935152677731\left(1-y\right)^3+9.391271827942\left(1-y\right)^4-\\&-10.669550079814\left(1-y\right)^5+1.090752834895\left(1-y\right)^6+18.035066773528\left(1-y\right)^7-30.846526371527\left(1-y\right)^8+\\&+25.547677957819\left(1-y\right)^9-11.871489087301\left(1-y\right)^{10}+2.01779572711\left(1-y\right)^{11}.\end{aligned} \tag{26}$$

5.2. In the last case, for $\beta = 3$, m = 3 and $p = r$ = 10, we obtain

$$\begin{aligned}\bar{u}(y) = {} & 0.5006021838046 + 1.693388059193(1-y)^2 - 5.276145236459(1-y)^3 + 15.803996128621(1-y)^4 - \\ & -37.019716793867(1-y)^5 + 64.574233173757(1-y)^6 - 81.501423102311(1-y)^7 + 71.920744556694(1-y)^8 - \\ & -41.9453960227434(1-y)^9 + 14.4915556210316(1-y)^{10} - 2.24183856772081(1-y)^{11}.\end{aligned} \quad (27)$$

To verify the accuracy of the obtained solutions, we compare these analytical results with numerical ones. From Tables 1 and 2 it can be seen that the analytical solutions of the problem (9) and (10) obtained using OAFM are very accurate. Our procedure does not depend upon small parameters.

Table 1

Comparison between the OAFM solution (26) with numerical solutions for m = 4 and $\beta = 1$

y	$u'_{\text{Numerical}}(y)$	$\bar{u}'(y)$ Eq. (26)	$\varepsilon(y) = \lvert u'_{\text{Num}}(y) - \bar{u}'(y)\rvert$
0	-1.12817389837567	-1.1281648632381	$9.1 \cdot \text{E}-06$
0.1	-1.08007201020173	-1.08007203203744	$2.2 \cdot \text{E}-08$
0.2	-1.02789810677876	-1.02798911439358	$7.3 \cdot \text{E}-09$
0.3	-0.97069990494008	-0.9706987450879	$1.16 \cdot \text{E}-06$
0.4	-0.90712039216312	-0.907120521578852	$1.28 \cdot \text{E}-07$
0.5	-0.835122348481375	-0.835122512467256	$1.64 \cdot \text{E}-07$
0.6	-0.751426477126471	-0.751426499955896	$2.29 \cdot \text{E}-08$
0.7	-0.650212114581314	-0.650212008248636	$1.06 \cdot \text{E}-07$
0.8	-0.519536046602793	-0.51953618755021	$1.33 \cdot \text{E}-07$
0.9	-0.328865037138871	-0.328865348065594	$3.11 \cdot \text{E}-07$
1	0	0	0

Table 2

Comparison between the OAFM solution (27) with numerical solutions for m = 3 and $\beta = 3$

y	$u'_{\text{Numerical}}(y)$	$\bar{u}'(y)$ Eq. (27)	$\varepsilon(y) = \lvert u'_{\text{Num}}(y) - \bar{u}'(y)\rvert$
0	-0.723902189531621	-0.7239024947717	$3.5 \cdot \text{E}-07$
0.1	-0.69404836540602	-0.69404854561906	$1.80 \cdot \text{E}-07$
0.2	-0.661695527325251	-0.6616957584503	$2.38 \cdot \text{E}-07$
0.3	-0.626262357730877	-0.626262402142876	$4.44 \cdot \text{E}-08$
0.4	-0.58692056853108	-0.586920612120674	$4.35 \cdot \text{E}-08$
0.5	-0.542425860024423	-0.54242591769214	$4.04 \cdot \text{E}-08$
0.6	-0.490770558886644	-0.490770578225402	$1.93 \cdot \text{E}-08$
0.7	-0.428368240200272	-0.428368185506899	$5.45 \cdot \text{E}-08$
0.8	-0.3477277827361	-0.367727761086688	$2.17 \cdot \text{E}-08$
0.9	-0.228457211360594	-0.22845722885654	$8.25 \cdot \text{E}-08$
1	0	0	0

6. CONCLUSIONS

In the present work, we applied a new method (OAFM) to determine analytic approximate solutions to thin film flow of a third grade fluid on a moving belt. Let us emphasize that our construction of the OAFM is different from any other procedures, especially referring to the fundamental functions and in consequence to the auxiliary functions $A_j(y, C_i)$ and to some parameters $C_1, C_2, \ldots$ which ensure a very rapid convergence of the solutions. Our procedure is valid even if the nonlinear equation does contain small or large parameters. Let us note that very good results are obtained after only one iteration and in only a few terms.

Our approach provides us with simple but rigorous way to control and adjust the convergence of the solutions by means of several convergence-control parameters C_i which are optimally determined. Optimal auxiliary functions method is very powerful, effective, efficient and easy to use.

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