THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 20, Number 1/2019, pp. 11–17

GENERAL SUM-CONNECTIVITY INDEX OF TREES AND UNICYCLIC GRAPHS WITH FIXED MAXIMUM DEGREE

Muhammad Kamran JAMIL [1], Ioan TOMESCU [2]

[1] GC University, Abdus Salam School of Mathematical Sciences, Lahore, Pakistan.
Riphah Institute of Computing and Applied Sciences,
Riphah International University, Department of Mathematics, Lahore, Pakistan.
[2] University of Bucharest, Faculty of Mathematics and Computer Science, Romaina
Corresponding author: Muhammad Kamran Jamil, Email: m.kamran.sms@gmail.com

Abstract. The general sum-connectivity index of a graph G is defined as $\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u)+d(v))^\alpha$, where $d(v)$ denotes the degree of the vertex v in G and α is a real number. In this paper it is deduced the maximum value for the general sum-connectivity index of n-vertex trees for $-1.7036 \le \alpha < 0$ and of n-vertex unicyclic graphs for $-1 \le \alpha < 0$ respectively, with fixed maximum degree Δ. The corresponding extremal graphs, as well as the n-vertex unicyclic graphs with the second maximum general sum-connectivity index for $n \ge 4$ are characterized. This extends the corresponding results by Du, Zhou and Trinajsti cʹ [arXiv.org/1210.5043] about sum-connectivity index.

Key words: Vertex degree, tree, unicyclic graphs, maximum degree, general sum-connectivity index.

1. INTRODUCTION

Let $G(V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ are the sets of vertices and of edges, respectively. For a vertex $v \in V(G)$, $d(v)$ denotes the degree of vertex v, $N(v)$ is the set of vertices adjacent to v and the maximum vertex degree of the graph G is denoted by $\Delta(G)$. If $uv \in E(G)$, $G-uv$ denotes the subgraph of G obtained by deleting the edge uv; similarly is defined the graph $G+uv$ if $uv \notin E(G)$. For $n \ge 3$ let $T(n,\Delta)$ be the set of trees with n vertices and maximum degree Δ and $U(n,\Delta)$ be the set of unicyclic graphs with n vertices and maximum degree Δ ($2 \le \Delta \le n-1$). Let P_n and C_n be the path and the cycle, respectively, on $n \ge 3$ vertices. For $\Delta = 2$, $T(n,\Delta) = \{P_n\}$ and $U(n,\Delta) = \{C_n\}$. Attaching a path P_r to a vertex v of a graph means adding an edge between v and a terminal vertex of the path. If $r = 1$, then we attach a pendant vertex.

The Randić index $R(G)$ was proposed by Randić [10]. This is one of the most used molecular descriptors in structure-property and structure-activity relationship studies [2, 6, 7, 8]. The Randić index is also called product-connectivity index and it is defined as

$$R(G) = \sum_{uv \in E(G)} (d(u)d(v))^{-1/2}.$$

Bollobás and Erdös [1] generalized the idea of Randić index and proposed the general Randić index, denoted as R_α. It is defined as

$$R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha,$$

where α is a real number.

The sum-connectivity index was proposed by Trinajstić et al. [13] and it was observed that sum-connectivity index and product-connectivity index correlate well among themselves and with the Π-electronic energy of benzenoid hydrocarbons [9]. This concept was extended to the general sum-connectivity index in [14] and defined as

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u)+d(v))^\alpha.$$

Then $\chi_{-1/2}$ is the sum-connectivity index [13].

Several extremal properties of the sum-connectivity and general sum-connectivity indices for trees, unicyclic graphs and general graphs were given in [3, 4, 11, 12, 13, 14].

In [5] Zhou et al. obtained the maximum sum-connectivity index of graphs in the set of trees and in the set of unicyclic graphs respectively, with a given number of vertices and maximum degree and determined the corresponding extremal graphs. They also found the n-vertex unicyclic graphs with the first two maximum sum-connectivity indices for $n \geq 4$. In this paper we extend these results for the general sum-connectivity index.

2. MAIN RESULTS

First we will discuss two lemmas that will be used in the proofs.

LEMMA 2.1 [4]. *Let Q be a connected graph with at least two vertices. For $a \geq b \geq 1$, let G_1 be the graph obtained from Q by attaching two paths P_a and P_b to $u \in V(Q)$ and G_2 the graph obtained from Q by attaching a path P_{a+b} to u. Then $\chi_\alpha(G_2) > \chi_\alpha(G_1)$, for $\alpha_1 \leq \alpha < 0$, where $\alpha_1 \approx -1.7036$ is the unique root of the equation $\dfrac{3^\alpha - 4^\alpha}{4^\alpha - 5^\alpha} = 2$.*

The following property is an extension of a transformation defined in [5].

LEMMA 2.2 [5]. *Let M be a connected graph with $|V(M)| \geq 3$ and u be a vertex of degree two of M. Let H be the graph obtained from M by attaching a path P_a to u. Denote by u_1 and u_2 the two neighbors of u in M, and by u' the pendant vertex of the path attached to u in H. If $d_H(u_2) \leq 3$, then for $H' = H - \{uu_2\} + \{u'u_2\}$ we have $\chi_\alpha(H') > \chi_\alpha(H)$, where $-1 \leq \alpha < 0$.*

Proof. If $d_H(u,u') = 1$, then for $\alpha < 0$ we have:

$$\chi_\alpha(H') - \chi_\alpha(H) = (d_H(u_1)+2)^\alpha + (d_H(u_2)+2)^\alpha - (d_H(u_1)+3)^\alpha - (d_H(u_2)+3)^\alpha > 0.$$

If $d_H(u,u') \geq 2$, then

$$\begin{aligned}\chi_\alpha(H') - \chi_\alpha(H) &= (d_H(u_1)+2)^\alpha - (d_H(u_1)+3)^\alpha + (d_H(u_2)+2)^\alpha - (d_H(u_2)+3)^\alpha + 2\cdot 4^\alpha - 3^\alpha - 5^\alpha \\ &> (d_H(u_2)+2)^\alpha - (d_H(u_2)+3)^\alpha + 2\cdot 4^\alpha - 3^\alpha - 5^\alpha.\end{aligned}$$

Since $(x+2)^\alpha - (x+3)^\alpha$ is decreasing for $x \geq 0$, we have $(d_H(u_2)+2)^\alpha - (d_H(u_2)+3)^\alpha \geq 5^\alpha - 6^\alpha$. Therefore

$$\chi_\alpha(H') - \chi_\alpha(H) > 2\cdot 4^\alpha - 3^\alpha - 6^\alpha.$$

The function $\eta(x) = 2\cdot 4^x - 3^x - 6^x$ has roots $x_1 = -1$ and $x_2 = 0$ and $\eta(x) > 0$ for $x \in (-1,0)$ [15]. It follows that $\chi_\alpha(H') > \chi_\alpha(H)$ for every $-1 \leq \alpha < 0$. For $\dfrac{n}{2} \leq \Delta \leq n-1$, let $T_{n,\Delta}$ be the tree obtained by attaching $2\Delta+1-n$ pendant vertices and $n-\Delta-1$ paths of length two to a vertex. For $\frac{n+2}{2}\le\Delta\le n-1$, let $U_{n,\Delta}$ be the unicyclic graph obtained by attaching $2\Delta-n-1$ pendant vertices and $n-\Delta-1$ paths of length two to the same vertex of a triangle.

For $\frac{n}{2}\le\Delta\le n-1$, let $T_{n,\Delta}$ be the tree obtained by attaching $2\Delta+1-n$ pendant vertices and $n-\Delta-1$ paths of length two to a vertex. For $\frac{n+2}{2}\le\Delta\le n-1$, let $U_{n,\Delta}$ be the unicyclic graph obtained by attaching $2\Delta-n-1$ pendant vertices and $n-\Delta-1$ paths of length two to the same vertex of a triangle.

THEOREM 2.3. *Let $G\in T(n,\Delta)$, where $2\le\Delta\le n-1$ and $\alpha_1\le\alpha<0$, where $\alpha_1\approx-1.7036$ is the unique root of the equation $\frac{3^\alpha-4^\alpha}{4^\alpha-5^\alpha}=2$. Then*

$$\chi_\alpha(G)\le\begin{cases}((\Delta+2)^\alpha-(\Delta+1)^\alpha+3^\alpha)(n-\Delta-1)+\Delta(\Delta+1)^\alpha & \text{if } \frac{n}{2}\le\Delta\le n-1\\ ((\Delta+2)^\alpha+3^\alpha-4^\alpha)\Delta+(n-\Delta-1)4^\alpha & \text{if } 2\le\Delta\le\frac{n-1}{2}\end{cases}$$

and equality holds if and only if $G=T_{n,\Delta}$ for $\frac{n}{2}\le\Delta\le n-1$, and G is a tree obtained by attaching Δ paths of length at least two to a unique vertex for $2\le\Delta\le\frac{n-1}{2}$.

Proof. The case $\Delta=2$ is clear since in this case $G=P_n$. Suppose that $\Delta\ge3$ and let G be a tree in $T(n,\Delta)$ having maximum general sum-connectivity index. Let v be a vertex of degree Δ in G. If there exists some vertex of degree greater than two in G different from v, then by Lemma 2.1, we may get a tree in $T(n,\Delta)$ with greater general sum-connectivity index, a contradiction. It follows that v is the unique vertex of degree greater than two in G. Let k be the number of neighbors of v with degree two. Since in $V(G)\setminus(\{v\}\cup N(v))$ there are $n-\Delta-1$ vertices, it follows that $k\le\min\{n-\Delta-1,\Delta\}$. If $n-\Delta-1\ge\Delta$, i.e., $\Delta\le\frac{n-1}{2}$, then $1\le k\le\Delta$. If $n-\Delta-1<\Delta$, i.e., $\Delta\ge\frac{n}{2}$, then $0\le k\le n-\Delta-1$. We get

$$\chi_\alpha(G)=(\Delta-k)(\Delta+1)^\alpha+k(\Delta+2)^\alpha+k\cdot3^\alpha+(n-\Delta-k-1)4^\alpha$$
$$=k((\Delta+2)^\alpha-(\Delta+1)^\alpha+3^\alpha-4^\alpha)+\Delta(\Delta+1)^\alpha+(n-\Delta-1)4^\alpha.$$

Since $(\Delta+2)^\alpha-(\Delta+1)^\alpha$ is increasing for $\Delta\ge3$ we obtain
$(\Delta+2)^\alpha-(\Delta+1)^\alpha+3^\alpha-4^\alpha\ge5^\alpha+3^\alpha-2\cdot4^\alpha>0$, the last inequality holding by Jensen's inequality. Consequently,

$$\chi_\alpha(G)\le\begin{cases}((\Delta+2)^\alpha-(\Delta+1)^\alpha+3^\alpha)(n-\Delta-1)+\Delta(\Delta+1)^\alpha & \text{if } \frac{n}{2}\le\Delta\le n-1\\ ((\Delta+2)^\alpha+3^\alpha-4^\alpha)\Delta+(n-\Delta-1)4^\alpha & \text{if } 2\le\Delta\le\frac{n-1}{2}.\end{cases}$$

For $\frac{n}{2} \le \Delta \le n-1$, the equality holds if and only if $k = n-\Delta-1$, i.e., each of the $n-\Delta-1$ neighbors of degree two of the vertex v is adjacent to exactly a pendant vertex, i.e., $G = T_{n,\Delta}$. For $2 \le \Delta \le \frac{n-1}{2}$ the equality holds for $k = \Delta$, i.e., G is a tree obtained by attaching Δ paths of length at least two to a unique vertex.

Now we obtain the maximum general sum-connectivity index of graphs in $U(n,\Delta)$ and deduce the extremal graphs. As a consequence, we deduce the n-vertex unicyclic graphs with the first and second maximum general sum-connectivity indices for $n \ge 4$.

THEOREM 2.4. *Let* $G \in U(n,\Delta)$, *where* $2 \le \Delta \le n-1$ *and* $-1 \le \alpha < 0$. *Then*

$$\chi_\alpha(G) \le \begin{cases} (n-\Delta-1)3^\alpha + (n-\Delta+1)(\Delta+2)^\alpha + (2\Delta-n-1)(\Delta+1)^\alpha + 4^\alpha & \text{if } \frac{n+2}{2} \le \Delta \le n-1 \\ (\Delta-2)3^\alpha + \Delta(\Delta+2)^\alpha + (n-2\Delta+2)4^\alpha 3 & \text{if } 2 \le \Delta \le \frac{n+1}{2}. \end{cases}$$

For $\frac{n+2}{2} \le \Delta \le n-1$ the equality holds if and only if $G = U_{n,\Delta}$. If $2 \le \Delta \le \frac{n+1}{2}$ the equality holds if and only if G is a unicyclic graph obtained by attaching $\Delta-2$ paths of length at least two to a fixed vertex of a cycle.

Proof. The case $\Delta = 2$ is trivial since in this case $G = C_n$. Suppose that $\Delta \ge 3$, G is a graph in $U(n,\Delta)$ with maximum general sum-connectivity index, and C is the unique cycle of G. Let v be a vertex of degree Δ in G.

If $\Delta = 3$ and there exists some vertex outside C with degree three, then by Lemma 2.1, we may get a graph in $U(n,3)$ with greater general sum-connectivity index, a contradiction. If there are at least two vertices on C with degree three, then by Lemma 2.2, we may deduce the same conclusion. Thus, $v \in V(C)$ and v is the unique vertex in G with degree three. Then either $\chi_\alpha(G) = (n-2)4^\alpha + 2\cdot 5^\alpha$ when v is adjacent to a vertex of degree one and two vertices of degree two for $n \ge 4$, or $\chi_\alpha(G) = (n-4)4^\alpha + 3\cdot 5^\alpha + 3^\alpha$ when v is adjacent to three vertices of degree two for $n \ge 5$. The difference of these two numbers equals $(n-2)4^\alpha + 2\cdot 5^\alpha - (n-4)4^\alpha - 3\cdot 5^\alpha - 3^\alpha = 2\cdot 4^\alpha - 5^\alpha - 3^\alpha < 0$ by Jensen's inequality. Hence, G is the graph obtained by attaching a pendant vertex to a triangle for $n = 4$, i.e., $G = U_{4,3}$, and a graph obtained by attaching a path of length at least two to a cycle for $n \ge 5$.

Now suppose that $\Delta \ge 4$. As for the case $\Delta = 3$ we deduce that the vertex of maximum degree is unique, otherwise G has not a maximum general sum-connectivity index in $U(n,\Delta)$. We will show that the vertex of maximum degree v lies on C. Suppose that v is not on C. Let w be the vertex on C such that $d_G(v,w) = \min\{d_G(v,x) : x \in V(C)\}$. If there is some vertex outside C with degree greater than two different from v, or if there is some vertex on C with degree greater than two different from w, then by Lemmas 2.1 and 2.2, we may get a graph in $U(n,\Delta)$ with greater general sum-connectivity index, a contradiction. Thus, v and w are the only vertices of degree greater than two in G, and $d_G(v) = \Delta$ and $d_G(w) = 3$. Let Q be the path connecting v and w. Suppose that $v_1, v_2, \cdots, v_{\Delta-1}$ are the neighbors of v outside Q. Let $d_i = d_G(v_i)$ for $i = 1,\ldots,\Delta-1$. Note that since G has maximum general sum-connectivity index, then $d_1,\ldots,d_{\Delta-1} \in \{1,2\}$, since otherwise we can apply Lemma 2.1 and obtain a graph having a greater general sum-connectivity index. Consider $G_1 = G - \{vv_3, \cdots, vv_{\Delta-1}\} + \{wv_3, \cdots, wv_{\Delta-1}\} \in U(n,\Delta)$. Note that $d_{G_1}(w) = \Delta$ and $d_{G_1}(v) = 3$. Then

$$\begin{aligned}\chi_\alpha(G_1) - \chi_\alpha(G) &= (d_1+3)^\alpha - (d_1+\Delta)^\alpha + (d_2+3)^\alpha - (d_2+\Delta)^\alpha + 2(\Delta+2)^\alpha - 2\cdot 5^\alpha \\ &> 5^\alpha - (2+\Delta)^\alpha + 5^\alpha - (2+\Delta)^\alpha + 2(\Delta+2)^\alpha - 2\cdot 5^\alpha = 0,\end{aligned}$$

since the function $(x+3)^\alpha - (x+\Delta)^\alpha$ is strictly decreasing in $x \ge 0$ for $\Delta \ge 4$. Because $d_{G_1}(v) = 3$, then by Lemma 2.1, we may get a graph G' in $U(n,\Delta)$ such that $\chi_\alpha(G') > \chi_\alpha(G_1) \ge \chi_\alpha(G)$, a contradiction. Hence, we have shown that v lies on C.

If there is some vertex outside C with degree greater than two, then by Lemma 2.1 we may obtain a graph in $U(n,\Delta)$ with greater general sum-connectivity index, a contradiction. If there is some vertex on C with degree three, then by Lemma 2.2, we may get a graph in $U(n,\Delta)$ with greater general sum-connectivity index, a contradiction. Thus, G is a graph obtained from C by attaching $\Delta-2$ paths to v. Let k be the number of neighbors of v with degree two. Then as above we get $k \le \min\{n-\Delta-1, \Delta-2\}$. If $n-\Delta-1 \ge \Delta-2$, i.e., $\Delta \le \frac{n+1}{2}$, then $0 \le k \le \Delta-2$. If $n-\Delta-1 < \Delta-2$, i.e., $\Delta \ge \frac{n+2}{2}$, then $0 \le k \le n-\Delta-1$. We get

$$\begin{aligned}\chi_\alpha(G) &= k3^\alpha + (k+2)(\Delta+2)^\alpha + (\Delta-k-2)(\Delta+1)^\alpha + (n-\Delta-k)4^\alpha \\ &= (3^\alpha + (\Delta+2)^\alpha - (\Delta+1)^\alpha - 4^\alpha)k + (\Delta-2)(\Delta+1)^\alpha + 2(\Delta+2)^\alpha + 4^\alpha(n-\Delta).\end{aligned}$$

We have

$$3^\alpha - 4^\alpha + (\Delta+2)^\alpha - (\Delta+1)^\alpha \ge 3^\alpha - 4^\alpha + 6^\alpha - 5^\alpha > 0$$

since the function $f(x) = (x+2)^\alpha - (x+1)^\alpha$ is strictly increasing for $\alpha < 0$, hence $f(\Delta) \ge f(4)$ and $f(4) > f(2)$. It follows that $\chi_\alpha(G)$ is bounded above by

$$\begin{cases}(3^\alpha + (\Delta+2)^\alpha - (\Delta+1)^\alpha - 4^\alpha)(n-\Delta-1) + (\Delta-2)(\Delta+1)^\alpha + 2(\Delta+2)^\alpha + 4^\alpha(n-\Delta) & \text{if } \frac{n+2}{2} \le \Delta \le n-1 \\ (3^\alpha + (\Delta+2)^\alpha - (\Delta+1)^\alpha - 4^\alpha)(\Delta-2) + (\Delta-2)(\Delta+1)^\alpha + 2(\Delta+2)^\alpha + 4^\alpha(n-\Delta) & \text{if } 2 \le \Delta \le \frac{n+1}{2}\end{cases}$$

$$= \begin{cases}(n-\Delta-1)3^\alpha + (n-\Delta+1)(\Delta+2)^\alpha + (2\Delta-n-1)(\Delta+1)^\alpha + 4^\alpha & \text{if } \frac{n+2}{2} \le \Delta \le n-1 \\ (\Delta-2)3^\alpha + \Delta(\Delta+2)^\alpha + (n-2\Delta+2)4^\alpha & \text{if } 2 \le \Delta \le \frac{n+1}{2}\end{cases}$$

Equality holds for $\frac{n+2}{2} \le \Delta \le n-1$ if and only if $k = n-\Delta-1$, i.e., $G = U_{n,\Delta}$; if $2 \le \Delta \le \frac{n+1}{2}$ then equality is reached if and only if $k = \Delta-2$, i.e., G is a unicyclic graph obtained by attaching $\Delta-2$ paths of length at least two to a unique vertex of a cycle.

THEOREM 2.5. *If $-1 \le \alpha < 0$, among the unicyclic graphs on $n \ge 4$ vertices, C_n is the unique graph with maximum general sum-connectivity index, which is equal to $n4^\alpha$. For $n = 4$, $U_{4,3}$ is the unique graph with the second maximum general sum-connectivity index, which is equal to $2\cdot 4^\alpha + 2\cdot 5^\alpha$. For $n \ge 5$,* *the graphs obtained by attaching a path of length at least two to a vertex of a cycle are the unique graphs with the second maximum general sum-connectivity index, which is equal to* $(n-4)4^{\alpha}+3\cdot 5^{\alpha}+3^{\alpha}$.

Proof. For $n=4$ we get

$$\chi_{\alpha}(U_{4,3})-\chi_{\alpha}(C_4)=2\cdot 5^{\alpha}-2\cdot 4^{\alpha}<0.$$

Now, suppose that $n\geq 5$ and G is a unicyclic graph on n vertices. Let Δ be the maximum degree of G, where $2\leq\Delta\leq n-1$. Let $f(x)=(x-2)3^{\alpha}+x(x+2)^{\alpha}+(n-2x+2)4^{\alpha}$ for $x\geq 2$. If $\frac{n+2}{2}\leq\Delta\leq n-1$, then by Theorem 2.4,

$$\begin{aligned}\chi_{\alpha}(G)&\leq(n-\Delta-1)3^{\alpha}+(n-\Delta+1)(\Delta+2)^{\alpha}+(2\Delta-n-1)(\Delta+1)^{\alpha}+4^{\alpha}\\&=f(\Delta)+(n-2\Delta+1)(3^{\alpha}-4^{\alpha}+(\Delta+2)^{\alpha}-(\Delta+1)^{\alpha})<f(\Delta)\end{aligned}$$

since the function $x^{\alpha}-(x+1)^{\alpha}$ is strictly decreasing for $x\geq 0$ and $\Delta\geq 4$.

If $2\leq\Delta\leq\frac{n+1}{2}$, then by Theorem 2.4, $\chi_{\alpha}\leq f(\Delta)$ and equality can be reached. We shall prove that $f'(x)<0$, which implies that $f(x)$ is strictly decreasing for $x\geq 2$. One deduces

$$f'(x)=3^{\alpha}+(x+2)^{\alpha}+\alpha x(x+2)^{\alpha-1}-2\cdot 4^{\alpha}.$$

Let

$$g(x)=(x+2)^{\alpha}+\alpha x(x+2)^{\alpha-1}.$$

We get

$$g'(x)=\alpha(x+2)^{\alpha-2}(x(\alpha+1)+4)<0.$$

So, $g(x)$ is strictly decreasing for $x\geq 2$, thus implying $g(x)\leq 4^{\alpha}+2\alpha 4^{\alpha-1}$. Consequently,

$$f'(x)\leq 3^{\alpha}-4^{\alpha}+2\alpha 4^{\alpha-1}=4^{\alpha}\left(\left(\frac{3}{4}\right)^{\alpha}+\frac{\alpha}{2}-1\right).$$

Considering the function $h(x)=\frac{x}{2}+(\frac{3}{4})^{x}$, we get $h''(x)=(ln(\frac{3}{4}))^{2}(\frac{3}{4})^{x}>0$, hence $h(x)$ is strictly convex. Since $h(-1)=5/6<1$ and $h(0)=1$, $h(x)$ being strictly convex on $[-1,0)$, it follows that $h(x)<1$ on this interval, or $f'(x)<0$ for every $-1\leq\alpha<0$, hence $f(x)$ is strictly decreasing for $x\geq 2$. It follows that for $3<\frac{n+2}{2}\leq\Delta\leq n-1$ we have $\chi_{\alpha}(G)<f(\Delta)<f(3)<f(2)$ and for $3\leq\Delta\leq\frac{n+1}{2}$ we obtain $\chi_{\alpha}(G)\leq f(\Delta)\leq f(3)<f(2)$. It follows that C_n is the unique n-vertex unicyclic graph with maximum general sum-connectivity index, equal to $f(2)$. Also the n-vertex unicyclic graphs with maximum degree $\Delta=3$ and general sum-connectivity index $f(3)$ are the n-vertex graphs with the second maximum general sum-connectivity index. By Theorem 2.4, these graphs consist from a cycle C_l of an arbitrary length l, $3\leq l\leq n-2$ and a path of length at least two attached to a vertex of C_l.

REFERENCES

1. B. Bollobás, P. Erdös, *Graphs of extremal weights*, Ars Combin., **50**, pp. 225-233 (1998).
2. R. Garcia-Domenech, J. Gálvez, J. V. de Julián-Ortiz, L. Pogliani, *Some new trends in chemical graph theory*, Chem. Rev., **108**, pp. 1127-1169 (2008).

3. Z. Du, B. Zhou, N. Trinajstić, *Minimum general sum-connectivity index of unicyclic graphs*, J. Math. Chem., 48, pp. 697-703 (2010).
4. Z. Du, Bo Zhou, N. Trinajstić, *On the general sum-connectivity index of trees*, Applied Mathematics Letters, **24**, pp. 402-405, (2011).
5. Z. Du, B. Zhou, N. Trinajstić, *Sum-connectivity indices of trees and unicyclic graphs of fixed maximum degree*, arXiv.org/1210.5043.
6. I. Gutman, B. Furtula (Eds.), *Recent Results in the Theory of Randić Index*, Univ. Kragujevac, Kragujevac, 2008.
7. L. B. Kier, L. H. Hall, *Molecular Connectivity in Structure-Activity Analysis*, Wiley, New York, 1986.
8. X. Li, I. Gutman, *Mathematical Aspects of Randić-type Molecular Structure Descriptors*, Univ. Kragujevac, Kragujevac, 2006.
9. B. Lučić, N. Trinajstić, B. Zhou, *Comparison between the sum-connectivity index and product-connectivity index for benzenoid hydrocarbons*, Chem. Phys. Lett., **475**, pp. 146-148 (2009).
10. M. Randić, *On characterization of molecular branching*, J. Am. Chem. Soc., **97**, pp. 6609-6615 (1975).
11. I. Tomescu, M. Arshad, *On the general sum-connectivity index of connected unicyclic graphs with k pendant vertices*, Discr. Appl. Math., **181**, pp. 306-309 (2015).
12. I. Tomescu, M. K. Jamil, *Maximum general sum-connectivity index for trees with given independence number*, MATCH Commun. Math. Comput. Chem., **72**, pp. 715-722 (2014).
13. B. Zhou, N. Trinajstić, *On a novel connectivity index*, J. Math. Chem., **46**, pp. 1252-1270 (2009).
14. B. Zhou, N. Trinajstić, *On general sum-connectivity index*, J. Math. Chem., **47**, pp. 210-218 (2010).
15. http:www.wolframalpha.com

Received August 13, 2018