THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 21, Number 2/2020, pp. 95–103

AN ANALOG OF MOLIEN'S FORMULA FOR GRADINGS

Piotr JĘDRZEJEWICZ

"Nicolaus Copernicus" University, Faculty of Mathematics and Computer Science, Toruń, Poland
E-mail: pjedrzej@mat.umk.pl

Abstract. We present an analog of Molien's formula for gradings, involving power series with coefficients in a group algebra. As an application we obtain a proof of a necessary condition for the homogeneous component of trivial degree to be generated by algebraically independent polynomials, analogous to the original proof of Chevalley-Shephard-Todd theorem.

Key words: group grading, graded ring, Chevalley-Shephard-Todd theorem, Molien's formula, Hilbert-Poincaré series.

1. INTRODUCTION

Throughout the paper k is a field and $A = k[x_1,\ldots,x_m]$ is the k-algebra of polynomials in m variables. By V we denote the k-linear space spanned by $x_1,\ldots,x_m$. Recall that a k-linear automorphism of V is called a pseudo-reflection if it has a finite order and its fixed subspace has codimension 1. Our motivation comes from a fundamental result of invariant theory, Chevalley-Shephard-Todd theorem.

THEOREM 1.1 (Shephard and Todd [8], Chevalley [1]). *Let G be a finite subgroup of* $\mathrm{GL}(V)$. *In the case of* $\operatorname{char} k > 0$, *assume additionally that* $|G|$ *is not divisible by* $\operatorname{char} k$. *The following conditions are equivalent:*

(i) *the subalgebra of invariants A^G is generated by algebraically independent polynomials,*
(ii) *the group G is generated by pseudo-reflections.*

In this paper we follow the exposition of Chevalley-Shephard-Todd theorem's proof given in [4] and [9]. For more information we refer the reader to a survey article [10].

Let G be a group with a multiplicative notation. Recall that a decomposition of the k-algebra A as a direct sum of k-linear subspaces $A = \sum_{g\in G} A^g$ is called a grading if $uw \in A^{gh}$ for every $u \in A^g$, $w \in A^h$ with $g,h \in G$. There are some basic analogies between group actions on algebras and group gradings, see the Introduction in [6] for a detailed discussion. In the situation of graded algebras, the analog of the subalgebra of invariants is the homogeneous component of trivial degree the neutral element of G. The action of the group G on A in the above theorem is *linear* in the sense that it is induced by linear transformations on V. An analog of a linear action for gradings is a *linear* grading in the sense that there is a k-linear basis of V consisting of elements, which are homogeneous with respect to this grading. For details we refer the reader to [3]. A grading $A = \sum_{g\in G} A^g$ is linear if and only if after a linear change of coordinates:

$$x_1 \in A^{g_1},\ldots,\ x_m \in A^{g_m} \qquad (*)$$

for some $g_1,\ldots,g_m \in G$. Gradings defined by (*) are called *good* gradings in [2]. In particular, see [2], Proposition 3.4 for the basic properties of such gradings.

The following is an analog of Chevalley-Shephard-Todd theorem for gradings.

THEOREM 1.2 ([3], Theorem 5.1). *Let G be a finite group with the neutral element e. Consider a grading*

$$A = \sum_{g \in G} A^g$$

of the polynomial algebra $A = k[x_1, \ldots, x_m]$ over a field k, such that $x_1 \in A^{g_1}, \ldots, x_m \in A^{g_m}$ for some $g_1, \ldots, g_m \in G$. Put $r_1 = |\langle g_1 \rangle|, \ldots, r_m = |\langle g_m \rangle|$. Moreover, put $G_0 = \cap_{i=1}^{m} \langle g_{1,\ldots,} g_{i-1,} g_{i+1,\ldots,} g_m \rangle$. Then the following conditions are equivalent:

(i) *A^e is generated by algebraically independent polynomials,*

(ii) *$A^e = k\left[x_1^{r_1}, \ldots, x_m^{r_m}\right]$,*

(iii) *the inner product of subgroups $\langle g_1 \rangle \cdot \ldots \cdot \langle g_m \rangle$ is direct,*

(iv) *$G_0 = \{e\}$.*

In the Chevalley-Shephard-Todd theorem for group actions (Theorem 1.1) the proof of implication $(\mathrm{i}) \to (\mathrm{ii})$ is based on properties of Hilbert-Poincaré series of the algebra of invariants. Molien's formula ([5], see [4], 17.2; [9], 4.1.3):

$$\sum_{n=0}^{\infty} \left(\dim_k A_n^G\right) T^n = \frac{1}{|G|} \sum_{\varphi \in G} \frac{1}{\det\left(\mathrm{Id}_V - \varphi T\right)}$$

plays a central role in the proof (T is an indeterminate, Id_V denotes the identity operator on V).

In this paper we present an analog of Molien's formula for G-gradings of polynomial algebras (Theorem 2.3) in which we combine Hilbert-Poincaré series of all components into a single series with coefficients in the group algebra $\mathbf{Z}G$ over the ring of integers. In Remark 2.5 we explain in detail why this is a generalization of the classical Hilbert-Poincaré series. In Section 3 we provide examples, which show that this formula allows to compute dimensions of all components at once.

The paper has a precise Hopf-algebraic motivation connected with the fact that group algebras and their duals serve as two main examples of Hopf algebras. Their Hopf algebra actions on algebras correspond to group actions and group gradings, respectively. Many properties of group actions can be generalized to Hopf algebra actions. A crucial test whether a property can be generalized is if it has an analog for gradings. In Section 4 we show that Chevalley-Shephard-Todd theorem passes this test. The key steps in the proof of the necessity part of Theorem 1.1 are two formulas involving degrees of generators of A^G ([4], 17.4, Theorem A, [9], 4.1.5, 4.1.6). In Section 4, using Molien's formula, we obtain analogs of these formulas for gradings (Propositions 4.2 and 4.3), and we perform the full proof of the necessity part of Theorem 1.2 via formal series. Let us point out that Theorem 1.2 can be proved directly (see [3]), without formal series, but everything we obtain in Section 4 is a clear argument in favor of the possibility to generalize the proof of the necessity part of Chevalley-Shephard-Todd Theorem for Hopf algebra actions. Let us stress however that the paper is not about Hopf algebras, they are only mentioned without any technicalities as a common generalization of group actions and group gradings.

2. AN ANALOG OF MOLIEN'S FORMULA

Recall that $A = k[x_1, \ldots, x_m]$ is the k-algebra of polynomials over a field k. We have a natural $\mathbf{N}_0$-grading of A defined by the degree of a polynomial:

$$A = \sum_{n=0}^{\infty} A_n ,$$

where $\mathbf{N}_0$ denotes the semigroup of non-negative integers and A_n is the k-linear space of homogeneous polynomials of degree n.

Let G be a finite group with a multiplicative notation. Consider a G-grading of the k-algebra A:

$$A=\sum_{g\in G}A^g$$

defined by $x_1\in A^{g_1},\dots, x_m\in A^{g_m}$ for some $g_1,\dots,g_m\in G$. Then, for any $g\in G$, we have a k-linear basis of A^g formed by the monomials $x_1^{l_1}\dots x_m^{l_m}$, which exponents $l_1,\dots,l_m$ satisfy the condition $g_1^{l_1}\dots g_m^{l_m}=g$. For simplicity we may assume that $g_1,\dots,g_m$ generate the group G. In this case G is a commutative group ([3], Proposition 2.1).

For $g\in G$ and $n=0,1,2,\dots$ we put $A_n^g=A^g\cap A_n$. Then we obtain a natural $\mathbf{N}_0$-grading as a vector space of each component A^g, where $g\in G$:

$$A^g=\sum_{n=0}^{\infty}A_n^g,$$

for details see [3], Propositions 3.2 and 3.3. Denote by $s_g(T)$ an arbitrary Hilbert-Poincaré series of the component A^g:

$$s_g(T)=\sum_{n=0}^{\infty}\left(\dim_k A_n^g\right)T^n\in\mathbf{Z}[[T]].$$

Now we can introduce a series with coefficients in the group algebra $\mathbf{Z}G$ that will be the main tool to work with our grading.

Definition 2.1. Put:

$$s(T)=\sum_{n=0}^{\infty}\sum_{g\in G}\left(\dim_k A_n^g\right)gT^n\in ZG[[T]],$$

and call $s(T)$ a Hilbert-Poincaré formal sum associated to the given G-grading of A.

For simplicity we consider $\mathbf{Z}[[T]]$ as a subring of $\mathbf{Z}G[[T]]$, that is, we identify the integer 1 with the neutral element of G. When we consider $\mathbf{Z}G[[T]]$ as a free $\mathbf{Z}[[T]]$-module with G as a basis, then the Hilbert-Poincaré series of all components are simply coefficients of a decomposition.

COROLLARY 2.2. *The following identity holds in* $\mathbf{Z}G[[T]]$:

$$s(T)=\sum_{g\in G}s_g(T)\cdot g.$$

The next theorem presents an analog of Molien's formula for gradings of the algebra of polynomials.

THEOREM 2.3. *The following identity holds in* $\mathbf{Z}G[[T]]$:

$$s(T)=\prod_{i=1}^{m}\frac{1}{1-g_iT}.$$

Proof. Monomials $x_1^{l_1}\dots x_m^{l_m}$ such that $l_1+\dots+l_m=n$ and $g_1^{l_1}\dots g_m^{l_m}=g$ form a k-linear basis of A_n^g. Hence,

$$\prod_{i=1}^{m}\frac{1}{1-g_iT}=\prod_{i=1}^{m}\sum_{l_i=0}^{\infty}(g_iT)^{l_i}=\sum_{l_1,\dots,l_m\ge 0}(g_1T)^{l_1}\cdot\ldots\cdot(g_mT)^{l_m}$$

$$=\sum_{n=0}^{\infty}\sum_{g\in G}\sum_{\substack{l_1,\dots,l_m\ge 0\\ l_1+\dots+l_m=n\\ g_1^{l_1}\dots g_m^{l_m}=g}}\left(g_1^{l_1}\cdot\ldots\cdot g_m^{l_m}\right)T^n=\sum_{n=0}^{\infty}\sum_{g\in G}\left(\dim_k A_n^g\right)gT^n=s(T).\ \square$$

As an immediate consequence of the above theorem we obtain an explicit presentation of $s(T)$ in the form of a quotient with a denominator belonging to $\mathbf{Z}[[T]]$.

THEOREM 2.4. *The following identity holds in* $\mathbf{Z}G[[T]]$:

$$s(T)=\prod_{i=1}^{m}\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1-T^{r_i}},$$

where $r_i=|\langle g_i\rangle|$ for $i=1,\ldots,m$.

Proof. In Theorem 2.3 we can transform each factor further:

$$\frac{1}{1-g_iT}=\frac{1-(g_iT)^{r_i}}{1-g_iT}\cdot\frac{1}{1-T^{r_i}}=\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1-T^{r_i}}.\ \square$$

Remark 2.5. Note that the series $s(T)$ introduced in Definition 2.1 is an adaptation of Hilbert-Poincaré series to the case we consider. Suppose we are given a k-algebra B and a semigroup G. The idea is to assign to a G-grading $B=\sum_{g\in G}B^g$ the formal sum $\sum_{g\in G}\left(\dim_k B^g\right)g$. In particular, when we take the semigroup $\{T^0,T^1,T^2,\ldots\}$ as a multiplicative realization of $\mathbf{N}_0$, we assign to an $\mathbf{N}_0$-grading $B=\sum_{n\in\mathbf{N}_0}B_n$ the formal infinite sum $\sum_{n\in\mathbf{N}_0}\left(\dim_k B_n\right)T^n$. In this paper we consider $G\times\mathbf{N}_0$-gradings (where G is a finite group), so we take $\{gT^n,\ g\in G,n\in\mathbf{N}_0\}$ as a multiplicative realization of $G\times\mathbf{N}_0$, and we assign to a grading $B=\sum_{(g,n)\in G\times\mathbf{N}_0}B_n^g$ the formal infinite sum $\sum_{(g,n)\in G\times\mathbf{N}_0}\left(\dim_k B_n^g\right)gT^n$.

3. EXAMPLES

In this section we provide examples of linear gradings and compute their Hilbert-Poincaré formal sums using Molien's formula. Recall that k is a field. Recall also that we identify integer 1 with the neutral element of G.

Example 3.1. Let $G=\langle g\mid g^2{=}1\rangle=\{1,g\}$. Consider a G-grading of $A=k[x,y]$ such that $x,y\in A^g$. Then

$$s(T)=\frac{1}{(1-gT)^2}=\left(\frac{1+gT}{1-T^2}\right)^2=\frac{1+2gT+g^2T^2}{(1-T^2)^2}=\frac{(1+T^2)+2T\cdot g}{(1-T^2)^2}.$$

When $\operatorname{char}k=2$, the above example presents a $\mathbf{Z}_2$-grading of $k[x,y]$ defined by the Euler derivation $d=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ (for details about gradings defined by derivations see e.g. [3], Section 8). In this case the neutral element's component is the kernel of d:

$$\{f\in k[x,y]: d(f)=0\}=k[x^2,y^2,xy].$$

It is a $k[x^2,y^2]$-algebra generated by a single polynomial, which illustrates Proposition 4.2 from [7].

Example 3.2. Let $G=\langle g\mid g^3=1\rangle=\{1,g,g^2\}$. Consider a G-grading of $A=k[x,y]$ such that $x,y\in A^g$. Then

$$s(T)=\frac{1}{(1-gT)^2}=\left(\frac{1+gT+g^2T^2}{1-T^3}\right)^2=\frac{1+g^2T^2+g^4T^4+2gT+2g^2T^2+2g^3T^3}{\left(1-T^3\right)^2}=$$
$$=\frac{\left(1+2T^3\right)+\left(2T+T^4\right)\cdot g+3T^2\cdot g^2}{\left(1-T^3\right)^2}.$$

When $\operatorname{char} k=3$, the above example presents a $\mathbf{Z}_3$-grading of $k[x,y]$ defined by the Euler derivation $d=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$. The kernel of d:

$$\{f\in k[x,y]: d(f)=0\}=k\left[x^3,y^3,x^2y,xy^2\right]$$

is not a $k\left[x^3,y^3\right]$-algebra generated by a single polynomial, which was observed in [7], Example 4.3.

Example 3.3. Let $G=\langle g\mid g^6=1\rangle=\{1,g,g^2,g^3,g^4,g^5\}$. Consider a G-grading of $A=k[x,y]$ such that $x\in A^{g^2}$, $y\in A^{g^3}$. Then

$$s(T)=\frac{1}{\left(1-g^2T\right)\left(1-g^3T\right)}=\frac{\left(1+g^2T+g^4T^2\right)\left(1+g^3T\right)}{\left(1-T^3\right)\left(1-T^2\right)}=$$
$$=\frac{1+gT^3+g^2T+g^3T+g^4T^2+g^5T^2}{\left(1-T^3\right)\left(1-T^2\right)}.$$

The above example presents a $\mathbf{Z}_6$-grading of $k[x,y]$ satisfying the conditions of Theorem 1.2. In particular, $k\left[x^3,y^2\right]$ is the neutral element's component.

Example 3.4. Let $G=\langle g\mid g^4=1\rangle=\{1,g,g^2,g^3\}$. Consider a G-grading of $A=k[x,y,z]$ such that $x,z\in A^g$, $y\in A^{g^2}$. Then

$$s(T)=\frac{1}{(1-gT)^2\left(1-g^2T\right)}=\frac{\left(1+gT+g^2T^2+g^3T^3\right)^2\left(1+g^2T\right)}{\left(1-T^4\right)^2\left(1-T^2\right)}=$$
$$=\frac{\left(\left(1+3T^4\right)+\left(2T+2T^5\right)g+\left(3T^2+T^6\right)g^2+4g^3T^3\right)\left(1+g^2T\right)}{\left(1-T^4\right)^2\left(1-T^2\right)}=$$
$$=\frac{\left(1+3T^3+3T^4+T^7\right)+\left(2T+4T^4+2T^5\right)g+\left(T+3T^2+3T^5+T^6\right)g^2+\left(2T^2+4T^3+2T^6\right)g^3}{\left(1-T^4\right)^2\left(1-T^2\right)}.$$

The above example presents a $\mathbf{Z}_4$-grading of $k[x,y,z]$ considered in [2], Example 3.2, where the authors proved that the neutral element's component is of the form

$$k\left[x^4,y^2,z^4,x^2y,yz^2,x^2z^2,x^3z,xz^3,xyz\right].$$

4. THE MAIN APPLICATION

In this section we use an analog of Molien's formula for gradings (Theorem 2.3) to provide a proof of a necessary condition for the neutral element's component to be generated by algebraically independent polynomials (implication $(\mathrm{i}) \to (\mathrm{iv})$ in Theorem 1.2). We follow as close as possible all elements of the respective proof of Chevalley-Shephard-Todd theorem for group actions given in [4] and [9].

Recall that $A = k[x_1, \ldots, x_m]$ is the algebra of polynomials over a field k, and G is a finite commutative group with multiplicative notation and neutral element 1. We consider a grading

$$A = \sum_{g \in G} A^g$$

such that $x_1 \in A^{g_1}, \ldots, x_m \in A^{g_m}$ for some $g_1, \ldots, g_m \in G$. We assume that $g_1, \ldots, g_m$ generate the group G and denote their ranks by $r_1, \ldots, r_m$, respectively. Moreover, we put $G_0 = \cap_{i=1}^m G_i$, where G_i is the subgroup generated by $g_{1,\ldots,} g_{i-1,} g_{i+1,\ldots,} g_m$, and we denote by t_i the index of G_i in G.

Assume that A^1 is generated by m algebraically independent homogeneous polynomials $u_1, \ldots, u_m$ of degrees $d_1, \ldots, d_m$, respectively. Moreover, A is integral over A^1 since $x_i^{r_i} \in A^1$ for $i = 1, \ldots, m$, therefore the number of generators is m. In this case the Hilbert-Poincaré series of the component A^1 can be expressed in the following way.

PROPOSITION 4.1 ([4, 17.1; 9, 2.5.5]). *The following identity holds in* $\mathbf{Z}[[T]]$:

$$s_1(T) = \prod_{i=1}^m \frac{1}{1 - T^{d_i}} .$$

In the proofs of the next two propositions we will use the following computational technique. We will transform a formal series from $\mathbf{Z}G[[T]]$ to a form $\frac{a(T)}{b(T)}$ with polynomials $a(T) \in \mathbf{Z}G[[T]]$ and $b(T) \in \mathbf{Z}[[T]]$ such that $b(1) \neq 0$. Then we will substitute $T = 1$ in $\frac{a(T)}{b(T)}$ and we will treat it as a "value" of $s(T)$ at $T = 1$:

$$s(T)\big|_{T=1} = \frac{a(1)}{b(1)} \in \mathbf{Q}G .$$

This definition is correct since it does not depend on the presentation as a quotient: if $\frac{a(T)}{b(T)} = \frac{c(T)}{d(T)}$ with $a(T), c(T) \in \mathbf{Z}G[[T]]$, $b(T), d(T) \in \mathbf{Z}[[T]]$, $b(1), d(1) \neq 0$, then $\frac{a(1)}{b(1)} = \frac{c(1)}{d(1)}$ in $\mathbf{Q}G$.

The following formula is an analog of [4], 17.4, Theorem A (i) and [9], 4.1.5.

PROPOSITION 4.2. $|G| = \prod_{i=1}^m d_i$.

Proof. From Corollary 2.2 and Theorem 2.4 we obtain

$$\prod_{i=1}^m \frac{1 + g_i T + \ldots + (g_i T)^{r_i - 1}}{1 - T^{r_i}} = \sum_{g \in G} g s_g(T) ,$$

so

$$\prod_{i=1}^m \frac{1 + g_i T + \ldots + (g_i T)^{r_i - 1}}{1 + T + \ldots + T^{r_i - 1}} = \sum_{g \in G} g (1 - T)^m s_g(T) . \tag{1}$$

We have

$$\prod_{i=1}^{m}\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1+T+\ldots+T^{r_i-1}}\Bigg|_{T=1}=\prod_{i=1}^{m}\frac{1+g_i+\ldots+g_i^{r_i-1}}{r_i}. \tag{2}$$

Observe that the following identity holds in the group algebra $\mathbf{Q}G$:

$$\prod_{i=1}^{m}\frac{1+g_i+\ldots+g_i^{r_i-1}}{r_i}=\frac{1}{|G|}\sum_{g\in G}g. \tag{3}$$

We can prove (3) by induction on m. For $m=1$ we have a trivial case of a cyclic group G. Assume the identity for m and consider a group G generated by $m+1$ elements $g_1,\ldots,g_{m+1}$. Let G' be a subgroup of G generated by $g_1,\ldots,g_m$, put $t=(G:G')$. Then:

$$\prod_{i=1}^{m+1}\frac{1+g_i+\ldots+g_i^{r_i-1}}{r_i}=\frac{1}{|G'|}\sum_{g\in G'}g\cdot\frac{1+g_{m+1}+\ldots+g_{m+1}{}^{r_{m+1}-1}}{r_{m+1}}=$$

$$=\frac{1}{r_{m+1}|G'|}\cdot\sum_{l=0}^{r_{m+1}-1}\sum_{g\in g_{m+1}^{l}G'}g=\frac{1}{r_{m+1}|G'|}\cdot\frac{r_{m+1}}{t}\cdot\sum_{g\in G}g=\frac{1}{|G|}\sum_{g\in G}g.$$

By Proposition 4.1, we have the following coefficient from $\mathbf{Z}[[T]]$ at the neutral element on the right-hand side of (1):

$$(1-T)^m s_1(T)=\prod_{i=1}^{m}\frac{1-T}{1-T^{d_i}}=\prod_{i=1}^{m}\frac{1}{1+T+\ldots+T^{d_i-1}},$$

so

$$(1-T)^m s_1(T)\big|_{T=1}=\prod_{i=1}^{m}\frac{1}{d_i}. \tag{4}$$

Finally, comparing coefficients at the neutral element on both sides of (1) and substituting $T=1$, by (2), (3) and (4) we obtain

$$\frac{1}{|G|}=\prod_{i=1}^{m}\frac{1}{d_i}. \ \square$$

The following formula is an analog of [4], 17.4, Theorem A (ii) and [9], 4.1.6.

PROPOSITION 4.3. $\sum_{i=1}^{m}(t_i-1)=\sum_{i=1}^{m}(d_i-1)$.

Proof. Denote $w_g(T)=(1-T)^m s_g(T)$. By (1) from the proof of Proposition 4.2 we have

$$\prod_{i=1}^{m}\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1+T+\ldots+T^{r_i-1}}=\sum_{g\in G}g\,w_g(T). \tag{5}$$

By differentiating the left-hand side and substituting $T=1$, we obtain:

$$\left(\prod_{i=1}^{m}\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1+T+\ldots+T^{r_i-1}}\right)'\Bigg|_{T=1}=$$

$$=\sum_{i=1}^{m}\frac{\left(g_i+2g_i^2T+\ldots+(r_i-1)g_i^{r_i-1}T^{r_i-2}\right)\cdot\left(1+T+\ldots+T^{r_i-1}\right)-\left(1+2T+\ldots+(r_i-1)T^{r_i-2}\right)\cdot\left(1+g_iT+\ldots+(g_iT)^{r_i-1}\right)}{\left(1+T+\ldots+T^{r_i-1}\right)^2}.$$

$$\cdot\prod_{\substack{j=1\\j\neq i}}^{m}\frac{1+g_jT+\ldots+(g_jT)^{r_j-1}}{1+T+\ldots+T^{r_j-1}}\Bigg|_{T=1}=\sum_{i=1}^{m}\frac{(1-r_i)+(3-r_i)g_i+\ldots+(r_i-1)g_i^{r_i-1}}{2r_i}\cdot\prod_{\substack{j=1\\j\neq i}}^{m}\frac{1+g_j+\ldots+g_j^{r_j-1}}{r_j}.$$

Applying the identity (3) from the proof of Proposition 4.2 for the group G_i we have:

$$\frac{(1-r_i)+(3-r_i)g_i+\ldots+(r_i-1)g_i^{r_i-1}}{2r_i}\cdot\prod_{\substack{j=1\\j\neq i}}^{m}\frac{1+g_j+\ldots+g_j^{r_j-1}}{r_j}=$$

$$=\frac{1}{2r_i}\cdot\sum_{l=0}^{r_i-1}\left(1-r_i+2l\right)g_i^{\,l}\cdot\frac{1}{|G_i|}\cdot\sum_{g\in G_i}g=\frac{1}{2r_i|G_i|}\cdot\sum_{l=0}^{r_i-1}\left(1-r_i+2l\right)\sum_{g\in g_i^{\,l}G_i}g=$$

$$=\frac{1}{2r_i|G_i|}\cdot\sum_{l=0}^{t_i-1}\frac{1-r_i+2l+1-r_i+2(l+r_i-t_i)}{2}\cdot\frac{r_i}{t_i}\cdot\sum_{g\in g_i^{\,l}G_i}g=\frac{1}{2|G|}\sum_{l=0}^{t_i-1}\left(1-t_i+2l\right)\sum_{g\in g_i^{\,l}G_i}g.$$

We have obtained the following:

$$\left(\prod_{i=1}^{m}\frac{1+g_iT+\ldots+(g_iT)^{r_i-1}}{1+T+\ldots+T^{r_i-1}}\right)'\Bigg|_{T=1}=\frac{1}{2|G|}\sum_{l=0}^{t_i-1}\left(1-t_i+2l\right)\sum_{g\in g_i^{\,l}G_i}g. \tag{6}$$

Now, by differentiating the coefficient at the neutral element on the right-hand side of (5) and substituting $T=1$, we obtain:

$$\left(w_1(T)\right)'\Big|_{T=1}=\left(\prod_{i=1}^{m}\frac{1}{1+T+\ldots+T^{d_i-1}}\right)'\Bigg|_{T=1}=$$

$$=\sum_{i=1}^{m}\frac{-\left(1+2T+\ldots+(d_i-1)T^{d_i-2}\right)}{1+T+\ldots+T^{d_i-1}}\cdot\prod_{\substack{j=1\\j\neq i}}^{m}\frac{1}{1+T+\ldots+T^{d_j-1}}\Bigg|_{T=1}=\sum_{i=1}^{m}\frac{-(d_i-1)}{2}\cdot\prod_{j=1}^{m}\frac{1}{d_j}.$$

Comparing with the respective coefficient on the left-hand side of (5), by (6) we have:

$$\frac{1}{2|G|}\sum_{i=1}^{m}\left(1-t_i\right)=\sum_{i=1}^{m}\frac{-\left(d_i-1\right)}{2}\cdot\prod_{j=1}^{m}\frac{1}{d_j}.$$

Recall that $|G|=\prod_{i=1}^{m}d_i$ (Proposition 4.2), so, finally

$$\sum_{i=1}^{m}\left(t_i-1\right)=\sum_{i=1}^{m}\left(d_i-1\right).\ \square$$

Now we can prove implication (i)$\to$(iv) of Theorem 1.2 by the analogy with [4, 18.5] and [9, 4.2.11]. Our motivation is connected with the problem of possible generalization for Hopf algebra actions, so in the proof below we don't use the equivalence (i)$\leftrightarrow$(ii) of Theorem 1.2, which is specific for the case of gradings.

Observe that $\cap_{i=1}^{m}G_i/G_0=G_0/G_0=\left\{\overline{1}\right\}$, where $\left\{\overline{1}\right\}$ is a class of 1 in G/G_0. For a G/G_0-grading obtained from the given G-grading we can apply implication (iv)$\to$(i) of Theorem 1.2. We infer that A^1 is generated by m algebraically independent homogeneous polynomials.

Homogeneous generators $u_1,\dots,u_m$ of A^1 have degrees $d_1,\dots,d_m$, respectively. Let $v_1,\dots,v_m$ be algebraically independent homogeneous generators of $A^{\overline{1}}$, with degrees $d_1',\dots,d_m'$. For the G/G_0 -grading we have $t_i' = \left(G/G_0 : G_i/G_0\right) = \left(G/G_i\right) = t_i$, so by Propositions 4.2 and 4.3 we obtain:

$$d_1' + \ldots + d_m' = t_1' + \ldots + t_m' = t_1 + \ldots + t_m = d_1 + \ldots + d_m$$

and

$$d_1' \cdot \ldots \cdot d_m' = \left|G/G_0\right| \le \left|G\right| = d_1 \cdot \ldots \cdot d_m .$$

Moreover, A^1 is a subalgebra of $A^{\overline{1}}$. Reasoning analogously like in [4], 18.5 or [9], 4.2.11, we obtain $d_i' = d_i$ for $i = 1,...,m$, so $\left|G/G_0\right| = \left|G\right|$, and $G_0 = \{1\}$.

ACKNOWLEDGEMENTS

The author would like to thank the referee for helpful remarks.

REFERENCES

1. C. CHEVALLEY, *Invariants of finite groups generated by reflections*, Amer. J. Math., **77**, pp. 778-782, 1955.
2. S. DĂSCĂLESCU, C. NĂSTĂSESCU, L. NĂSTĂSESCU, *Group gradings on polynomial algebras*, Comm. Algebra, **44**, pp. 3340-3348, 2016.
3. P. JĘDRZEJEWICZ, *Linear gradings of polynomial algebras*, Cent. Eur. J. Math., **6**, pp. 13-24, 2008.
4. R. KANE, *Reflection groups and invariant theory*, CMS Books Math., **5**, Springer-Verlag, New York, 2001.
5. T. MOLIEN, *Über die Invarianten der linearen Substitutionsgruppen*, Berliner Sitzungsber., 1898, pp. 1152-1156.
6. C. NĂSTĂSESCU, F. VAN OYSTAEYEN, *Methods of graded rings*, Lecture Notes in Math., **1836**, Springer-Verlag, Berlin, Heidelberg, New York, 2004.
7. A. NOWICKI, M. NAGATA, *Rings of constants for k-derivations in* $k[x_1,...,x_n]$, J. Math. Kyoto Univ., **28**, pp. 111-118, 1988.
8. G.C. SHEPHARD, J.A. TODD, *Finite unitary reflection groups*, Canad. J. Math., **6**, pp. 274-304, 1954.
9. T.A. SPRINGER, *Invariant theory*, Lecture Notes in Math., **585**, Springer-Verlag, Berlin, Heidelberg, New York, 1977.
10. R.P. STANLEY, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (N.S.), **1**, pp. 475-511, 1979.

Received November 21, 2018