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# SECOND ORDER DIFFERENTIAL EQUATION WITH PERIODIC FUNDAMENTAL MATRIX

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**Abstract**. A class of linear differential equations having a coefficient depending on two real parameters is considered. The dependence on the parameters ensures the existence of an explicit periodical analytical solution. For a certain dependence of the coefficient parameters, the second solution of the differential equation is also periodic. Parameter dependence is specified by an integral with parameters. We deduce the analytical expression of the second solution of the equation.

***Key words:*** linear differential equation, dynamic system, parametric resonance.

## 1. PROBLEM FORMULATION

Let us consider the following second order periodic linear differential equation with respect to real dimensionless time $t$. The coefficient of the equation depends on the real small $p$ and $k$ parameters

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + Q(t,p,k)\, z = 0 \tag{1}$$

In what follows we assume that the coefficient $Q(t,p,k)$ is a real positive bounded continuous periodic function with respect to $t$. By the working hypothesis we admit the particular abstract case

$$Q(t,p,k) = 1 + 4\frac{3p - \left[4p + 5(p^2-k)\right]\cos^2 t + 6(p^2-k)\cos^4 t}{1 - 2p\cos^2 t + (p^2-k)\cos^4 t} d\,. \tag{2}$$

The set of solutions of linear equation is the two dimensional real space [1,2]. The function $x$ is a periodic solution [3,4]

$$x(t) = \frac{\cos t}{(1-p)^2 - k}\left[1 - 2p\cos^2 t + (p^2-k)\cos^4 t\right], \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + Qx = 0, \;\; x(0) = 1, \;\; \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,. \tag{3}$$

The second solution is described by *Floquet* theory [5,7].

Let $u$ be the derivative of solution $x$. The $x$ and $u$ periodic functions check the next system

$$u(t) = \frac{-\sin t}{(1-p)^2 - k}\left[1 - 6p\cos^2 t + 5(p^2-k)\cos^4 t\right], \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u\,, \;\; \frac{\mathrm{d}u}{\mathrm{d}t} = -Qx\,, \;\; u(0) = 0\,. \tag{4}$$

If $k$ is different from zero, then by composing, the real or complex following constants depend on the $p$ and $k$ parameters

$$a = \sqrt{1 - p - \sqrt{k}}\,, \quad b = \sqrt{1 - p + \sqrt{k}}\,, \quad A = p + \frac{p^2 + k}{2\sqrt{k}}, \quad B = 2p - A, \quad C = A\left(2 + \frac{B}{\sqrt{k}} + \frac{A}{2a^2}\right), \tag{5}$$

$$D = B\left(2 - \frac{A}{\sqrt{k}} + \frac{B}{2b^2}\right), \quad \beta_1 = \left[(1-p)^2 - k\right]^2 \frac{C}{a}, \quad \beta_2 = \left[(1-p)^2 - k\right]^2 \frac{D}{b}, \quad \sigma = \beta_1 + \beta_2 . \tag{6}$$

The unknown $y_{\mathrm{p}}$ and $v_{\mathrm{p}}$ functions represent the periodic solution of the next non-homogeneous system

$$\frac{\mathrm{d}y_{\mathrm{p}}}{\mathrm{d}t} = v_{\mathrm{p}}, \quad \frac{\mathrm{d}v_{\mathrm{p}}}{\mathrm{d}t} = -Q\,y_{\mathrm{p}} - 2\sigma u, \quad y_{\mathrm{p}}(0) = 0, \quad v_{\mathrm{p}}(0) = 1 - \sigma . \tag{7}$$

The function $y$ and its derivative $v$ have the following expressions

$$\begin{aligned} &y = y_{\mathrm{p}} + \sigma t x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v = v_{\mathrm{p}} + \sigma x + \sigma t u, \\ &\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v_{\mathrm{p}}}{\mathrm{d}t} + 2\sigma u + \sigma t \frac{\mathrm{d}u}{\mathrm{d}t} = -Q y_{\mathrm{p}} - \sigma t Q x. \end{aligned} \tag{8}$$

This function and its derivative represent a solution of the homogeneous system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -Q\,y, \quad y(0) = 0, \quad v(0) = 1 . \tag{9}$$

Since the derivative of following expression is zero, the functions $x$, $u$, $y$, $v$ have the integral property

$$x v - u y = 1 . \tag{10}$$

Consequently, the *Floquet*'s expression of the fundamental matrix follows (see [5,6,7])

$$\Phi(t) = \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} x & y_{\mathrm{p}} \\ u & v_{\mathrm{p}} + \sigma x \end{bmatrix} + \sigma t \begin{bmatrix} 0 & x \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & y_{\mathrm{p}} \\ u & v_{\mathrm{p}} + \sigma x \end{bmatrix} \exp\left(\begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix} t\right). \tag{11}$$

The problem is to prove the expression of the coefficient $\sigma$ and to obtain analytically the periodic solution $y_{\mathrm{p}}$, $v_{\mathrm{p}}$. Let $y_{\mathrm{p}1}$ be the first term of the $y_{\mathrm{p}}$ expression

$$\begin{aligned} &K_1 = \frac{1}{2}\left(\frac{A^2}{a^2} + \frac{B^2}{b^2}\right) - 2p, \quad K_2 = p^2 - k - \frac{1}{2}\left[\left(p - \sqrt{k}\right)\frac{A^2}{a^2} + \left(p + \sqrt{k}\right)\frac{B^2}{b^2}\right], \\ &y_{\mathrm{p}1}(t) = \left(1 - 2p + p^2 - k\right)\left(1 + K_1 \cos^2 t + K_2 \cos^4 t\right) \sin t . \end{aligned} \tag{12}$$

Let $\gamma$ be the periodic function

$$\gamma(t) = \frac{\beta_1 a}{1 + (a^2 - 1)\cos^2 t} + \frac{\beta_2 b}{1 + (b^2 - 1)\cos^2 t} . \tag{13}$$

The second term $y_{p2}$ of the $y_p$ expression follows as

$$y_{\mathrm{p}2}(t) = x(t)\,C_2(t), \quad C_2(t) = \int_0^t [\,\gamma(t) - \sigma\,]\,\mathrm{d}t, \quad y_{\mathrm{p}}(t) = y_{\mathrm{p}1}(t) + y_{\mathrm{p}2}(t) . \tag{14}$$

Thus

$$\begin{aligned} &v_{\mathrm{p}2}(t) = \frac{\mathrm{d}y_{\mathrm{p}2}}{\mathrm{d}t} = u(t)\,C_2(t) + x(t)[\gamma(t) - \sigma, \qquad v_{\mathrm{p}}(t) = v_{\mathrm{p}1}(t) + v_{\mathrm{p}2}(t) \\ &v_{\mathrm{p}1}(t) = \frac{\mathrm{d}y_{\mathrm{p}1}}{\mathrm{d}t} = (1 - 2p + p^2 - k)\left[1 - 2K_1 + \left(3K_1 - 4K_2\right)\cos^2 t + 5K_2 \cos^4 t\right]\cos t. \end{aligned} \tag{15}$$

The $y_{\mathrm{p}}$ and $v_{\mathrm{p}}$ functions represent the periodic solution of the non-homogeneous system (7).

## 2. EXPLICIT CHARACTERISTIC COEFFICIENT

From formulas (7) and (9) result in the non-homogeneous equation for $y_p$

$$x\left(v_{\mathrm{p}}+\sigma x+\sigma t u\right)-u\left(y_{\mathrm{p}}+\sigma t x\right)=x\left(\frac{\mathrm{d} y_{\mathrm{p}}}{\mathrm{d} t}+\sigma x\right)-u y_{\mathrm{p}}=1 . \tag{16}$$

Let $C$ be the variable constant of integration

$$y_{\mathrm{p}}=C x \quad \Rightarrow \quad \frac{\mathrm{d} C}{\mathrm{d} t}=\frac{1}{x^{2}}-\sigma, \quad C(0)=0, \tag{17}$$

where

$$\frac{1}{x^{2}}=\frac{(1-2 p+p^{2}-k)^{2}}{\cos ^{2} t}\left[1+\frac{1}{[1-2 p \cos ^{2} t+(p^{2}-k) \cos ^{4} t]^{2}}-1\right] . \tag{18}$$

The derivative of the next function $C_0$ it is a singularity of function $x^{-2}$. The product $C_0\,x$ will be a periodic bounded function

$$C_0=[(1-p)^2-k] \tan t, \quad C_0\, x=[(1-p)^2-k]\,[1-2 p \cos^2 t+(p^2-k)\cos^4 t\,] \sin t, \quad C_\sigma=C-C_0. \tag{19}$$

Consequently

$$y_{\mathrm{p}}=C_{0} x+C_{\sigma} x \quad \Rightarrow \quad \frac{\mathrm{d} C_{\sigma}}{\mathrm{d} t}=\left[\frac{1}{x^{2}}-\frac{(1-2 p+p^{2}-k)^{2}}{\cos ^{2} t}\right]-\sigma, \quad C_{\sigma}(0)=0 . \tag{20}$$

The first expression of characteristic coefficient will be

$$\sigma=\frac{2}{\pi} \int_{0}^{\frac{\pi}{2}}\left[\frac{1}{x^{2}}-\frac{(1-2 p+p^{2}-k)^{2}}{\cos ^{2} t}\right] \mathrm{d} t, \qquad \xi(t)=1-2 p \cos ^{2} t+(p^{2}-k) \cos ^{4} t,$$
$$\sigma=\frac{2}{\pi}(1-2 p+p^{2}-k)^{2} \int_{0}^{\frac{\pi}{2}} \frac{2 p-(p^{2}-k) \cos ^{2} t}{\xi(t)}\left(\frac{1}{\xi(t)}+1\right) \mathrm{d} t. \tag{21}$$

We will obtain the second expression of σ.

## 3. SECOND EXPLICIT SOLUTION

If $k$ is different from zero, then by composition, the real or complex constants depend on the $p$ and $k$ parameters, according to formula (5)

$$a^2 b^2=(1-p)^2-k, \qquad a^2+b^2=2(1-p), \qquad b^2-a^2=2\sqrt{k} . \tag{22}$$

We make the change of the variable of integration

$$s=\tan t, \qquad \frac{\mathrm{d} s}{\mathrm{d} t}=\frac{1}{\cos ^{2} t}, \qquad \cos ^{2} t=\frac{1}{s^{2}+1} \tag{23}$$

$$\frac{1}{x}=\frac{(1-p)^{2}-k}{\cos t} \frac{1}{1-2 p \cos ^{2} t+(p^{2}-k) \cos ^{4} t}=\frac{(1-p)^{2}-k}{\cos t} \frac{(s^{2}+1)^{2}}{(s^{2}+1)^{2}-2 p(s^{2}+1)+p^{2}-k} \tag{24}$$

The last denominator has the equivalent expression

$$(s^2+1)^2-2p(s^2+1)+p^2-k=(s^2+a^2)\,(s^2+b^2)$$

$$\frac{(s^{2}+1)^{2}}{(s^{2}+1)^{2}-2 p(s^{2}+1)+p^{2}-k}=1+\frac{2 p(s^{2}+1)-p^{2}+k}{(s^{2}+a^{2})(s^{2}+b^{2})}=1+\frac{A}{s^{2}+a^{2}}+\frac{B}{s^{2}+b^{2}} \tag{25}$$

From equation (17) we deduce the equation of function $h(s)$

$$C(t) = h(s), \qquad \frac{\mathrm{d}C}{\mathrm{d}t} = \frac{1}{x^2} - \sigma = \frac{\mathrm{d}h}{\mathrm{d}s}\frac{1}{\cos^2 t} = \frac{[(1-p)^2 - k]^2}{\cos^2 t}\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 - \sigma$$

$$\frac{\mathrm{d}h}{\mathrm{d}s} = (1-2p+p^2-k)^2\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 - \frac{\sigma}{s^2+1}, \qquad h(0) = 0\,. \tag{26}$$

For $k$ different from zero there are the following identities

$$\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 = 1 + \frac{A^2}{(s^2+a^2)^2} + \frac{B^2}{(s^2+b^2)^2} + \frac{A\left(2+\frac{B}{\sqrt{k}}\right)}{s^2+a^2} + \frac{B\left(2-\frac{A}{\sqrt{k}}\right)}{s^2+b^2} \tag{27}$$

$$\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 = 1 + \frac{A^2}{(s^2+a^2)^2} + \frac{B^2}{(s^2+b^2)^2} + \frac{2A}{s^2+a^2} + \frac{2B}{s^2+b^2} + \frac{2AB}{b^2-a^2}\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right) \tag{28}$$

On the other hand

$$\frac{A^2}{(s^2+a^2)^2} = \frac{A^2}{2a^2}\left[\frac{a^2-s^2}{(s^2+a^2)^2} + \frac{1}{s^2+a^2}\right] = \frac{A^2}{2a^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+a^2}\right) + \frac{A^2}{2a^2}\frac{1}{s^2+a^2}\,. \tag{29}$$

Consequently

$$\left(1 + \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}\right)^2 = \frac{\mathrm{d}}{\mathrm{d}s}\left\{s + \frac{A^2}{2a^2}\left(\frac{s}{s^2+a^2}\right) + \frac{B^2}{2b^2}\left(\frac{s}{s^2+b^2}\right)\right\} +$$
$$+ A\left(2 + \frac{B}{\sqrt{k}} + \frac{A}{2a^2}\right)\frac{1}{s^2+a^2} + B\left(2 - \frac{A}{\sqrt{k}} + \frac{B}{2b^2}\right)\frac{1}{s^2+b^2}\,. \tag{30}$$

The unknown $h(s)$ has two $h_1(s)$ and $h_2(s)$ terms

$$h(\mathrm{s}) = h_1(s) + h_2(s), \qquad h_1(s) = (1-2p+p^2-k)^2\left(s + \frac{A^2}{2a^2}\cdot\frac{s}{s^2+a^2} + \frac{B^2}{2b^2}\cdot\frac{s}{s^2+b^2}\right)$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}s} = (1-2p+p^2-k)^2\left(\frac{C}{s^2+a^2} + \frac{D}{s^2+b^2}\right) - \frac{\sigma}{s^2+1}. \tag{31}$$

Thus

$$y_{\mathrm{p1}}(t) = x(t)\,h_1(\tan t) = (1-2p+p^2-k)^2\,[1 - 2p\cos^2 t + (p^2-k)\cos^4 t + E\,]\sin t, \tag{32}$$

where

$$E = 0.5\,(\cos^4 t)[A^2 a^{-2}(\tan^2 t + b^2) + B^2 b^{-2}(\tan^2 t + a^2)]$$
$$E = 0.5\,(\cos^2 t)\ \{A^2 a^{-2}[\,1 + (-p + k^{1/2})\cos^2 t\,] + B^2 b^{-2}[\,1 - (p + k^{1/2})\cos^2 t\,]\}. \tag{33}$$

Finally

$$y_{\mathrm{p1}}(t) = (1-2p+p^2-k)\,(1 + K_1\cos^2 t + K_2\cos^4 t)\sin t,$$
$$K_1 = 0.5\,(A^2a^{-2} + B^2b^{-2}) - 2p, \qquad K_2 = p^2 - k - 0.5\,[A^2a^{-2}(p - k^{1/2}) + B^2b^{-2}\ (p + k^{1/2})\,]. \tag{34}$$

Formula (12) is correct. From formulas (26), (29) and (30) it follows

$$C = A\left(2 + \frac{B}{\sqrt{k}} + \frac{A}{2a^2}\right), \quad D = B\left(2 - \frac{A}{\sqrt{k}} + \frac{B}{2b^2}\right),$$

$$\beta_1 = (1 - 2p + p^2 - k)^2\, C/a, \quad \beta_2 = (1 - 2p + p^2 - k)^2\, D/b \tag{35}$$

$$\frac{\mathrm{d}h_2}{\mathrm{d}s} = \left(\beta_1 \frac{a}{s^2 + a^2} + \beta_2 \frac{b}{s^2 + b^2}\right) - \frac{\sigma}{s^2 + 1}.$$

The $C_2(t)$ function equation is

$$C_2(t) = h_2(\tan t), \quad \frac{\mathrm{d}C_2}{\mathrm{d}t} = \frac{\mathrm{d}h_2}{\mathrm{d}s}\frac{1}{\cos^2 t} = \left(\beta_1 \frac{a}{\sin^2 t + a^2\cos^2 t} + \beta_2 \frac{b}{\sin^2 t + b^2 \cos^2 t}\right) - \sigma. \tag{36}$$

According to formula (13), the parenthesis has the expression $\gamma(t)$. Formulas (14) and (15) give the periodic solution $y_{\mathrm{p}}$ and $v_{\mathrm{p}}$ of the non-homogeneous system (7). The solution $(y, v)$ of the system (9) will be:

$$y(t) = y_{\mathrm{p1}}(t) + y_{\mathrm{p2}}(t) + t\,\sigma\, x(t), \qquad v(t) = v_{\mathrm{p1}}(t) + v_{\mathrm{p2}}(t) + \sigma\, x(t) + t\,\sigma\, u(t). \tag{37}$$

The second solution $y(t)$ is the periodic function if and only if $\sigma(p, k) = 0$. The graph of the function $p(k)$ for which the coefficient $\sigma$ is zero is comprised between two segments

$$\sigma(p(k), k) = 0 \quad \Rightarrow \quad -k/4 - 0.002 < p(k) < -k/4 + 0.00001, \quad -0.2 < k < 0.2. \tag{38}$$

## 4. RESULTS

The analytical solutions of the differential system of formulas (4), (7) and (9) have the expressions specified in the MATHCAD program [8,9], below. The graphs in Figure 1 show the $y_{\mathrm{p}}$ and $v_{\mathrm{p}}$ periodic solution of the non-homogeneous system (6) and also the two additive components $y_{\mathrm{p1}}$, $y_{\mathrm{p2}}$ and $v_{\mathrm{p1}}$, $v_{\mathrm{p2}}$. For the chosen $k$ and $p$ values the coefficient $\sigma$ is practically zero, so $y \cong y_{\mathrm{p}}$ and $v \cong v_{\mathrm{p}}$ are periodic function.

The program's instructions are the following:

$k := 0.2 \quad p := -0.051213008$

$$Q(t) := 1 + 4\cdot\frac{3\cdot p - \left[4\cdot p + 5\cdot(p^2 - k)\right]\cdot\cos(t)^2 + 6\cdot(p^2 - k)\cdot\cos(t)^4}{1 - 2\cdot p\cdot\cos(t)^2 + (p^2 - k)\cdot\cos(t)^4}$$

$$x(t) := \frac{\cos(t)}{(1-p)^2 - k}\cdot\left[1 - 2\cdot p\cdot\cos(t)^2 + (p^2 - k)\cdot\cos(t)^4\right] \qquad a := \sqrt{1 - p - \sqrt{k}}$$

$$u(t) := \frac{-\sin(t)}{(1-p)^2 - k}\cdot\left[1 - 6\cdot p\cdot\cos(t)^2 + 5\cdot(p^2 - k)\cdot\cos(t)^4\right] \qquad b := \sqrt{1 - p + \sqrt{k}}$$

$$A := p + \frac{p^2 + k}{2\sqrt{k}} \qquad B := 2\cdot p - A \qquad C := A\cdot\left(2 + \frac{B}{\sqrt{k}} + \frac{A}{2\cdot a^2}\right) \qquad D := B\cdot\left(2 - \frac{A}{\sqrt{k}} + \frac{B}{2\cdot b^2}\right)$$

$$\beta_1 := \left[(1-p)^2 - k\right]^2\cdot\frac{C}{a} \qquad \beta_2 := \left[(1-p)^2 - k\right]^2\cdot\frac{D}{b} \qquad \sigma := \beta_1 + \beta_2$$

$$\gamma(t) := \frac{\beta_1\cdot a}{1 + (a^2 - 1)\cdot\cos(t)^2} + \frac{\beta_2\cdot b}{1 + (b^2 - 1)\cdot\cos(t)^2} \qquad C_2(t) := \int_0^t \left[\gamma(t) - \sigma\right]\mathrm{d}t$$

$$K_1 := \frac{1}{2}\cdot\left(\frac{A^2}{a^2}+\frac{B^2}{b^2}\right)-2\cdot p, \qquad K_2 := p^2-k-\frac{1}{2}\cdot\left[\left(p-\sqrt{k}\right)\cdot\frac{A^2}{a^2}+\left(p+\sqrt{k}\right)\cdot\frac{B^2}{b^2}\right]$$

$$y_{\mathrm{p1}}(t) = \left(1-2\cdot p+p^2-k\right)\cdot\left(1+K_1\cdot\cos(t)^2+K_2\cdot\cos(t)^4\right)\cdot\sin(t)$$

$$v_{\mathrm{p1}}(t) := (1-2\cdot p+p^2-k)\cdot\left[1-2\cdot K_1+(3\cdot K_1-4\cdot K_2)\cdot\cos(t)^2+5\cdot K_2\cdot\cos(t)^4\right]\cdot\cos(t)$$

$$y_{\mathrm{p2}}(t) := x(t)\cdot C2(t) \qquad v_{\mathrm{p2}}(t) := u(t)\cdot C_2(t) + x(t)\cdot[\,\gamma(t)-\sigma\,]$$

$$y_{\mathrm{p}}(t) := y_{\mathrm{p1}}(t)+y_{\mathrm{p2}}(t) \qquad v_{\mathrm{p}}(t) := v_{\mathrm{p1}}(t)+v_{\mathrm{p2}}(t)$$

$$\sigma = -8.557\cdot 10^{-10}$$

$$y(t) := y_{\mathrm{p}}(t)+t\cdot\sigma\cdot x(t)$$

$$v(t) := v_{\mathrm{p}}(t)+\sigma\cdot x(t)+t\cdot\sigma\cdot u(t).$$

This determines the terms of the periodic component of the second solution. The graphs of these terms are given in Fig. 1, when $\sigma$ is null the fundamental matrix is periodic

$$\sigma(p(k),k)=0 \qquad \Rightarrow \qquad \Phi(t+2\pi)=\Phi(t). \tag{39}$$

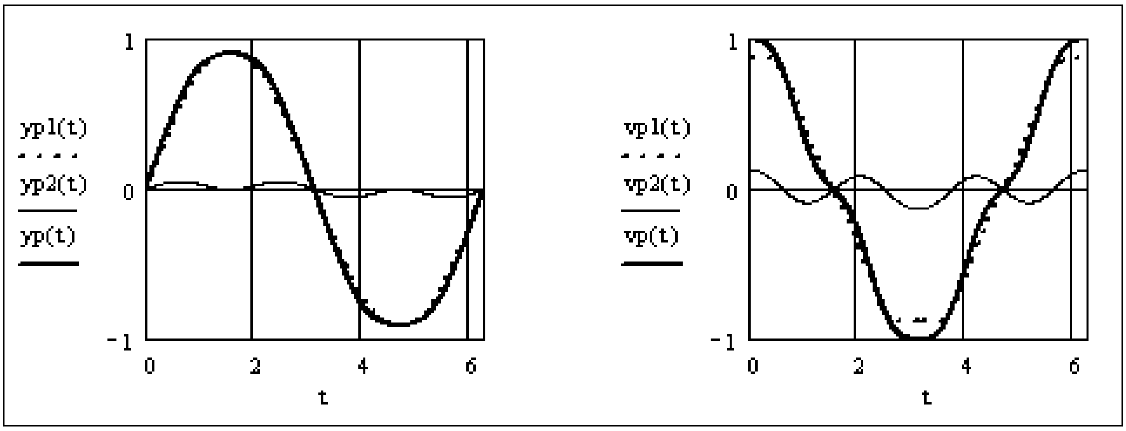


Fig. 1 – The graphs of the periodic solution $y_{\mathrm{p}}$ and $v_{\mathrm{p}}$ of system.

Numerical $x$, $u$, $y_{\mathrm{p}}$, $v_{\mathrm{p}}$, $y$, $v$ solutions of the (4), (7) and (9) systems are denominated in capital letters. These solutions check the next system, where the integration interval is divided into $N=2048$ parts,

$$\mathbf{u}_0 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1-\sigma \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{D}(t,u) := \begin{bmatrix} u_1 \\ -Q(t)\cdot u_0 \\ u_3 \\ -Q(t)\cdot u_2 - 2\cdot\sigma\cdot u_1 \\ u_5 \\ -Q(t)\cdot u_4 \end{bmatrix} \qquad \mathbf{S} := \mathrm{rkfixed}(\mathbf{u}_0, 0.4\cdot\pi, N, \mathbf{D}).$$

Solution matrix columns **S** represent the values of the variable $t$ and the corresponding values of unknown $X$, $U$, $Y_{\mathrm{p}}$, $V_{\mathrm{p}}$, $Y$, $V$

$$t := \mathbf{S}^{<0>} \quad X := \mathbf{S}^{<1>} \quad U := \mathbf{S}^{<2>} \quad Y_{\mathrm{p}} := \mathbf{S}^{<3>} \quad V_{\mathrm{p}} := \mathbf{S}^{<4>} \quad Y := \mathbf{S}^{<5>} \quad V := \mathbf{S}^{<6>}$$

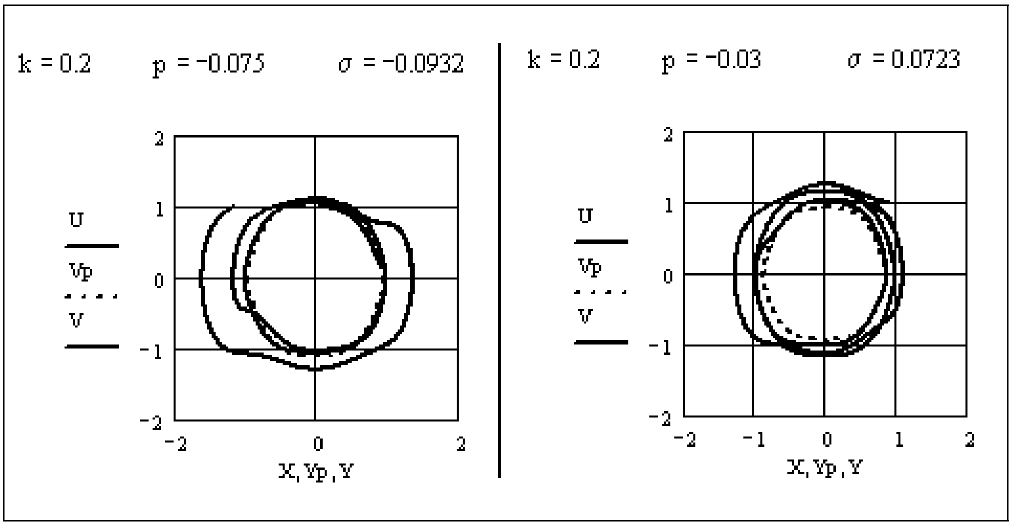


Fig. 2 – The graphs ($U$, $X$) and ($V_p$, $Y_p$) show periodicity.

The amplitude of the oscillations of the solution $y$ increases proportionally to time $t$. It is important to integrate on the interval [0, 2π], since $y(t+2n\pi) = y(t) + 2n\pi\,\sigma x(t)$.

## 5. CONCLUSIONS

The constructive demonstration of following propositions is the main contribution of the work. If $Q(t,p,k)$ is a set of functions according to formula (2), then differential equation (1) has a periodic solution $x(t)$ given by formula (3). If the function $\sigma(p,k)$ has the algebraic expression (6), then the differential system (7) has a periodic solution $y_p(t)$, $v_p(t)$. The second fundamental solution $y(t)$ of equation (1) has the expression $y(t) = y_p(t) + t\cdot\sigma(p,k)\cdot x(t)$, and the fundamental matrix of the general system (9) has the expression (11). It is also shown that the algebraic function $\sigma(p,k)$ is the value of an integral with the real $p$ and $k$ parameter. Explicit expressions of the fundamental matrix components are defined in MATCAD programs (Figures 1 and 2). Numerical calculations test the truth of the above propositions, especially for cases $k=0.2$ and $p=-0.051213008$ respectively $p=-0.075$, $p=-0.03$. The proof is in the second section. The wronskian determinant of solutions $x$ and $y$ does not explicitly depend on $Q$. Given $u=\mathrm{d}x/\mathrm{d}t$ and also $v=\mathrm{d}y/\mathrm{d}t$ we will obtain for $y(t)$ the equation (10) $x\cdot v - u\cdot y = 1$ or the linear equation (16) for $y_p(t)$. The homogeneous equation has the solution $C\cdot x(t)$. The method of variation of the integration constant requires $y_p(t) = C(t)\cdot x(t)$. The unknown function $C(t)$ checks a singular equation (17) in the sense that $x(\pi/2) = 0$. The singularity is isolated, resulting in the expression $C(t) = C_0(t) + C_\sigma(t)$. Although $C_0(t)$ is periodic, unbounded according to (19), the product $x(t)\,C_0(t)$ is a periodic bounded function. The term $C_\sigma(t)$ must be periodic; otherwise $y_p(t)$ would not be the periodic component of $y(t)$. The derivative $\mathrm{d}C_\sigma/\mathrm{d}t$ is a periodic function bounded according to (20). But it is known that the integral of a periodic function is periodic, if and only if the mean value of the function is zero. Imposing this condition it follows the expression (21) of the coefficient $\sigma(p,k)$. This is an integral with $p$ and $k$ as parameters. In the third section the expression of the periodic component $y_p(t)$ is determined. By the change of the integration variable $C(t) = h(s)$, $s = \tan t$, the equation (26) follows. The function $h(s)$ is the primitive of a rational function. By a decomposition in certain simple fractions, we need to express the $h(s) = h_1(s) + h_2(s)$ function. Function $h_1(s)$ has the explicit expression (31). The first periodic $y_{p1}(t)$ term of the $y_p(t)$ component is derived according to formula (34). In the differential equation (35) of function $h_2(s)$ the coefficient $\sigma(p,k)$ appears. To have $h_2(s)$ bounded at the infinite point the algebraic expression (6) of the coefficient $\sigma(p,k)$ is got. According to formula (36), the function $h_2(s)$ corresponds to the restriction to the interval $[0, \pi/2)$ of function $C_2(t)$. The differential equation (35) will correspond to the equation (36) of the function $C_2(t)$ for the positive variable $t$. Solution $C_2(t)$ of equation (36) is given in expression (13) of $\gamma(t)$ function and integral function (14). Knowing $C_2(t)$ we determine the second component $y_{p2}(t)$ of the $y_p(t)=y_{p1}(t)+y_{p2}(t)$ function.

In the fourth section there is a numerical program for testing the formulas given above. For the exemplified case $k=0.2$ the theoretical formulas are consistent with the numerical results. In the case of negative $k$ values, the coefficients $a$ and $b$ have complex conjugate values. However, both $\sigma(p,k)$ and $\gamma(t)$ functions have real values. Finally $C_2(t)$ and $y_p(t)$ have real values. It is important to note that the graphs in Figure 2 are plotted by knowing only $Q(t)$ and the algebraic expression of the $\sigma$ coefficient. The particular case $k=p^2, p=2q/(1+3q)$ was detailed in the paper [9]. The criterion of integrity $\sigma=0$ can be generalized if the periodic solution depends on several parameters.

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