THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 20, Number 3/2019, pp. 295–302

CONTROLLED FORCED FRACTIONAL VIBRATING SYSTEM

Adel AGILA[1,*], Dumitru BALEANU [2,3]

[1] Mechanical Engineering Department, Omar Al-Mukhtar University, Al-Bayda, Libya
[2] Department of Mathematics, the Faculty of Arts and Sciences, Cankaya University, Ankara, Turkey,
[3] Institute of Space Sciences, Magurele-Bucharest, Romania
[*] Corresponding author: Adel AGILA, E-mail: adelagila@gmail.com

Abstract. The reliability of dynamic systems is enhanced by vibration control. Many types of controllers are used to control the dynamic systems' vibrations. The integer and fractional PID controllers are used to control the fractional and integer dynamic systems. Different techniques are utilized to model the controlled systems. In this study, the discrete integer proportional integral derivative (PID) controller is used to control a forced damped variable-order fractional oscillatory systems. The objectives of this study are the analysis of controlled fractional system responses, and the investigation of controller gains' effects on system response characteristics. The Caputo-Fabrizio fractional derivative is used to model the system fractional dissipating force. The system responses are approximated by numerical and time discretization techniques. In order to verify the feasibility and effectiveness of the introduced methods, the fractional system response and integer system response are compared at fractional order close to one. The controlled responses of the fractional system are obtained for different fractional derivative order values. The results demonstrate same effects of PID gains on the fractional and integer oscillatory system responses' metrics. However, the system responses are varying based on the fractional derivative order values. The study shows that the integer response and the fractional responses have same behaviors and different instantaneous values.

Key words: discrete-time controller, PID, fractional oscillatory system, Caputo-Fabrizio fractional derivative.

1. INTRODUCTION

Recently, there has been a shift toward applying fractional calculus to different branches of pure and applied sciences [1–14]. Different systems in various disciplines are modeled as fractional systems due to their elements' materials behaviors, such as modeling of some organics tissues in Biomedical Engineering [15,16], viscoelastic materials [17,18], and anomalous diffusion of mass transfer or liquid transport through porous media [19,20]. Analytical and numerical techniques are used to obtain the solutions of the fractional order models [21–23]. Many studies were done in the control and optimal control of fractional models' responses [24,25]. The proportional integral derivative (PID) controller is applied as fractional PID to control integer and fractional systems [26,27]. An analytical solution based on Mittag-Leffler function is used to obtain the response of a nonsingular fractional forced mass-spring-damper system [28]. Hypothetical uncontrolled damped single and multi-degree of freedom fractional vibrating systems were studied in [29,30], respectively. The fractional PID controller is utilized to control a vibrating constant order fractional system. This system is modeled using Caputo fractional derivative. The controlled responses system is obtained by an analytical technique [31].

In our study, a forced damped variable order fractional vibrating system is controlled by using the discrete-time PID controller. The motivations behind the study are the investigation of the introduced controlled fractional system responses, and the analysis of the effects of fractional variable order $\alpha(t)$ and PID gains on the system responses. The novelties of the work are the application of non-singular fractional derivative, the studying of variable order fractional controlled dynamic system, and the applied of discrete-time PID controller to control the system response. Continuous and discrete-time PID controllers are introduced in Section 2. Some needed preliminary definitions in fractional calculus are presented in Section 3.

These definitions include the Caputo fractional derivative, and Caputo-Fabrizio (C-F) variable order fractional derivative. In Section 4 the considered dynamic system is modeled. The fractional damping force is modeled by using variable order C-F fractional derivative. In Section 5 numerical and time discretization techniques are used to approximate the open loop system responses. In Section 6 the closed system responses are obtained by applying the discrete-time PID controller. The effects of controller gains on the characteristics of the closed loop fractional system responses are investigated in Section 7. We study in Section 8 the effect of the real variable order derivative $\alpha(t)$ on the controlled and uncontrolled fractional system responses. The integer model response and fractional model response at α close to one, are compared. The comparison is done to verify the effectiveness of the introduced modeling procedure and the feasibility of responses' obtaining techniques.

2. DISCRETE-TIME PID CONTROLLER

The mechanism of feedback control basically depends on the measured responses of controlled systems. The term feedback describes the connection manner of two or more dynamic system [32]. The feedback that is used in different disciplines is generally connected to closed-loop control systems. There are different types of controller starting from on-off controller to more advanced control such as PID controller up to Programmable Logic Controller (PLC). The PID controller is basically introduced as continuous or discrete controller forms. A continuous PID controller applied to a plant or a system is shown in Fig. 1. The continuous form of PID controller is given by equation Eq. (1) as follows:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,\mathrm{d}\tau + K_d \frac{\mathrm{d}e(t)}{\mathrm{d}t}, \tag{1}$$

where $u(t)$ is the output of the controller and the input of the controlled system, $e(t)$ represents the continuous error between the continuous desired output $y_d(t)$ and the actual continuous output $y(t)$. The coefficients K_p, K_i and K_d are the proportional, integral, and derivative controller gains, respectively.

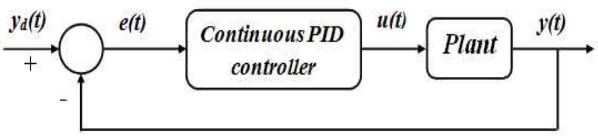

Fig. 1 – A block diagram of a continuous-time PID controlled plant.

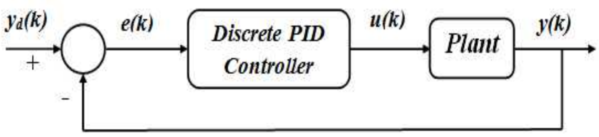

Fig. 2 – A block diagram of a discrete-time PID controlled plant.

The continuous form of the PID controller that is given by equation Eq. (1) can be implemented in a discrete form [33] as follows. The discrete time proportional term is defined as $u_p(k) = K_p e(k)$, where k is an index of the discrete signals sample points. The discrete time integral term is $u_i(k) \approx u_i(k-1) + K_i \frac{h\left[e(k)+e(k-1)\right]}{2}$, where h is the time increment. The discrete time derivative term is given by $u_d(k) \approx K_d \frac{h\left[e(k)+e(k-1)\right]}{h}$. The block diagram that is shown in Fig. 2 illustrates a discrete-time PID controlled plant. The discrete controller output control $u(k)$ is expressed by the summation of the discrete-time controller terms as follows:

$$u(k) = u_p(k) + u_i(k) + u_d(k), \tag{2}$$

The introduced discrete-time PID controller is applied to the fractional dynamic system that was described in Section 4. Some primarily concepts of fractional calculus need to be defined before the system description.

3. PRELIMINARY DEFINITIONS IN FRACTIONAL DERIVATIVES

There are some formulas that define different types of fractional derivative: the left and right Riemann-Liouville (R-L) fractional derivatives, the left and right Caputo fractional derivatives, the left and right Coimbra fractional derivative, which is defined as a variable order fractional derivative, and the non-singular Caputo Fabrizio (C-F) fractional derivatives. In this Section, we introduce the Caputo and C-F fractional derivatives' definitions. The left and right side first orders Caputo fractional derivatives are introduced in [34]. The left side first order variable fractional order Caputo fractional derivative can be defined as [35]

$$ {}_{0+}^{C}D_t^{\alpha(t)}\, y(t) = \frac{1}{\Gamma[1-\alpha(t)]}\int_{0+}^{t}(t-\zeta)^{-\alpha(t)}\frac{\mathrm{d}y(\zeta)}{\mathrm{d}\zeta}\mathrm{d}\zeta, \qquad 0 \le \alpha(t) < 1\,. \tag{3} $$

A non-singular variable order C-F fractional derivative is defined as [36,37]

$$ {}_{0+}^{CF}D_t^{\alpha(t)}\, y(t) = \frac{P[\alpha(t)]}{\Gamma[1-\alpha(t)]}\int_{0+}^{t}\frac{\mathrm{d}}{\mathrm{d}\zeta}y(\zeta)\exp\left[\frac{\alpha(t)(t-\zeta)}{1-\alpha(t)}\right]\mathrm{d}\zeta, \qquad 0 < \alpha(t) < 1\,. \tag{4} $$

where $P[\alpha(t)]$ is a normalized function such that $P(0)=P(1)=1$ [36].

4. THE DYNAMIC SYSTEM MODEL

The Kelvin-Voigt model [38] that is shown in Fig. 3a is applied to a mass M, see Fig. 3b, to investigate its response due to controlled external force $u(t)$.

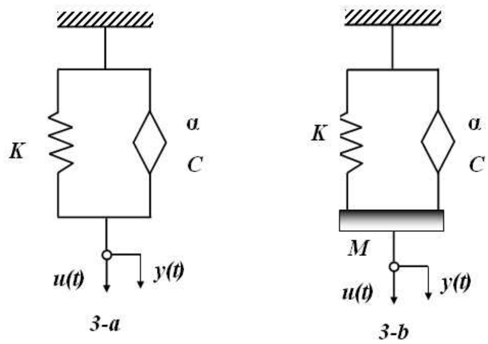

Fig. 3 – a) The Kelvin-Voigt model; b) mass spring dashpot system.

The equation of motion of the generated system, in Fig. 3b, is given by

$$ M\,\ddot{y}(t) + C\,{}_{0+}^{CF}D_t^{\alpha}\, y(t) + K\,y(t) = u(t)\,, \tag{5} $$

where C is the damping coefficient of the dashpot element and K represents the stiffness of the spring. In this study the model (5) is extended to be a variable order fractional oscillatory system as follows:

$$ M\,\ddot{y}(t) + C\,{}_{0+}^{CF}D_t^{\alpha(t)}\, y(t) + K\,y(t) = u(t). \tag{6} $$

The introduced discrete-time PID controller is applied to the fractional dynamic model that is given by Eq. (6). Numerical and discretization techniques are utilized to obtain the controlled system responses.

5. OPEN LOOP SYSTEM RESPONSE

The response $y(t)$ of the uncontrolled fractional dynamic system is approximated numerically before applying the discrete PID controller. The forward finite difference method is applied to the first and second derivatives of the system output $y(t)$. The non-singular variable order C-F fractional derivative that is defined by Eq. (4) is substituted into the damping term of the system model in Eq. (6) to generate the following model:

$$M\,\ddot{y}(t)+C\left[\frac{\mathrm{P}\big(\alpha(t)\big)}{\big[1\text{-}\alpha(t)\big]}\right]\int_{0+}^{t}\frac{\mathrm{d}}{\mathrm{d}\zeta}y(\zeta)\exp\left[-\frac{\alpha(t)\big(t-\zeta\big)}{1-\alpha(t)}\right]\mathrm{d}\zeta+K\,y(t)=u(t)\,. \tag{7}$$

The integral in Eq. (7) can be approximated by discretization technique as follows

$$\left[\frac{P\big[\alpha(t)\big]}{\big[1\text{-}\alpha(t)\big]}\right]\int_{0+}^{t}\frac{\mathrm{d}}{\mathrm{d}\zeta}y(\zeta)\exp\left[-\frac{\alpha(t)\big(t-\zeta\big)}{1-\alpha(t)}\right]\mathrm{d}\zeta=\left[\frac{P\big[\alpha(t)\big]}{\big[1-\alpha(t)\big]}\right]\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\frac{\mathrm{d}}{\mathrm{d}\zeta}y\big(\zeta\big)\exp\left[-\frac{\alpha\big(t_k\big)\big(t_k-\zeta\big)}{1-\alpha\big(t_k\big)}\right]\mathrm{d}\zeta\,. \tag{8}$$

The forward finite difference method is used to approximate the derivative $\frac{\mathrm{d}y(\zeta)}{\mathrm{d}\zeta}$ at the point k to obtain to following expression:

$$\left[\frac{\mathrm{P}\big[\alpha(t)\big]}{\big[1\text{-}\alpha(t)\big]}\right]\int_{0+}^{t}\frac{\mathrm{d}}{\mathrm{d}\zeta}y(\zeta)\exp\left[-\frac{\alpha(t)\big(t-\zeta\big)}{1-\alpha(t)}\right]\mathrm{d}\zeta=\left[\frac{P\big[\alpha(t)\big]}{\big[1-\alpha(t)\big]}\right]\sum_{j=0}^{k-1}\frac{y_{k+1}-y_k}{h}\int_{t_j}^{t_{j+1}}\exp\left[-\frac{\alpha\big(t_k\big)\big(t_k-\zeta\big)}{1-\alpha\big(t_k\big)}\right]\mathrm{d}\zeta\,. \tag{9}$$

The second derivative $\ddot{y}(t)$ in the first term of the model in Eq. (7) can be expressed by using the finite differences method at the point *k*. We substitute the approximated integral that is obtained by Eq. (9) into model Eq. (7) to obtain the following expression:

$$M\,\frac{y_{k+1}-2y_k+y_{k-1}}{h^2}+C\left[\frac{P\big[\alpha(t)\big]}{\big[1-\alpha(t)\big]}\right]\sum_{j=0}^{k-1}\frac{y_{k+1}-y_k}{h}\int_{t_j}^{t_{j+1}}\exp\left[-\frac{\alpha\big(t_k\big)\big(t_k-\zeta\big)}{1-\alpha\big(t_k\big)}\right]\mathrm{d}\zeta+K\,y_k=u_k\,. \tag{10}$$

From Eq. (10) for arbitrary system parameters and external exciting force, the introduced dynamic system open loop responses can be obtained by solving the integral in Eq. (10) and substituting the initial conditions $y(t_0)=y_0$, and $\dot{y}\big(t_0\big)=\dot{y}_0$. For more details we refer to [39]. The unit step uncontrolled responses of the introduced forced fractional dynamic system are shown in Fig. 4. The comparison between the un-damped classical integer system response and the un-damped fractional system response, for $\alpha=0.99$, is shown in Fig. 4a. The integer system response is compared to the fractional one for under-damped model with $\alpha=0.99$ in Fig. 4b. The fractional dynamic system responses, for $\alpha=0.5$ and $\alpha=0.8$ are shown, respectively, in Fig. 4c and Fig. 4d.

6. CLOSED LOOP SYSTEM RESPONSE

The considered uncontrolled damped fractional system responses can be controlled to match desired responses by using the introduced discrete-time PID controller. The numerical response y_k that is obtained from Eq. (10) is taken as a feedback signal of the closed loop control system. A reference signal r_k is assigned to represent the desired response of the controlled system. The error control signal e_k is generated as $e_k=r_k-y_k$. For arbitrary controller gains, the generated error signal e_k is substituted into the controller outputs $u_p(k)$, $u_i(k)$, $u_d(k)$ that generate the controlled input signal $u(k)$, see Eq. (2). The controlled system response can be generated by means of Eq. (10). Unit step responses of controlled and uncontrolled systems are shown in Fig. 5. The responses of controlled and uncontrolled damped fractional system for $\alpha=0.5$ and $\alpha=0.8$ are demonstrated in Fig. 5a, and Fig. 5b, respectively.

It is deduced from Fig. 4a and Fig. 4b that the responses of integer system representation are identical to the fractional ones, when the value of fractional order α is close to one. Based on the system models that are given by Eq. (5) and Eq. (6) the fractional models reach the classical integer representation as the fractional order α be close to one. Moreover, we can infer from Fig. 4c and Fig. 4d that the settling time of the system response decreases as α goes to one for the same system parameters. It is inferred from Fig. 5 that the response of the fractional controlled system reaches the steady state faster and demonstrates lower system response characteristics. The controller effects on the fractional system are similar to those on the conventional integer system. These effects are discussed in more details in the next Section.

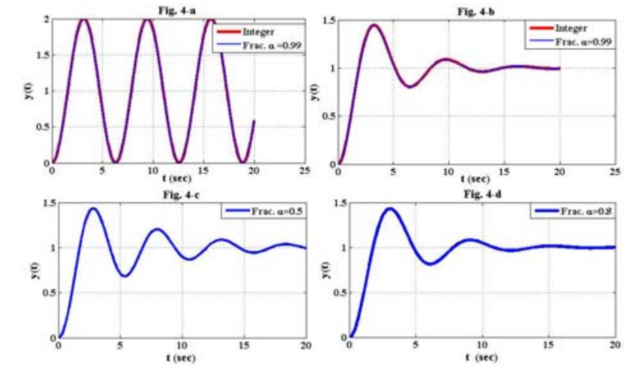

Fig. 4 – Unit step uncontrolled responses: a) un-damped fractional α = 0.99 *vs.* un-damped integer; b) damped fractional α = 0.99 *vs.* damped integer; c) damped fractional α = 0.5; d) damped fractional α = 0.8.

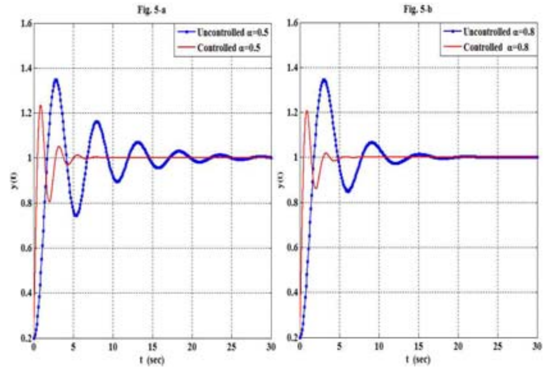

Fig. 5 – Unit step controlled *vs.* uncontrolled responses: a) damped fractional α = 0.5; b) damped fractional α = 0.8.

7. EFFECTS OF CONTROLLER GAINS ON SYSTEM RESPONSES

In the classical integer models, the rise time, the overshoot percentage, the settling time, peak time, and the steady state error that represent the system response characteristics are affected by the PID controller gains [40]. In this Section, we investigate the effects of these gains on the fractional system response characteristics.

For $k_i = 3$ and $k_d = 2$ the effect of the proportional controller gain k_p on the characteristics of integer and fractional systems' responses are shown in Fig. 6. In which the fractional order of the fractional system damping force is taken to be close to one ($\alpha = 0.99$). The effect of k_p on the fractional system response characteristics is shown in Fig. 7 for two different fractional system damping force orders, namely, $\alpha = 0.5$ and $\alpha = 0.8$.

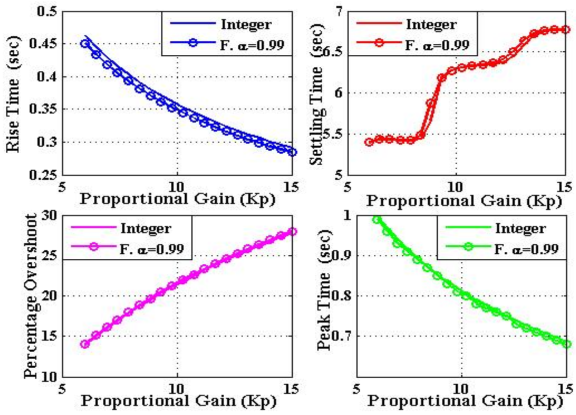

Fig. 6 – The effects of k_p on the characteristics of integer and fractional systems' responses.

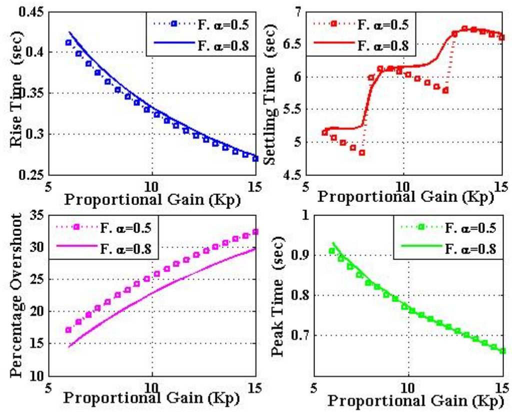

Fig. 7 – The effects of k_p on the characteristics fractional systems' responses at $\alpha = 0.5$ and $\alpha = 0.8$.

The behaviors of integer and fractional systems' responses against k_p are similar for constant values of k_i and k_d. This is deduced from Fig. 6 and Fig. 7. The effects of the controller integral gains k_i and k_d on the characteristics of integer and fractional systems' responses are illustrated in Fig. 8 and Fig. 9, respectively.

It is clear from Fig. 6 through Fig. 9 that, regardless the characteristics values, the results demonstrate similar responses of the integer and fractional systems, when controller gains vary. For instance, the rise time decreases in both integer and fractional systems' responses, as k_i increases. However, the rise time values of the integer system are different from the values of the fractional one. Moreover, when the k_p increases, the overshoot and settling time are increased in both integer and fractional systems' responses. Furthermore, as k_d increases, the characteristics of both integer and fractional systems' responses are decreased.

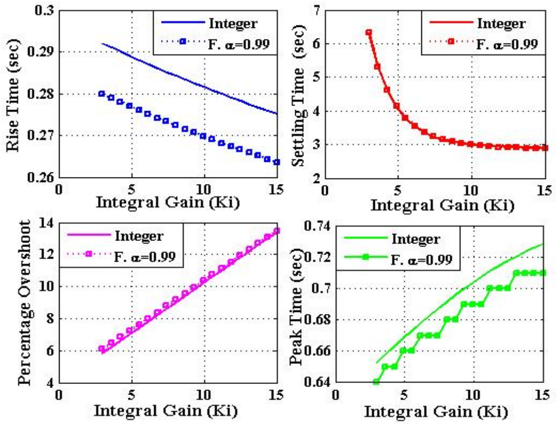

Fig. 8 – The effects of k_i on the characteristics of integer and fractional systems' responses.

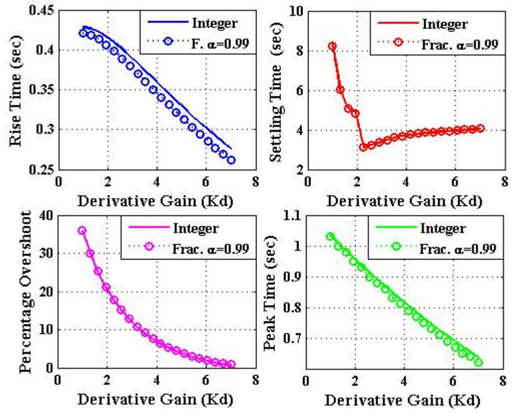

Fig. 9 – The effects of k_d on the characteristics of integer and fractional systems' responses.

8. EFFECT OF THE FRACTIONAL DERIVATIVE VARIABLE ORDER ON SYSTEM RESPONSES

The damping force variable fractional order $\alpha(t)$ of the fractional model given by equation Eq. (6) is introduced as a linear function of time. In this Section the effects of $\alpha(t)$ on the uncontrolled and controlled fractional systems' responses are investigated. Figure 10a shows the uncontrolled fractional systems' responses $y(\alpha(t),t)$ against the linear function of $\alpha(t)$ and t. The introduced discrete-time PID controller is applied to the fractional system whose responses are shown in Fig. 10a. The controller gains' values are chosen to be relatively more effective with respect to the system parameters (M, C and K), see Eq. 5. Figure 10b shows the effects of $\alpha(t)$ on the controlled system responses $y(\alpha(t),t)$, when relatively more effective controller gains' values are applied. In order to investigate the effects of $\alpha(t)$ on the controlled fractional system responses, the PID controller gains' values are chosen to be relatively low, see Fig. 10c.

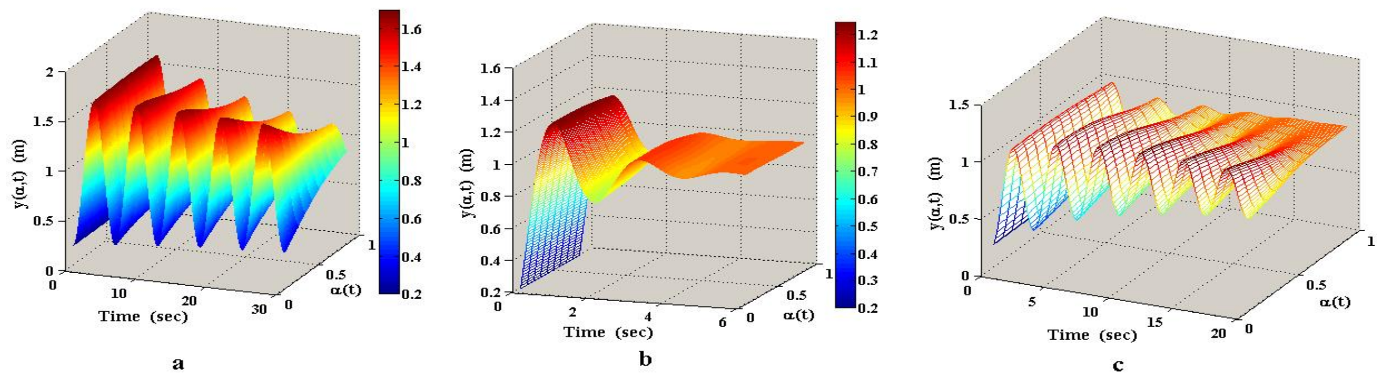

Fig. 10 – The effects of $\alpha(t)$ on the: a) uncontrolled, b) relatively high controlled, c) relatively low controlled fractional system response.

It is deduced from Fig. 10a that the uncontrolled fractional system response reaches the steady state faster, as $\alpha(t)$ is close to one. It is concluded from Fig. 10 that the effects of the fractional derivative variable order $\alpha(t)$ on the uncontrolled and controlled fractional system are similar. However, the effects of $\alpha(t)$ on the controlled fractional system responses are decreased, when the controller gains' values are chosen to be relatively more effective with respect to the system parameters, see Fig. 10b.

9. CONCLUSION

A forced damped variable order fractional oscillatory system is modeled and controlled by the discrete-time PID controller. The damped fractional force of the system is modeled by the C-F fractional derivative. The system responses are obtained by applying numerical and discretization techniques. Uncontrolled and controlled fractional system responses are compared to the conventional integer model responses. The results demonstrate that the fractional system responses at $\alpha(t)=0.99$ matches the integer system response for the same systems' parameters. This comparison shows the feasibility and effectiveness of the applied numerical and discretization techniques. The effects of the PID controller gains on the fractional system responses are investigated. The results show that both fractional system and integer system responses' characteristics have similar behaviors versus the variations of the controller gains. The effect of the fractional derivative variable order $\alpha(t)$ is studied. The study shows that the steady state of the fractional system response reaches faster, as $\alpha(t)$ is close to one. In summary, this study has illustrated that the same parameters integer and fractional oscillatory systems have similar responses' behaviors. However their responses' characteristics are different, being based on the choice of different $\alpha(t)$ values. That result is due to the properties of visco-elastic material of the dashpot element in the fractional system.

REFERENCES

1. M. HERZALLAH, D. BALEANU, *Fractional-order Euler-Lagrange equations and formulation of Hamiltonian equations*, Nonlinear Dynamics, **58**, pp. 385–391, 2009.
2. T. AVKAR, D. BALEANU, *Fractional Euler-Lagrange equations for constrained systems,* AIP Conference Proceedings Ankara, Turkey, **729**, *1*, pp. 84–90, 2004.
3. M. ALQURAN, I. JARADAT, D. BALEANU, R. ABDEL-MUHSEN, *An analytical study of (2+1)-dimensional physical models embedded entirely in fractal space*, Romanian Journal of Physics, **64**, *1-2*, art. 103, 2019.
4. J.F. GOMEZ AGUILAR, D. BALEANU, *Solutions of the telegraph equations using a fractional calculus approach,* Proceedings of the Romanian Academy, Series A, **15**, *1*, pp. 27–34, 2014.
5. G. ALOTTA, S. LORENZO, M. PAOLA, A. PIRROTTA, *Earthquake ground motion artificial simulations through fractional Tajimi-Kanai model*, AIMETA Turin, Italy, pp. 17–20, 2013.
6. S.I. MUSLIH, D. BALEANU, *Fractional Euler-Lagrange equations of motion in fractional space*, Journal of Vibration and Control, **13**, *9-10*, pp. 1209–1216, 2007.
7. Y.H. YOUSSRI, W.M. ABD-ELHAMEED, *Numerical spectral Legendre-Galerkin algorithm for solving time fractional telegraph equation*, Romanian Journal of Physics, **63**, *3-4*, art. 107, 2018.
8. A.A. EL-KALAAWY, E.H. DOHA, S.S. EZZ-ELDIEN, M.A. ABDELKAWY, R.M. HAFEZ5, A.Z.M. AMIN, D. BALEANU, M.A. ZAKY, *A computationally efficient method for a class of fractional variational and optimal control problems using fractional Gegenbauer functions*, Romanian Reports in Physics, **70**, *2*, art. 109, 2018.
9. D. KUMAR, J. SINGH, D. BALEANU, *A fractional model of convective radial fins with temperature-dependent thermal conductivity*, Romanian Reports in Physics, **69**, *1*, art. 103, 2017.
10. XIAO-JUN YANG, FENG GAO, H.M. SRIVASTAVA, *New rheological models within local fractional derivative*, Romanian Reports in Physics, **69**, 3, art. 113, 2017.
11. X.-J. YANG, J.A. TENREIRO MACHADO, D. BALEANU, *Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler type functions*, Romanian Reports in Physics, **69**, *4*, art. 115, 2017.
12. XIAO-JUN YANG, *New general fractional-order rheological models with kernels of Mittag-Leffler functions*, Romanian Reports in Physics, **69**, *4*, art. 118, 2017.
13. D. BALEANU, J.H. ASAD, A. JAJARMI, *The fractional model of spring pendulum: new features within different kernels*, Proceedings of the Romanian Academy, Series A, **19**, *3*, pp. 447–454, 2018.
14. XIAO-JUN YANG, *New rheological problems involving general fractional derivatives with nonsingular power-law kernels*, Proceedings of the Romanian Academy, Series A, **19**, *1*, pp. 45–52, 2018.
15. L.S. TAYLOR, A.L. LERNER, D.J. RUBENS, K.J. PARKER, *A Kelvin-Voight fractional derivative model for viscoelastic characterization of liver tissue*, ASME 2002 International Mechanical Engineering Congress and Exposition, New Orleans, Louisiana, 2002, pp. 447–448.
16. L. DESERI, M.D. PAOLA, M. ZINGALES, P. POLLACI, *Power-law hereditariness of hierarchical fractal bones*, International Journal for Numerical Methods in Biomedical Engineering, **29**, *12*, pp. 1338–1360, 2013.
17. A. FREED, K. DIETHELM, Y. LUCHKO, *Fractional-order viscoelasticity (FOV): Constitutive development using the fractional calculus*, First annual report NASA/TM-2002-211914, NASA's Glenn Research Center, Ohio, USA, 2002.
18. K.D. PAPOULIA, V.P. PANOSKALTSIS, N.V. KURUP, I. KOROVAJCHUK, *Rheological representation of fractional order viscoelastic material models,* Rheologica Acta, **49**, *4*, pp. 381–400, 2010.
19. H.G. SUN, W. CHEN., Y.Q. CHEN, *Variable-order fractional differential operators in anomalous diffusion modeling*, Physica A, **388**, *21*, pp. 4586–4592, 2009.

20. S. SHEN., F. LIU, J. CHEN, I. TURNER, V. ANH, *Numerical techniques for the variable order time fractional diffusion equation,* Applied Mathematics & Computation, **218**, *22*, pp. 10861–10870, 2012.
21. M. MERDAN, *On the solutions of time-fractional generalized Hirota-Satsuma coupled-KDV equation with modified Riemann-Liouville derivative by an analytical technique*, Proceedings of the Romanian Academy, Series A, **16**, *1*, pp. 3–10, 2015.
22. J.F. GÓMEZ AGUILAR, J.R. RAZO HERNÁNDEZ, R.F. ESCOBAR JIMÉNEZ, C.M. ASTORGA-ZARAGOZA, V.H. OLIVARES PEREGRINO, T. CÓRDOVA FRAGA, *Fractional electromagnetic waves in plasma*, Proceedings of the Romanian Academy, Series A, **17**, *1*, pp. 31–38, 2016.
23. A. BHRAWY, *A new spectral algorithm for time-space fractional partial differential equations with subdiffusion and superdiffusion*, Proceedings of the Romanian Academy, Series A, **17**, *1*, pp. 39–47, 2016.
24. A.H. BHRAWY, E.H. DOHA, D. BALEANU, S.S. EZZ-ELDIEN, M. ABDELKAWY, *An accurate numerical technique for solving fractional optimal control problems,* Proceedings of the Romanian Academy, Series A, **16**, *1*, pp. 47–54, 2015.
25. D. BALEANU, A.C. MACHADO, A.C. LOU, *Fractional Dynamics and Control*, Springer Science & Business Media, 2012.
26. M. ZAMANI, M. KARIMI-GHARTEMANI, N. SADATI, M. PARNIANI, *Design of a fractional order PID controller for an AVR using particle swarm optimization,* Control Engineering Practice, **17**, *12*, pp. 1380–1387, 2009.
27. B. KHALFA, C. ABDELFATEH, *Optimal tuning of fractional order* $PI^{\lambda}D^{\mu}A$ *controller using Particle Swarm Optimization algorithm*, IFAC-Papers On Line, **50**, *1*, pp. 8084–8089, 2017.
28. J.F. GOMEZ-AGUILAR, H. YÉPEZ-MARTÍNEZ, C. CALDERÓN-RAMÓN, I. CRUZ-ORDUÑA, R.F. ESCOBAR-JIMÉNEZ, V.H. OLIVARES-PEREGRINO, *Modeling of a mass-spring-damper system by fractional derivatives with and without a singular kernel*, Entropy, **17**, *9*, pp. 6289–6303, 2015.
29. A. AGILA, D. BALEANU, R. EID, B. IRFANOGLU, *A freely damped oscillating fractional dynamic system modeled by fractional Euler–Lagrange equations,* Journal of Vibration & Control, **24**, *7*, pp. 1228–1238, 2018.
30. A. AGILA, D. BALEANU, *A study of a multi-degree of freedom fractional order damped oscillatory system*, U.P.B. Scientific Bulletin, Series D, **80**, *2*, pp. 51-62, 2018.
31. I. PODLUBNY, *Fractional-order systems and fractional-order controllers*, Institute of Experimental Physics, Slovak Academy of Sciences, **12,** *3*, pp. 1–18, 1994.
32. K.J. ASTROM, R.M. MURRAY, *Feedback systems an introduction for scientists and engineers,* Princeton University Press, 2008.
33. F. HAUGEN, *Advanced dynamics and control*, TechTeach, 2010.
34. A.A. KILBAS, H.M. SRIVASTAVA, J.J. TRUJILLO, *Theory and applications of fractional differential equations,* Elsevier Science Limited, vol. 204, 2006.
35. J. WANG, LI LIU, LE LIU, Y. CHEN, *Numerical solution for the variable order fractional partial differential equation with Bernstein polynomials*, International Journal of Advancements in Computing Technology, **6**, *3*, pp. 22–38, 2014.
36. M. CAPUTO, M. FABRIZIO, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation and Applications, **1**, *2*, pp. 1–13, 2015.
37. J. LOSADA, J.J. NIETO, *Properties of a new fractional derivative without singular kernel,* Progress in Fractional Differentiation and Applications, **1**, *2*, pp. 87–92, 2015.
38. R. LEWANDOWSKI, B. CHORAZYCZEWSKI, *Identification of the parameters of the Kelvin-Voigt and the Maxwell fractional models, used to modeling of viscoelastic dampers*, Computers & Structures, **88**, *1-2*, pp. 1–17, 2010.
39. A. AGILA D. BALEANU, R. EID, B. IRFANOGLU, *Applications of the extended fractional Euler-Lagrange equations model to freely oscillating dynamical systems*, Romanian Journal of Physics, **61**, *3-4*, pp. 350–359, 2016.
40. K.A. TEHRANI, A. MPANDA, *PID control theory* (Ch. 9), in: *Introduction to PID controllers– Theory, tuning and application to frontier areas* (ed. R.C. Panda), IntechOpen, 2012, pp. 213–228.

Received November 15, 2018