THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 21, Number 2/2020, pp. 105–114

ON A FRACTIONAL PROBLEM WITH VARIABLE EXPONENT

Mounir HSINI[1], Nawal IRZI[1], Khaled KEFI[1,2]

[1] University of Tunis El Manar, Faculty of Sciences of Tunis
[2] Northern Border University, Faculty of Computer Sciences and Information Technology of Rafha, Kingdom of Saudi Arabia
E-mails: mounir.hsini@ipeit.rnu.tn, nawalirzi15@gmail.com, khaled_kefi@yahoo.fr

Abstract. In this paper, we study the fractional $p(x)$-Laplacian problem with variable exponents

$$\begin{cases} (-\Delta)^s_{p(.)}u(x) + |u(x)|^{q(x)-2}u(x) = \lambda \frac{\partial F}{\partial u}(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Where $\Omega \subset \mathbb{R}^N$, $N > 2$ is a bounded smooth domain, $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ while λ is a positive parameter and q is a continuous function on $\overline{\Omega}$.

Key words: Fractional Laplacian, Sobolev space, Gagliardo seminorm, variational methods.

1. INTRODUCTION

Problem involving fractional Laplace operator has been given considerable attention since they are arises in many fields of sciences, notably the fields of physics, finance, electromagnetism, astronomy, fluid potentials and fluid dynamics, see [1, 3]. The result about the fractional Sobolev spaces with variable exponent and the fractional p(x)-Laplacian is obtained in [8, 11].

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. The aim of this work is to study the existence of solutions of the following nonlocal fractional p(x)-Laplacian problem

$$\begin{cases} \mathscr{L}u(x) + |u(x)|^{q(x)-2}u(x) = \lambda \dfrac{\partial F}{\partial u}(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

where $q \in C(\Omega)$ a continuous function, $F \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, $\lambda > 0$ and the operator $\mathscr{L}$ is given by

$$\mathscr{L}u(x) := (-\Delta)^s_{p(.)}u(x) = P.V. \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+s\cdot p(x,y)}}dy, \quad x \in \Omega,$$

where P.V. is a commonly used abbreviation in the principal value sense, $0 < s < 1$ and $p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)$ is a continuous function with $s \cdot p(x,y) < N$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. In order to have a well defined trace on $\partial\Omega$, for simplicity, we just restrict ourselves to $s \cdot p^- > 1$, as in the paper [8], where $p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y)$.

Next, we make the following assumptions

($\mathbf{H}_1$) $F:\overline{\Omega}\times\mathbb{R}\longrightarrow\mathbb{R}$ is homogeneous of degree r, that is,

$$F(x,tu)=t^{r(x)}F(x,u)\ \text{ for all }\ t>0, x\in\overline{\Omega},\ u\in\mathbb{R}.$$

($\mathbf{H}_2$) $\left|\frac{\partial F}{\partial t}(x,t)\right|\leq CV(x)|t|^{r(x)-2}t$, for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$, where C is a positive constant, $V\in L^{l(x)}(\Omega)$, $l,\ r\in C(\overline{\Omega})$ are such that for all $x\in\overline{\Omega}$, we have

$$1<r(x)<p(x,x)<\frac{N}{s}<l(x)\ \text{and}\ \ p(x,x)\leq q(x)<p^*(x):=\frac{Np(x,x)}{N-s\cdot p(x,x)}.$$

($\mathbf{H}_3$) There exists an $\Omega_0\subset\subset\Omega$ with $|\Omega_0|>0$ such that $F(x,t)>0$ for all $(x,t)\in\Omega_0\times\mathbb{R}^*$.

Remark 1. Due to assumption ($\mathbf{H_1}$), F leads to the so-called Euler identity,

$$t\frac{\partial F}{\partial t}(x,t)=r(x)F(x,t),\ \text{for all}\ t\in\mathbb{R}\ \text{ and }\ x\in\Omega. \tag{2}$$

The fractional $p(x)$-Laplacian operator was first introduced by Kaufmann, Rossi and Vidal in [8], in which they established a compact embedding result and they proved the existence and the uniqueness of weak solution for the following problem

$$\begin{cases}\mathscr{L}u(x)+|u(x)|^{q(x)-2}u(x)=\lambda f(x), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega.\end{cases}$$

Provided $f\in L^{\alpha(x)}(\Omega)$ for some $\alpha(x)>1$.

For a smooth bounded domain Ω in $\mathbb{R}^N$, we consider two continuous functions $p:\overline{\Omega}\times\overline{\Omega}\to(1,\infty)$ and $q:\overline{\Omega}\to\mathbb{R}$. We assume that p is symmetric and

$$1<p^-=\min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}p(x,y)\leq p(x,y)\leq p^+=\max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}p(x,y)<\infty,$$

and

$$1<q^-=\min_{x\in\overline{\Omega}}q(x)\leq q(x)\leq q^+=\max_{x\in\overline{\Omega}}q(x)<\infty.$$

For $s\in(0,1)$, the fractional Sobolev space with variable exponents via the Gagliardo approach $E=W^{s,q(x),p(x,y)}(\Omega)$ is defined as follows

$$E=\left\{u\in L^{q(x)}(\Omega),\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+s\cdot p(x,y)}}dxdy<\infty,\ \text{for some}\ \mu>0\right\}.$$

Let

$$[u]_{s,p(x,y)}=\inf\left\{\mu>0,\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+s\cdot p(x,y)}}dxdy<1\right\},$$

be the variable exponent Gagliardo seminorm and define

$$\|u\|_E=[u]_{s,p(x,y)}+|u|_{q(x)},$$

then $(E,\|\cdot\|_E)$ is a Banach space. We could get the following properties

LEMMA 1 (See Lemma 2.1 in [13]).
1. If $1 \leq [u]_{s,p(x,y)} < \infty$, *then*

$$([u]_{s,p(x,y)})^{p_-} \leq \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s\cdot p(x,y)}} dxdy \leq ([u]_{s,p(x,y)})^{p^+};$$

2. If $[u]_{s,p(x,y)} \leq 1$, *then*

$$([u]_{s,p(x,y)})^{p^+} \leq \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s\cdot p(x,y)}} dxdy \leq ([u]_{s,p(x,y)})^{p_-}.$$

We denote $E_0 = W_0^{s,q(x),p(x,y)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in E, then E_0 is a Banach space with the norm $\|\cdot\|_{E_0} = [u]_{s,p(x,y)}$. For $u \in W^{s,q(x),p(x,y)}(\Omega)$, we set

$$\rho(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s\cdot p(x,y)}} dxdy + \int_\Omega |u|^{q(x)} dx \tag{3}$$

and we have

$$\|u\|_E = \inf\Big\{\mu > 0,\ \rho(\frac{u}{\mu}) \leq 1\Big\}. \tag{4}$$

It's well known that $\|\cdot\|_E$ is a norm which is equivalent to the norm $\|\cdot\|_{W^{s,q(x),p(x,y)}(\Omega)}$.

Due to the Lemma 2.2 in [13], $(W^{s,q(x),p(x,y)}(\Omega), \|\cdot\|_E)$ is uniformly convex and $W^{s,q(x),p(x,y)}(\Omega)$ is a reflexive Banach space.

In the following Lemma, we give a compact embedding result into the variable exponent Lebesgue spaces. For the proof we refer to [8].

LEMMA 2. *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain and* $s \in (0,1)$. *Let* $q(x)$, $p(x,y)$ *be continuous variable exponents with* $s \cdot p(x,y) < N$ *for* $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ *and* $q(x) \geq p(x,x)$ *for* $x \in \overline{\Omega}$. *Assume that* $\gamma : \overline{\Omega} \longrightarrow (1,\infty)$ *is a continuous function such that*

$$p^*(x) = \frac{Np(x,x)}{N - s\cdot p(x,x)} > \gamma(x) \geq \gamma^- = \inf_{x\in\Omega} \gamma(x) > 1, \ \textit{for}\ x \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N,s,p,q,r,\Omega)$ *such that for every* $u \in W^{s,q(x),p(x,y)}$, *it holds that*

$$|u|_{\gamma(x)} \leq C\|u\|_E.$$

That is, the space $W^{s,q(x),p(x,y)}(\Omega)$ *is continuously embedded in* $L^{\gamma(x)}$. *Moreover, this embedding is compact. In addition, if* $u \in W_0^{s,q(x),p(x,y)}$, *the following inequality holds*

$$|u|_{\gamma(x)} \leq C\|u\|_{E_0}.$$

Remark 2. Suppose that $q(x) = \overline{p}(x,x) := p(x,x)$. Since

$$1 < p^- < \overline{p} < \frac{Np(x,x)}{N - s\cdot p(x,x)}.$$

It follows that $[u]_{s,p(x,y)}$ is a norm on $W_0^{s,q(x),p(x,y)}(\Omega)$, which is equivalent to the norm $\|\cdot\|_E$ on $W^{s,q(x),p(x,y)}$.

2. BASICS AND TERMINOLOGY

In this section we recall some definitions about the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Sobolev spaces $W^{k,q(x)}(\Omega)$, see [7, 9, 10, 12]. Set

$$C_+(\overline{\Omega}) = \Big\{h;\ \ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for any } x \in \overline{\Omega}\Big\}.$$

For $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ by

$$L^{q(x)}(\Omega) = \Big\{u;\ \ u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{q(x)}dx < \infty\Big\}.$$

The Luxemburg norm, on this space is given by the formula

$$|u|_{q(x)} = \inf\Big\{\alpha > 0,\ \int_\Omega \Big|\frac{u(x)}{\alpha}\Big|^{q(x)} dx \le 1\Big\}.$$

It's well known, that $(L^{q(x)}(\Omega); |\cdot|_{q(x)})$ is a is a separable, uniformly convex Banach space.

$(L^{q(x)}(\Omega); |\cdot|_{L^{q(x)}})$ is called a generalized Lebesgue space. Moreover, its conjugate space is $L^{q'(x)}(\Omega)$, where $\frac{1}{q'(x)} + \frac{1}{q(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, one has the following Hölder type inequality

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le \Big(\frac{1}{q^-} + \frac{1}{q'^-}\Big)|u|_{q(x)}|v|_{q'(x)} \le 2|u|_{q(x)}|v|_{q'(x)}. \tag{5}$$

An important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{q(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{q(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{q(x)}(u) = \int_\Omega |u|^{q(x)}dx.$$

If $(u_n), u \in L^{q(x)}(\Omega)$ and $q^+ < \infty$, then the following relations hold true.

$$|u|_{q(x)} > 1 \Rightarrow |u|^{q^-}_{q(x)} \le \rho_{q(x)}(u) \le |u|^{q^+}_{q(x)}. \tag{6}$$

$$|u|_{q(x)} < 1 \Rightarrow |u|^{q^+}_{q(x)} \le \rho_{q(x)}(u) \le |u|^{q^-}_{q(x)}. \tag{7}$$

$$|u_n - u|_{q(x)} \to 0 \text{ if and only if } \rho_{q(x)}(u_n - u) \to 0. \tag{8}$$

Another interesting property of the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ is the following

PROPOSITION 1 (see [5]) . *Let p and q be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x)q(x) \le \infty$, for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$. Then*

$$\min\Big(|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}\Big) \le ||u|^{p(x)}|_{q(x)} \le \max\Big(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)}\Big).$$

If k is a positive integer number and $q \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{k,q(x)}(\Omega) = \Big\{u \in L^{q(x)}(\Omega) : D^\alpha u \in L^{q(x)}(\Omega),\ \text{ for all } |\alpha| \le k\Big\}.$$

Here $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index, $|\alpha| = \Sigma_{i=1}^{N} \alpha_i$ and

$$D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{|\alpha_1|} \ldots \partial_{x_N}^{|\alpha_N|}}.$$

On $W^{k,q(x)}(\Omega)$ we consider the following norm

$$\|u\|_{k,q(x)} = \Sigma_{|\alpha| \le k} |D^{\alpha} u|_{p(x)}.$$

In this paper, we denote by $W_0^{k,q(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{k,q(x)}(\Omega)$.
Then $W^{k,q(x)}(\Omega)$ is a reflexive and separable Banach spaces. On the other hand if $q \in C_+(\overline{\Omega})$ satisfies $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, the imbedding from $W^{k,q(x)}(\Omega)$ into $L^{p(x)}(\Omega)$ is compact and continuous.

LEMMA 3 (see [2]). *For all $u, v \in E_0$, we consider the following functional $I : E_0 \to E_0^*$ such that*

$$\langle I(u), v \rangle = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + s \cdot p(x,y)}} dxdy$$

then

1. *I is a bounded and strictly monotone operator.*
2. *I is a mapping of type (S_+), that is, if $u_n \rightharpoonup u \in E_0$ and $\limsup_{n \to +\infty} I(u_n)(u_n - u) \le 0$, then $u_n \to u \in E_0$.*
3. *$I : E_0 \to E_0^*$ is a homeomorphism.*

3. MAIN RESULTS

Throughout the paper, the letters $C, C_i, i = 1, 2, ...$ denote positive constants which may change from line to line. In the sequel, denote by $l'(x)$ the conjugate exponent of the function $l(x)$ and put $\alpha(x) := \dfrac{l(x)r(x)}{l(x) - r(x)}$ for any $x \in \Omega$, we have:

Remark 3. Assume assumption $(\mathbf{H_2})$ is fulfilled, then $l'(x)r(x) < p^*(x)$, $\alpha(x) < p^*(x)$ for all $x \in \Omega$, so the embedding $E_0 \hookrightarrow L^{l'(x)r(x)}(\Omega)$ and $E_0 \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

Definition 1. *We say that $u \in W_0^{s,q(x),p(x,y)}$ is a weak solution of (1) if*

$$\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + s \cdot p(x,y)}} dxdy + \int_{\Omega} |u|^{q(x)-2} u(x) v(x) \, \mathrm{d}x$$
$$= \lambda \int_{\Omega} \frac{\partial F}{\partial u}(x, u) v(x) \, \mathrm{d}x,$$

for every $v \in W_0^{s,q(x),p(x,y)}$.

The first result in this paper is the following.

THEOREM 1. *Assume hypothesis $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$ are fulfilled. Then, there exists $\lambda^* > 0$, such that for any $\lambda \in (0, \lambda^*)$ problem (1) has a weak solution.*

In the second, we establish that the Euler-Lagrange functional associated to problem (1), has a global minimizer.

THEOREM 2. *Assume that the hypothesis* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *are fulfilled. Then for any* $\lambda > 0$ *problem (1) has a weak solution.*

In order to formulate the variational approach of problem (1), we introduce the Euler Lagrange functional $\Psi_\lambda : E_0 \to \mathbb{R}$, defined by

$$\Psi_\lambda(u) := \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s\cdot p(x,y)}}dxdy + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \lambda\int_\Omega F(x,u)dx.$$

Standard arguments showed that $\Psi_\lambda \in C^1(E_0,\mathbb{R})$ and for any $v\in E_0$

$$\begin{aligned}\langle \Psi'_\lambda(u), v\rangle &= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s\cdot p(x,y)}}dxdy \\ &\quad + \int_\Omega |u|^{q(x)-2}u(x)v(x)\,\mathrm{d}x - \lambda\int_\Omega \frac{\partial F}{\partial u}(x,u)v(x)\,\mathrm{d}x.\end{aligned}$$

We start with the following auxiliary property.

LEMMA 4. *Suppose we are under hypotheses of Theorem 1. Then for all* $\rho \in (0,1)$, *there exist* $\lambda^* > 0$ *and* $b > 0$ *such that for all* $u \in E_0$ *with* $\|u\|_{E_0} = \rho$

$$\Psi_\lambda(u) \geq b > 0 \qquad \text{for all} \qquad \lambda \in (0,\lambda^*).$$

Proof. Since the embedding $E_0 \hookrightarrow L^{l'(x)r(x)}(\Omega)$ is continuous, then

$$|u|_{l'(x)r(x)} \leq C_1\|u\|_{E_0}, \text{ for all } u\in E_0. \tag{9}$$

Let us assume that $\|u\|_{E_0} < \min(1, 1/C_1)$, where C_1 is positive constant of(9). Then, we have $|u|_{l'(x)r(x)} < 1$, using Hölder inequality (5), Proposition 1, remark 1 and inequality (9), we deduce that for any $u\in E_0$ with $\|u\|_{E_0} = \rho$, we obtain

$$\begin{aligned}\Psi_\lambda(u) &= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s\cdot p(x,y)}}dxdy + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \lambda\int_\Omega F(x,u)dx \\ &\geq \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s\cdot p(x,y)}}dxdy - \lambda C|V|_{l(x)}||u|^{r(x)}|_{l'(x)} \\ &\geq \frac{1}{p^+}\|u\|_{E_0}^{p^+} - \lambda C|V|_{l(x)}|u|^{r^-}_{l'(x)r(x)} \\ &\geq \frac{1}{p^+}\|u\|_{E_0}^{p^+} - \lambda C|V|_{l(x)}C_1^{r^-}\|u\|_{E_0}^{r^-} \\ &\geq \frac{1}{p^+}\rho^{p^+} - \lambda C|V|_{l(x)}C_1^{r^-}\rho^{r^-} = \rho^{r^-}\left(\frac{1}{p^+}\rho^{p^+-r^-} - \lambda C|V|_{l(x)}C_1^{r^-}\right).\end{aligned}$$

By the above inequality, we remark that if we define

$$\lambda^* = \frac{\rho^{p^+-r^-}}{2p^+C|V|_{l(x)}C_1^{r^-}}, \tag{10}$$

then for any $\lambda\in(0,\lambda^*)$ and $u\in E_0$ with $\|u\|_{E_0} = \rho$ there exists $b = \dfrac{\rho^{p^+}}{2p^+} > 0$ such that $\Psi_\lambda(u) \geq b > 0$. The proof of Lemma 4 is complete. □

The following result asserts the existence of a valley for Ψ_λ near the origin.

LEMMA 5. *There exists $\phi \in E_0$ such that $\phi \geq 0$, $\phi \neq 0$ and $\Psi_\lambda(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Since the embedding $E_0 \hookrightarrow L^{q(x)}(\Omega)$ is continuous, then

$$|u|_{q(x)} \leq c\|u\|_{E_0}, \text{ for all } u \in E_0. \tag{11}$$

Moreover, assumption $(\mathbf{H_2})$ implies $r(x) < p(x,x)$, for all $x \in \Omega_0$. In the sequel, denote by $r_0^- = \inf_{\Omega_0} r(x)$, $p_0^- = \inf_{\Omega_0\times\Omega_0} p(x,y)$. and $q_0^- = \inf_{\Omega_0} q(x)$. Let ε_0 such that $r_0^- + \varepsilon_0 < p_0^-$. On the other hand, since $r \in C(\overline{\Omega_0})$, there exists an open set $B_0 \subset\subset \Omega_0$ such that $|r(x) - r_0^-| < \varepsilon_0$, for all $x \in B_0$. It follows that $r(x) \leq r_0^- + \varepsilon_0 < p_0^-$, for all $x \in B_0$.
Let $\phi \in C_0^\infty(\Omega)$ such that $supp(\phi) \subset B_0 \subset \Omega_0$, $\phi = 1$ in a subset $\Omega_1 \subset supp(\phi)$, $0 \leq \phi \leq 1$ in B_0. we obtain

$$\begin{aligned}
\Psi_\lambda(t\phi) &= \int_{\Omega\times\Omega} \frac{|t\phi(x) - t\phi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s\cdot p(x,y)}}dxdy + \int_\Omega \frac{|t\phi|^{q(x)}}{q(x)}dx - \lambda\int_\Omega F(x,t\phi)dx \\
&\leq \frac{t^{p_0^-}}{p_0^-}\int_{\Omega\times\Omega} \frac{|\phi(x) - \phi(y)|^{p(x,y)}}{|x-y|^{N+s\cdot p(x,y)}}dxdy + \frac{t^{q_0^-}}{q_0^-}\int_\Omega |\phi|^{q(x)}dx - \lambda\int_{B_0} t^{r(x)}F(x,\phi)dx \\
&\leq \frac{t^{p_0^-}}{p_0^-}\left[\max(\|\phi\|_{E_0}^{p_0^+}, \|\phi\|_{E_0}^{p_0^-}) + \max(c^{q_0^+}\|\phi\|_{E_0}^{q_0^+}, c^{q_0^-}\|\phi\|_{E_0}^{q_0^-})\right] - \lambda t^{r_0^- + \varepsilon_0}\int_{B_0} F(x,\phi)dx.
\end{aligned}$$

Therefore $\Psi_\lambda(t\phi) < 0$, for $t < \delta^{1/\left(p_0^- - r_0^- - \varepsilon_0\right)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda p_0^- \int_{B_0} F(x,\phi)dx}{\max(\|\phi\|_{E_0}^{p_0^+}, \|\phi\|_{E_0}^{p_0^-}) + \max(c^{q_0^+}\|\phi\|_{E_0}^{q_0^+}, c^{q_0^-}\|\phi\|_{E_0}^{q_0^-})}\right\}.$$

Since $\phi = 1$ in Ω_1, then $\|\phi\|_{E_0} > 0$, so the proof of Lemma 5 is complete. □

Proof of Theorem 1 completed. Let $\lambda^* > 0$ be defined as in (10) and $\lambda \in (0,\lambda^*)$. By Lemma 4 it follows that on the boundary of the ball centered at the origin and of radius ρ in E_0, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Psi_\lambda > 0. \tag{12}$$

On the other hand, by Lemma 5, there exists $\phi \in E_0$ such that $\Psi_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, using Hölder inequality (5), Proposition 1 and inequality (9), we deduce that for any $u \in B_\rho(0)$ we have

$$\Psi_\lambda(u) \geq \frac{1}{p^+}\|u\|_{E_0}^{p^+} - \lambda C|V|_{l(x)}|u|_{l'(x)r(x)}^{r^-}.$$

It follows that $-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} \Psi_\lambda < 0$. Let $0 < \varepsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Using the above information, the functional $\Psi_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle, there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Psi_\lambda(u_\varepsilon) \leq \underline{c} + \varepsilon \\ \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\varepsilon) + \varepsilon\cdot \| u - u_\varepsilon \|_{E_0}, \quad u \neq u_\varepsilon. \end{cases}$$

Since $\Psi_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} \Psi_\lambda + \varepsilon \leq \inf_{B_\rho(0)} \Psi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda$, we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $\xi_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $\xi_\lambda(u) = \Psi_\lambda(u) + \varepsilon \cdot \parallel u - u_\varepsilon \parallel_{E_0}$. It is clear that u_ε is a minimum point of ξ_λ and thus

$$\frac{\xi_\lambda(u_\varepsilon + t\cdot v) - \xi_\lambda(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\varepsilon + t\cdot v) - \Psi_\lambda(u_\varepsilon)}{t} + \varepsilon \cdot \parallel v \parallel_{E_0} \geq 0.$$

Letting $t \to 0$ it follows that $\langle \Psi'_\lambda(u_\varepsilon), v\rangle + \varepsilon \cdot \parallel v \parallel_{E_0} \geq 0$, we infer that $\parallel \Psi'_\lambda(u_\varepsilon) \parallel_{E_0} \leq \varepsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(w_n) \longrightarrow \underline{c} < 0 \quad \text{and} \quad \Psi'_\lambda(w_n) \longrightarrow 0_{E_0^*}. \tag{13}$$

It is clear that $\{w_n\}$ is bounded in E_0. Thus, there exists w in E_0 such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E_0. Since $q(x), \alpha(x) < p^*(x)$ (see remark 3), we deduce that there exists a compact embedding $E_0 \hookrightarrow L^{q(x)}(\Omega)$ and $E_0 \hookrightarrow L^{\alpha(x)}(\Omega)$. Furthermore, we have

$$\{w_n\} \to w \text{ stongly in } L^{q(x)}(\Omega) \text{ as } n \to \infty$$

and

$$\{w_n\} \to w \text{ stongly in } L^{\alpha(x)}(\Omega) \text{ as } n \to \infty.$$

For the strong convergence of $\{w_n\}$ in E_0, we need the following proposition.

PROPOSITION 2.

i. $\lim_{n\to\infty} \int_\Omega \frac{\partial F}{\partial u}(x, w_n)(w_n - w)dx = 0.$

ii. $\lim_{n\to\infty} \int_\Omega |w_n|^{q(x)-2} w_n(w_n - w)dx = 0.$

Proof.

i. Using Hölder inequality (5) we have

$$\begin{aligned} \int_\Omega |\frac{\partial F}{\partial u}(x, w_n)|(w_n - w)dx &\leq C_1 |V|_{l(x)} ||w_n|^{r(x)-2} w_n(w_n - w)|_{l'(x)} \\ &\leq C_1 |V|_{L^{l(x)}(\Omega)} ||w_n|^{r(x)-2} w_n|_{\frac{r(x)}{r(x)-1}} |w_n - w|_{\alpha(x)}. \end{aligned}$$

Now if $||w_n|^{r(x)-2} w_n|_{\frac{r(x)}{r(x)-1}} > 1$, by Proposition 1, we get $||w_n|^{r(x)-2} w_n|_{\frac{r(x)}{r(x)-1}} \leq |w_n|^{r^+}_{r(x)}$. The compact imbedding $E \hookrightarrow L^{r(x)}(\Omega)$ ends the proof.

ii. Using Hölder inequality (5), one has

$$\int_\Omega |w_n|^{q(x)-2} w_n(w_n - w)dx \leq ||w_n|^{q(x)-2} w_n|_{\frac{q(x)}{q(x)-1}} |w_n - w|_{q(x)}.$$

Now if $||w_n|^{q(x)-2} w_n|_{\frac{q(x)}{q(x)-1}} > 1$, by Proposition 1, we get $||w_n|^{q(x)-2} w_n|_{\frac{q(x)}{q(x)-1}} \leq |w_n|^{q^+}_{q(x)}$. The compact imbedding $E_0 \hookrightarrow L^{q(x)}(\Omega)$ ends the proof.

$\square$

Since $\Psi'_\lambda(w_n) \to 0$ and w_n is bounded in E_0 we have

$$\begin{aligned} |\langle \Psi'_\lambda(w_n), w_n - w\rangle| &\leq |\langle \Psi'_\lambda(w_n), w_n\rangle| + |\langle \Psi'_\lambda(w_n), w\rangle| \\ &\leq \|\Psi'_{\lambda,}(w_n)\|_{E_0^*}\|w_n\|_{E_0} + \|\Psi'_\lambda(w_n)\|_{E_0^*}\|w\|_{E_0}. \end{aligned}$$

Moreover, using Proposition 1, we get

$$\lim_{n\to\infty}\langle \Psi'_\lambda(w_n), w_n - w\rangle = 0.$$

There for,

$$\lim_{n\to\infty} I(w_n)(w_n - w) = 0.$$

Where I was defined in Lemma 3 which ensures that $\{w_n\}$ converges strongly to w in E_0. Since $\Psi_\lambda \in C^1(E_0, \mathbb{R})$, we conclude

$$\Psi'_\lambda(w_n) \to \Psi'_\lambda(w), \quad \text{as} \quad n \to \infty. \tag{14}$$

Relations (13) and (14) show that $\Psi'_\lambda(w) = 0$ and thus w is a weak solution for problem (1). Moreover, by relation (13), it follows that $\Psi_\lambda(w) < 0$ and thus, w is a nontrivial weak solution for (1). The proof of Theorem 1 is complete.

Proof of Theorem 2. Using Hölder inequality (5) for $\|u\|_{E_0} > 1$, we have

$$\begin{aligned} \Psi_\lambda(u) &= \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s\cdot p(x,y)}}dxdy + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \lambda\int_\Omega F(x,u)dx \\ &\geq \frac{1}{p^+}\|u\|_{E_0}^{p^-} - \lambda \max\Big(C|V|_{l(x)}|u|^{r^-}_{l'(x)r(x)}, C|V|_{l(x)}|u|^{r^+}_{l'(x)r(x)}\Big) \\ &\geq \frac{1}{p^+}\|u\|_{E_0}^{p^-} - \lambda \max\Big(C|V|_{l(x)}C_1^{r^-}\|u\|^{r^-}_{E_0}, C|V|_{l(x)}C_1^{r^+}\|u\|^{r^+}_{E_0}\Big). \end{aligned}$$

Since $r^+ < p^-$, so $\Psi_\lambda(u) \to +\infty$, as $\|u\|_{E_0} \to +\infty$. As a conclusion, since Ψ_λ is weakly lower semi-continuous then it has a global minimizer which is solution of problem (1), moreover Lemma 5 ensures that this minimizer is nontrivial, which ends the proof.

ACKNOWLEDGEMENTS

The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

REFERENCES

1. D. APPLEBAUM, *Lévy processes and stochastic calculus*, 2nd edition, Camb. Stud. Adv. Math. 116, Cambridge University Press, 2009.
2. A. BAHROUNI, V.D. RADULESCU, *On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent*, Discrete Contin. Dyn. Syst. Ser. S., **11**, *3*, pp. 379–389, 2018.
3. R. CONT, P. TANKOV, *Financial modelling with jump processes*, Chapman and Hall/CRC Financ. Math. Ser., Chapman and Hall/CRC, Boca Raton, FL, 2004.

4. P. DRABEK, S.I. POHOZAEV, *Positive solutions for the p-Laplacian: application of the fibering method*, Proc. Roy. Soc. Edinburgh Sect. A., **127**, pp. 703–726, 1997.
5. D.E. EDMUNDS, J. RÁKOSNIK, *Sobolev embeddings with variable exponent*, Studia Math., **143**, pp. 267–293, 2000.
6. X.L. FAN, D. ZHAO, *On the generalized Orlicz-Sobolev space* $W^{k,p(x)}(\Omega)$, J. Gansu Edu. College, **12**, pp. 1–6, 1998.
7. M. HSINI, N. IRZI, K. KEFI, *Nonhomogeneous* $p(x)$*-Laplacian Steklov problem with weights*, Complex Variables and Elliptic Equations, **65**, *3*, pp. 440–454, 2020.
8. U. KAUFMANN, J.D. ROSSI, R. VIDAL, *Fractional Sobolev spaces with variable exponents and fractional* $p(x)$*-Laplacians*, Electron. J. Qual. Theory Differ. Equ., **76**, pp. 1–10, 2017.
9. O. KOVÁCIK, J. RÁKBOSNIK, *On spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$, Czechoslovak Math. J., **41**, pp. 592–618, 1991.
10. J. MUSIELAK, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., **1034**, Springer, Berlin, 1983.
11. L.M. PEZZO, J.D. ROSSI, *Traces for fractional Sobolev spaces with variable exponents*, Adv. Oper. Theory, **2**, pp. 435–446, 2017.
12. D. ZHAO, W.J. QIANG, X.L. FAN, *On generalized Orlicz spaces* $L^{p(x)}(\Omega)$, J. Gansu Sci., **9**, *2*, pp. 1–7 ,1997.
13. C. ZHANG, X. ZHANG, *Renormalized solutions for the fractional* $p(x)$*-Laplacian equation with* L^1 *data*, Nonlinear Analysis, **190**, p. 111610, 2020.

Received January 8, 2019