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INTEGRAL FORMULATION FOR STABILITY AND VIBRATION ANALYSIS OF BEAMS ON ELASTIC FOUNDATION

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Abstract. This work presents a simple approximate integral method based on the use of flexibility influence functions (Green's functions) for numerical calculations of either critical buckling loads or natural frequencies in the case of Euler-Bernoulli type beams resting on two-parameters elastic foundation. Several numerical applications demonstrate good agreement with available results from literature obtained analytically or by other methods. The theoretical approach presented in this work leads to an eigenvalue matrix form efficient for numerical solutions.

Key words: integral method, Green's functions, buckling, vibration, beams, elastic foundation.

1. INTRODUCTION

Main design issues for beam like structures are the critical buckling loads calculation and also the natural frequencies estimation in order to avoid possible resonances. In the last decades a large number of technical works have been dedicated to this subject. The analysis of dynamics of beams resting on elastic foundation is of particular interest for industrial applications such as the study of concrete structures or pipelines resting on elastic soil, railway applications or the study of some parts of machinery resting on isolation members. One of the first attempts to analyse the buckling of beams on elastic foundation is described in the work [1], where Hetényi presented a trial approach for calculation of critical buckling loads. An analytical formula for the critical buckling loads, in the case of uniform simply supported beams on elastic foundation is also given in [2]. The paper [3] presents the Recursive Differentiation Method (RDM) to find analytical solutions for critical buckling loads and natural frequencies of non-uniform beams resting on elastic foundations. Other numerical methods, such as Differential Quadrature Method (DQM) and Variational Iteration Method (VIM), are described in [4], respectively [5].

A general form of the equation describing bending deflection $w(x,t)$ of a non-uniform beam with bending stiffness $EI(x)$ resting on a two parameters elastic foundation and subjected to constant axial compression force P and transverse distributed force $p(x,t)$ (see Fig. 1), is according to [3]:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)+\left(P-k_2\right)\frac{\partial^2 w}{\partial x^2}+k_1 w+\rho A(x)\frac{\partial^2 w}{\partial t^2}=p(x,t)\,, \tag{1}$$

In this equation ρ and $A(x)$ represent the beam material mass density and the cross-section area. A common situation is the Winkler foundation model with $k_2=0$ and with the elastic coefficient k_1 constant or variable along the beam axis [6–8]. The Winkler model was first presented in the paper [9] in 1867, representing a linear algebraic relation between the bending displacement of the beam and the contact pressure at foundation, and it was originally used for the analysis of deflections and stress states of railroad tracks. The coefficient k_1 is commonly called the Winkler foundation modulus or modulus of subgrade reaction while k_2 is the shear foundation modulus, known as the Pasternak effect. In fact some of the best known models for the two parameter foundation are the Pasternak one [10] and the Vlasov and Leontiev one [11]. These two parameters have different expressions as presented for example in the paper [12].

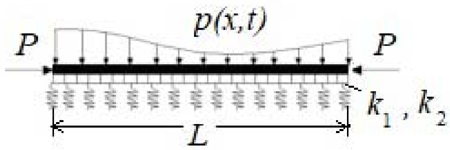

Fig. 1 – Beam resting on two parameter elastic foundation.

2. PROBLEM FORMULATION

2.1. Integral form for the differential equation of beam bending behavior

The differential equation governing the static bending response of a straight beam subjected to the transverse load $p(x)$, in term of the bending displacement $w(x)$, is given by:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)=p(x)\,. \tag{2}$$

The integral form of this equation, based on the use of flexibility influence functions (Green's functions) [13] can be written as:

$$w(x)=\int_0^L G_w(x,\xi)\,p(\xi)\,\mathrm{d}\xi\,. \tag{3}$$

In this relation $G_w(x,\xi)$ is the Green function representing the bending deflection w at location x on the beam due to a transverse unit force applied at location ξ (Fig. 2). The Green's function values represent flexibility coefficients and they can be numerically calculated using specific methods such as the Mohr-Maxwell method, Castigliano's theorem or the principle of virtual work.

The equation (3) has been used in many applications, and we reference here as a representative example, the works [14, 15] in which the authors developed an aeroelastic analysis for large aspect ratio wings considered as cantilever beams.

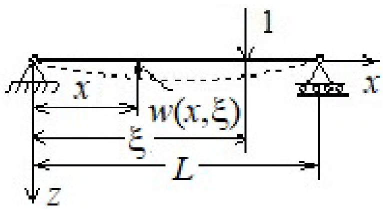

Fig. 2 – Physical significance of the Green function for a simply supported beam.

The reference [16] starts from this integral form in order to calculate the natural frequencies for transverse vibrations of rotating beam highlighting the stiffening effect of the rotation. A more general case of pretwisted rotating beam coupled vibration analysis was presented in [17].

2.2. Integral matrix forms of the equation for a beam resting on two parameters elastic foundation

In the present paper, an appropriate form, to find the natural circular frequencies ω, is developed using the equation (1) when the external distributed load is $p(x,t)=0$.This equation is re-written as:

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)=\left(-P+k_2\right)\frac{\partial^2 w}{\partial x^2}-k_1 w+\rho A(x)\,\omega^2 w\,. \tag{4}$$

The right hand side terms in (4) will be regarded as a distributed transverse force $p(x)$ allowing the integral form (3) to be developed using appropriate Green's functions for the case of the simply supported beam. The integrals involved in such formulations can computed by using the Simpson's method, choosing of an even number $n=2m$ of equally spaced collocation (sampling) points on beam axis. The integral is then calculated as:

$$\int_0^L f(\xi)\,\mathrm{d}\xi = \frac{L}{n}\left[f(\xi_1) + 2\sum_{k=1}^{m-1} f(\xi_{2k}) + 4\sum_{k=1}^{m} f(\xi_{2k-1}) + f(\xi_n) \right] = \sum_{i=1}^{n} f_i \cdot W_i \,, \tag{5}$$

where f_i are the values of $f(\xi_i)$ in the collocation points ξ_i and W_i are the corresponding weighting numbers. The relation (3) can be written in matrix form:

$$\mathbf{w} = \mathbf{G}_w \mathbf{W} \mathbf{p} \,, \tag{6}$$

where:

– $\mathbf{G}_w$ is a (n,n) dimension matrix containing the Green's functions values,

– $\mathbf{W}$ is a (n,n) diagonal weighting matrix containing the weighting numbers W_i,

– $\mathbf{w}$ and $\mathbf{p}$ are column vectors of the bending deflections $w(\xi)$ and of the distributed forces $p(\xi)$ in the chosen n collocation points respectively.

For the case of constant values k_1, k_2, the equation (4) in matrix form becomes:

$$\mathbf{w} = (-P + k_2)\mathbf{G}_w \mathbf{W} \mathbf{D}_2 \mathbf{w} - k_1 \mathbf{G}_w \mathbf{W} \mathbf{w} + \omega^2 \mathbf{G}_w \mathbf{W} \mathbf{M} \mathbf{w} \,. \tag{7}$$

The new matrices in the previous relation are:

– $\mathbf{D}_2$ a (n,n) differentiation matrix used to obtain the second derivative of the bending deflection w and

– $\mathbf{M}$ a (n,n) diagonal matrix containing the beam distributed mass $m(x) = \rho A(x)$ in the collocation points.

A different approach is the projection (expansion) of the bending displacement $w(x)$ using a suitable basis of p known functions $f_k(x)$ for the simply supported boundary conditions:

$$w(x) = \sum_{k=1}^{p} C_k f_k(x) \,, \tag{8}$$

where C_k are constant coefficients. In this manner the second derivative of the bending deflection can be directly obtained without the need of a differentiation matrix. For the n collocation points one can obtain relations of the form:

$$\mathbf{w} = \mathbf{F}\mathbf{C}, \quad \mathbf{w}'' = \mathbf{F}_2 \mathbf{C} \,, \tag{9}$$

which give the values of the bending deflections and of theirs second derivatives. The matrices $\mathbf{F}$ and $\mathbf{F}_2$, containing the values $f_k(x)$ respectively $f_k''(x)$ at the collocation points, are of dimension (n,p) and the column vector $\mathbf{C}$ contains p coefficients C_k. Using the relations (9), equation (7) becomes:

$$\mathbf{F}\mathbf{C} = (-P + k_2)\mathbf{G}_\mathbf{w} \mathbf{W} \mathbf{F}_2 \mathbf{C} - k_1 \mathbf{G}_\mathbf{w} \mathbf{W} \mathbf{F} \mathbf{C} + \omega^2 \mathbf{G}_\mathbf{w} \mathbf{W} \mathbf{M} \mathbf{F} \mathbf{C} \,. \tag{10}$$

After left multiplication with the transpose $\mathbf{F}^\mathrm{T}$, the previous relation takes the form:

$$\mathbf{A}_1 \mathbf{C} = (-P + k_2)\mathbf{B}_1 \mathbf{C} - k_1 \mathbf{B}_2 \mathbf{C} + \omega^2 \mathbf{B}_3 \mathbf{C} \,, \tag{11}$$

where all the matrices $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ are now of dimension (p,p):

$$\mathbf{A}_1 = \mathbf{F}^\mathrm{T}\mathbf{F}, \quad \mathbf{B}_1 = \mathbf{F}^\mathrm{T}\mathbf{G}_w \mathbf{W}\mathbf{F}_2, \quad \mathbf{B}_2 = \mathbf{F}^\mathrm{T}\mathbf{G}_w \mathbf{W}\mathbf{F}, \quad \mathbf{B}_3 = \mathbf{F}^\mathrm{T}\mathbf{G}_w \mathbf{W}\mathbf{M}\mathbf{F} \,. \tag{12}$$

The general matrix relations (7), determined using n collocation points, can be put in a form of a n dimensional eigenproblem allowing the calculation of the critical buckling loads P_{cr} (for $\omega=0$), of the natural circular frequencies ω for $P=0$ or of p values below the critical buckling load. The general matrix relations

(11), in the case of the use on n collocation points and p collocation functions can also be put in a form of an eigenproblem of dimension $p<n$, allowing the same calculations.

For the critical buckling loads calculations, that is for $\omega=0$, the relation (7) takes the form:

$$\mathbf{w}=(-P+k_2)\mathbf{G}_w\mathbf{W}\mathbf{D}_2\mathbf{w}-k_1\mathbf{G}_w\mathbf{W}\mathbf{w}\,. \tag{13}$$

After left multiplication with the inverse $\mathbf{G}_1$ of the matrix $\mathbf{G}_w\mathbf{W}\mathbf{D}_2$ one can obtain the following relation:

$$\mathbf{G}_1\mathbf{w}=(-P+k_2)\mathbf{I}_n\mathbf{w}-k_1\mathbf{G}_2\mathbf{w}\,, \tag{14}$$

where $\mathbf{I}_n$ is the (n,n) unit matrix and:

$$\mathbf{G}_1=\left(\mathbf{G}_w\mathbf{W}\mathbf{D}_2\right)^{-1},\quad \mathbf{G}_2=\mathbf{G}_1\mathbf{G}_w\mathbf{W}\,. \tag{15}$$

Relation (14) is an n dimensional eigenproblem of the form:

$$\left(\mathbf{G}_1-k_2\mathbf{I}_n+k_1\mathbf{G}_2\right)\mathbf{w}=\mathbf{A}_n\mathbf{w}=-P\mathbf{w}\,. \tag{16}$$

The eigenvalues λ of the matrix $\mathbf{A}_n$ give the critical buckling loads $\lambda=-P_{\mathrm{cr}}$.
When the relation (11) is used for the buckling loads calculations (the case $\omega=0$), this relation becomes:

$$\mathbf{A}_1\mathbf{C}=(-P+k_2)\mathbf{B}_1\mathbf{C}-k_1\mathbf{B}_2\mathbf{C}\,. \tag{17}$$

After left multiplication with the inverse of the matrix $\mathbf{B}_1$ one can obtain a relation of the form:

$$\mathbf{D}_1\mathbf{C}=(-P+k_2)\mathbf{I}_p\mathbf{C}-k_1\mathbf{D}_2\mathbf{C}\,, \tag{18}$$

where $\mathbf{I}_p$ is the (p,p) unit matrix and:

$$\mathbf{D}_1=\mathbf{B}_1^{-1}\mathbf{A}_1,\quad \mathbf{D}_2=\mathbf{B}_1^{-1}\mathbf{B}_2\,. \tag{19}$$

This represents a p dimensional eigenvalue problem of the form:

$$\left(\mathbf{D}_1-k_2\mathbf{I}_p+k_1\mathbf{D}_2\right)\mathbf{C}=\mathbf{A}_p\mathbf{C}=-P\mathbf{C}\,, \tag{20}$$

for which the eigenvalues λ of the matrix $\mathbf{A}_p$ provide the critical buckling loads $\lambda=-P_{\mathrm{cr}}$.

For the calculations of the natural circular frequencies ω, left multiplication of equation (7) with the inverse $\mathbf{G}_3$ of the matrix $\mathbf{G}_w\mathbf{W}\mathbf{M}$, leads to the equation:

$$\mathbf{G}_3\mathbf{w}=(-P+k_2)\mathbf{G}_4\mathbf{w}-k_1\mathbf{G}_5\mathbf{w}+\omega^2\mathbf{I}_n\mathbf{w}\,, \tag{21}$$

where $\mathbf{I}_n$ is the (n,n) unit matrix and:

$$\mathbf{G}_3=\left(\mathbf{G}_w\mathbf{W}\mathbf{M}\right)^{-1},\quad \mathbf{G}_4=\mathbf{G}_3\mathbf{G}_w\mathbf{W}\mathbf{D}_2,\quad \mathbf{G}_5=\mathbf{G}_3\mathbf{G}_w\mathbf{W}\,. \tag{22}$$

Equation (21) represents an n dimensional eigenvalue problem of the form:

$$\left(\mathbf{G}_3-k_2\mathbf{G}_4+P\mathbf{G}_4+k_1\mathbf{G}_5\right)\mathbf{w}=\mathbf{A}_n\mathbf{w}=\omega^2\mathbf{w}\,, \tag{23}$$

where the eigenvalues $\lambda=\omega^2$ of the matrix $\mathbf{A}_n$ give the natural circular frequencies. When the relation (11) is used for calculations of the natural circular frequencies ω, left multiplication with the inverse of the matrix $\mathbf{B}_3$, leads to the equation:

$$\mathbf{D}_3\mathbf{C}=(-P+k_2)\mathbf{D}_4\mathbf{C}-k_1\mathbf{D}_5\mathbf{C}+\omega^2\mathbf{I}_p\mathbf{C}\,, \tag{24}$$

where $\mathbf{I}_p$ is the (p,p) unit matrix and:

$$\mathbf{D}_3=\mathbf{B}_3^{-1}\mathbf{A}_1,\ \ \mathbf{D}_4=\mathbf{B}_3^{-1}\mathbf{B}_1,\ \ \mathbf{D}_5=\mathbf{B}_3^{-1}\mathbf{B}_2\,. \tag{25}$$

This represents a p dimensional eigenproblem of the form:

$$\left(\mathbf{D}_3 - k_2\mathbf{D}_4 + P\mathbf{D}_4 + k_1\mathbf{D}_5\right)\mathbf{C} = \mathbf{A}_p\mathbf{C} = \omega^2\mathbf{C}\,, \tag{26}$$

where the eigenvalues $\lambda=\omega^2$ of the matrix $\mathbf{A}_p$ provide the natural circular frequencies.

3. NUMERICAL ANALYSES AND DISCUSSIONS

3.1. Stability analysis

The first example concerning the buckling load calculation is taken from [5]. The results for uniform beams ($EI=$const.) on Winkler foundation (with $k_2=0$) are presented using two non-dimensional coefficients namely β (for k_1) and α representing a critical buckling load parameter:

$$\beta = \frac{k_1L^4}{EI}, \quad \alpha = \frac{P_{\text{cr}}L^4}{EI}. \tag{27}$$

A comparison of results obtained using this formulation and those presented in [5] is shown in Table 1 in the case of a simply-supported beam, using an increasing number of collocation points.

Table 1

Results for α parameter obtained with n collocation points and relation (16)

β	$n=10$	$n=20$	$n=40$	$n=60$	$n=100$	Results [5]
0	11.051	10.378	10.108	10.025	9.961	9.8696
50	16.756	15.712	15.298	15.172	15.075	14.9357
100	22.416	21.032	20.485	20.318	20.188	20.0017

For the formulation based on collocation functions, a set of p sinusoidal functions representing the real buckling modes for the uniform simply supported beam are used in relation (8), written as:

$$w(x) = \sum_{k=1}^{p} C_k \sin\left(\frac{k\pi x}{L}\right). \tag{28}$$

Table 2 shows the comparison between the results obtained using collocation functions, with $p=2$ and $n=100$ points, and the results obtained using only collocation points for the same example. One can observe that an improved accuracy is obtained using the collocation function approach.

Table 2

Results for α parameter obtained with collocation functions and relation (20)

β	$n=100$	$p=2,\ n=100$	Results [5]
0	9.961	9.868	9.8696
50	15.075	14.9341	14.9357
100	20.188	20.0001	20.0017

In order to verify the present model for the fundamental buckling load calculation in the case of an uniform simply supported beam resting on a two parameter elastic foundation, a case test from the reference [18] has been considered. Two non-dimensional parameters of the foundation and a stability parameter λ are defined as:

$$\overline{k}_1 = \frac{k_1L^4}{EI}, \quad \overline{k}_2 = \frac{k_2L^2}{\pi^2 EI}, \quad \lambda = \sqrt{\frac{P_{\text{cr}}L^2}{EI}}. \tag{29}$$

Table 3 shows good agreement of the results obtained with $n=100$ collocation points and $p=2$ collocation functions, in comparison with FEM results reported in Table 1 from [18].

Table 3

Results for stability parameter λ obtained with $n=100$ collocation points and relation (20)

$\bar{k}_1$	$\bar{k}_2=0$		$\bar{k}_2=1$		$\bar{k}_2=2.5$	
	FEM [18]	Present	FEM [18]	Present	FEM [18]	Present
0	3.1415	3.1413	4.4428	4.4427	5.8774	5.8772
100	4.4723	4.4721	5.4654	5.4653	6.6840	6.6838

3.2. Free vibrations analysis

The first example concerning the natural circular frequencies calculation is taken from [19]. In this reference a benchmark is presented for a simply-supported uniform beam with $E = 24.82$ GPa, $\rho A = 446.3$ kg/m, $I = 1.439\text{x}10^{-3}$ m^4, on Winkler foundation with the length $L = 6.096$ m, $k_2=0$ and constant distributed stiffness $k_1 = K = 16.55$ MN/m^2. Comparison is carried out by using the analytical solutions presented in [20]:

$$\omega_i = \sqrt{\frac{EI}{\rho A}}\sqrt{\frac{i^4\pi^4}{L^4}+\frac{K}{EI}}\,. \tag{30}$$

Table 4 shows the convergence of results ω_i (in Hz) for this example, obtained with the collocation points approach in comparison with the analytical results given by (30).

Table 4

Results for ω_i obtained with n collocation points and relation (23)

ω_i	$n = 10$	$n = 20$	$n = 40$	$n = 60$	$n = 100$	Results (30)
ω_1	32.866	32.889	32.896	32.897	32.898	32.898
ω_2	55.604	56.492	56.726	56.771	56.794	56.808
ω_3	104.74	110.07	111.43	111.68	111.82	111.90
ω_4	168.57	187.80	192.24	193.08	193.51	193.76

Another example concerning the natural circular frequencies calculation is taken from the paper [21]. In this reference results are obtained for an uniform simply-supported beam resting on a two-parameter elastic foundation using several non-dimensional parameters:

$$\bar{k}_1 = \frac{k_1 L^4}{EI},\ \bar{k}_2 = \frac{k_2 L^2}{\pi^2 EI},\ \gamma = \frac{P}{P_{\text{cr}}},\ \lambda^4 = \frac{\rho A L^2 \omega^2}{EI}\,. \tag{31}$$

The beam has constant values I, A and constant foundation parameters k_1 and k_2. Using the formulae (31), the critical buckling force can be calculated as, [22]:

$$P_{\text{cr}} = \frac{EI\pi^4 + k_1 L^4 + k_2\pi^2 L^2}{\pi^2 L^2}\,. \tag{32}$$

Table 5 shows the results for this example using $n=100$ collocation points, compared with the FEM results [21].

Table 5

Results for frequency factor λ obtained with $n=100$ collocation points and relation (23)

$\bar{k}_1$	γ	$\bar{k}_2=0$		$\bar{k}_2=1$		$\bar{k}_2=2.5$	
		FEM [21]	Present	FEM [21]	Present	FEM [21]	Present
0	0	3.1415	3.1415	3.7306	3.7315	4.2970	4.2896
	0.4	2.7705	2.7691	3.2947	3.2867	3.7893	3.7970
	0.8	2.1257	2.1201	2.5270	2.5070	2.9050	2.8763
100	0	3.7483	3.7483	4.1437	4.1404	4.5824	4.5763
	0.4	3.3055	3.3041	3.6541	3.6478	4.0408	4.0305
	0.8	2.5350	2.5296	2.8014	2.7862	3.0964	3.0732

The next numerical test regarding the free vibration analysis is an example from [22] where the following data has been considered: beam with $L = 4$ m, having square cross-section $b = h = 0.3$ m, Young's modulus $E = 2.1\cdot 10^{11}$ Pa, mass density $\rho = 7860$ kg/m^3. To describe the two foundation parameters the following non-dimensional parameters have been considered:

$$\bar{k}_1 = \frac{k_1 L^4}{EI}, \quad \bar{k}_2 = \frac{k_2 L^2}{EI}. \tag{33}$$

The first three natural frequencies $f_i = \omega_i/(2\pi)$ and f_{10} have been obtained for a compression force $P = 40$ kN. Table 6 gives the results obtained with the collocation function approach with $n = 100$ points, $p = 20$ in equation (26), results being compared with those obtained with the spectral finite element method SFEM [22].

Table 6

Results for frequency factor λ obtained with relation (26)

Case	Results from	Natural frequencies [Hz]			
		f_1	f_2	f_3	f_{10}
$\bar{k}_1 = 0$, $\bar{k}_2 = 0$, $P = 40$ kN	[22]	43.96	175.80	395.53	4394.71
	Present	43.93	175.72	395.22	4359.01
$\bar{k}_1 = 10$, $\bar{k}_2 = 0$, $P = 40$ kN	[22]	46.16	176.36	395.78	4394.73
	Present	46.13	176.28	395.47	4359.03
$\bar{k}_1 = 10$, $\bar{k}_2 = 25$, $P = 40$ kN	[22]	83.80	225.10	447.97	4450.04
	Present	83.79	225.04	447.69	4414.79

The results show good agreement especially for the lower modes.

The last example considered here concerns a non-uniform tapered beam free vibration analysis with data from [23]. The distribution of the bending stiffness and cross-section area are given by the following linear relations:

$$EI(x) = EI_0\left(1-\alpha\cdot x/L\right), \quad A(x) = A_0\left(1-\alpha\cdot x/L\right), \tag{34}$$

where EI_0 and A_0 are the values at $x = 0$. The next non-dimensional parameters have been also used:

$$\bar{k}_1 = \frac{k_1 L^4}{EI_0} = 1, \quad \bar{k}_2 = 0, \quad \varpi = \omega\sqrt{\frac{\rho A_0}{k_1}}. \tag{35}$$

The calculations are carried out for unit values of the parameters E, L, k_1, ρ, A_0, I_0 so that the normalized frequency parameter $\varpi = \omega$.

Table 7 shows the results obtained with the present integral formulation with $n = 100$ points, $p = 10$ functions and using relation (26), compared with the results presented in [23] obtained with the Adomian Decomposition Method.

Table 7

Results for ω_i obtained with the relation (26)

ω_i	Results from	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$
ω_1	[23]	9.9217	9.9170	9.8932
	Present	9.9209	9.9162	9.8924
ω_2	[23]	39.4928	39.5047	39.5340
	Present	39.4799	39.4918	39.5211
ω_3	[23]	88.8340	88.8511	88.8986
	Present	88.7689	88.7860	88.8335
ω_4	[23]	157.9189	157.9389	157.9966
	Present	157.7133	157.7333	157.7909
ω_5	[23]	246.7443	246.7661	246.8302
	Present	246.2424	246.2642	246.3282

One can observe that the approach using collocation functions is especially useful in the case of non-uniform beams or for calculations of the natural frequencies for higher modes of vibration.

4. CONCLUSIONS

This paper presents a simple approximate integral method for stability and free vibration analysis of beams resting on Winkler or Pasternak type elastic foundations. The numerical applications have been demonstrated for the case of the simply-supported boundary conditions. The equations of motion governing the bending behavior has been written in integral form using appropriate Green's functions, the numerical integration being carried out using the collocation method. The values of these functions are flexibility influence coefficients representing bending displacements at prescribed collocation points on the beam due to unit forces placed at other collocation points.

The presented approach leads to a matrix formulation using an integration matrix, a differentiation matrix used to obtain the second derivative of the bending displacement, and diagonal matrices taking into account distributed parameters for the non-uniform beam such as the mass distribution. The final equation represents an eigenvalue problem whose solutions are the critical buckling loads or the squares of natural frequencies of the beam. The method has been shown to be very efficient for numerical calculations, the results obtained for different benchmarks being in good agreement with reported data. The accuracy increases with the number of collocation (sampling) points or the number of collocation functions. For the non-uniform beam configurations, the collocation functions representing the real sinusoidal buckling modes of the uniform simply-supported beam, were shown to be effective in improving the accuracy. This approach can be also used for other beam boundary conditions if appropriate Green's functions are replaced in this formulation.

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