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Binding numbers for fractional (a,b,k)-critical covered graphs

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Abstract: A graph G is said to be fractional (a,b,k)-critical covered if after deleting any k vertices of G, the remaining graph of G is fractional $[a,b]$-covered. In this article, we gain a binding number condition for a graph to be fractional (a,b,k)-critical covered, which is an improvement and extension of Yuan and Hao's previous result [Y. Yuan and R. Hao, Neighborhood union conditions for fractional $[a,b]$-covered graphs, Bull. Malays. Math. Sci. Soc., 43, pp. 157167, 2020, https://doi.org/10.1007/s40840-018-0669-y].

Key words: graph; binding number; fractional $[a,b]$-factor; fractional $[a,b]$-covered graph; fractional (a,b,k)-critical covered graph.

1. INTRODUCTION

We discussed only finite, undirected and simple graphs. For a graph G, the vertex set of G is denoted by $V(G)$, and the edge set of G is denoted by $E(G)$. For a vertex v of G, we use $d_G(v)$ for the degree of v in G, and $N_G(v)$ for the set of vertices adjacent to v in G. Let $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. Let X be a vertex subset of G. We write $G[X]$ for the subgraph of G induced by X, and $G-X = G[V(G)\setminus X]$. A subset $X \subseteq V(G)$ is called independent if $G[X]$ does not possess edges. Let $N_G(X) = \bigcup_{v\in X} N_G(v)$. The binding number of G is defined by

$$bind(G) = min\Big\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\Big\}.$$

Let a and b be two integers with $0 \le a \le b$. Then an $[a,b]$-factor of G is a spanning subgraph F of G satisfying $a \le d_F(x) \le b$ for any $x \in V(G)$. An r-factor is an $[r,r]$-factor. A fractional $[a,b]$-factor of G is a function h from $E(G)$ to $[0,1]$ such that $a \le d_G^h(v) \le b$ for each $v \in V(G)$, where $d_G^h(v) = \sum_{e\in E(v)} h(e)$ and $E(v)$ is the set of edges incident with v. A fractional r-factor is a fractional $[r,r]$-factor. A graph G is said to be fractional $[a,b]$-covered if for any $e \in E(G)$, G possesses a fractional $[a,b]$-factor h such that $h(e) = 1$. A fractional $[a,b]$-covered graph is called a fractional r-covered graph when $a = b = r$. For a nonnegative integer k, a graph G is said to be fractional (a,b,k)-critical covered if after deleting any k vertices, the obtained graph is fractional $[a,b]$-covered. A fractional (r,k)-critical covered graph is a fractional (r,r,k)-critical covered graph.

There are rich results on the problems of factors and fractional factors in graphs [1–22]. Furthermore, a lot of results on the relationship between binding number and factors as well as fractional factors in graphs are derived. For example, Katerinis and Woodall [23] pointed out a binding number condition for the existence of r-factors in graphs; Kano and Tokushige [24] demonstrated a result on the relationship between the binding number and the existence of f-factors in graphs; Zhou and Sun [25,26] derived some binding number conditions for graphs to possess [1,2]-factors with given properties; Chen [27] put forward a binding number condition for

$[a,b]$-factor; Zhou, Xu and Duan [28] studied the relationship between the binding number and the existence of fractional r-factors in graphs; Yuan and Hao [29] gained a binding number condition for graphs being fractional r-covered graphs.

The following results on fractional r-factors and fractional r-covered graphs depending on binding numbers are known.

THEOREM 1 [28]. *Let $r \geq 2$ be an integer, and let G be a graph of order n such that $n \geq 4r-6$. Then (1) G possesses a fractional r-factor if rn is even and $bind(G) > \frac{(2r-1)(n-1)}{r(n-2)+3}$; (2) G possesses a fractional r-factor if rn is odd and $bind(G) > \frac{(2r-1)(n-1)}{r(n-2)+2}$.*

THEOREM 2 [29]. *Let r be an integer with $r \geq 2$, and let G be a graph of order n satisfying $\delta(G) \geq r+1$ and $n > 4r+1$. Then G is fractional r-covered if $bind(G) > \frac{(2r-1)(n-1)}{r(n-2)}$.*

In this article, we generalise Theorems 1 and 2 to fractional (a,b,k)-critical covered graphs, and point out the relationship between binding numbers and fractional (a,b,k)-critical covered graphs. Furthermore, our main result is claimed as follows.

THEOREM 3. *Let a, b and k be nonnegative integers with $a \geq 1$ and $b \geq \max\{a,2\}$, and let G be a graph of order n with $n \geq \frac{(a+b-1)(a+b-2)+1}{b} + \frac{bk}{b-1}$. Then G is fractional (a,b,k)-critical covered if $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$.*

We easily gain the following result when $k = 0$ in Theorem 3.

COROLLARY 1. *Let a, b be integers with $a \geq 1$ and $b \geq \max\{a,2\}$, and let G be a graph of order n with $n \geq \frac{(a+b-1)(a+b-2)+1}{b}$. Then G is fractional $[a,b]$-covered if $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b}$.*

If $a = b = r$ in Corollary 1, then we possess the following result.

COROLLARY 2. *Let r be an integer with $r \geq 2$, and let G be a graph* of order n with $n \geq 4r-6+\frac{3}{r}$. Then G is fractional r-covered if $bind(G) > \frac{(2r-1)(n-1)}{rn-2r}$.

We easily see that the result of Corollary 2 is stronger that one of Theorem 2. Hence, our main result (Theorem 3) is an improvement and generalization of Yuan and Hao's result (Theorem 2). If $a = b = r$ in Theorem 3, then we deduce the following corollary.

COROLLARY 3. *Let r and k be nonnegative integers with $r \geq 2$, and let G be a graph of order n with $n \geq \frac{(2r-1)(2r-2)+1}{r} + \frac{rk}{r-1}$. Then G is fractional (r,k)-critical covered if $bind(G) > \frac{(2r-1)(n-1)}{rn-2r-rk}$.*

2. THE PROOF OF THEOREM 3

Li, Yan and Zhang [30] acquired a criterion for graphs being fractional $[a,b]$-covered, which plays an important role in the proof of Theorem 3.

LEMMA 1 [30]. *Let a and b be two integers with $b \geq a \geq 0$. Then a graph G is fractional $[a,b]$-covered if and only if*

$$\gamma_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geq \varepsilon(S,T)$$

for any subset S of $V(G)$, where $T = \{t : t \in V(G) \setminus S, d_{G-S}(t) \leq a\}$ and $\varepsilon(S,T)$ is defined by

$$\varepsilon(S,T) = \begin{cases} 2, & if\ S\ is\ not\ independent, \\ 1, & if\ S\ is\ independent,\ and\ there\ is\ an\ edge\ joining\ V(G) \setminus (S \cup T)\ and\ S,\ or \\ & there\ is\ an\ edge\ e = uv\ joining\ T\ and\ S\ such\ that\ d_{G-S}(v) = a\ for\ v \in T, \\ 0, & otherwise. \end{cases}$$

LEMMA 2 [31]. *Let G be a graph of order n, and let c be a positive real number. If $bind(G) > c$, then $\delta(G) > n - \frac{n-1}{c}$.*

Proof of Theorem 3. Let $Q \subseteq V(G)$ with $|Q| = k$. Setting $H = G - Q$. It suffices to demonstrate that H is fractional $[a,b]$-covered. Suppose, to the contrary, that H is not fractional $[a,b]$-covered. Then it follows from Lemma 1 that

$$\gamma_H(S,T) = b|S| + d_{H-S}(T) - a|T| \leq \varepsilon(S,T) - 1 \tag{1}$$

for some subset S of $V(H)$, where $T = \{u : u \in V(H) \setminus S, d_{H-S}(u) \leq a\}$.

CLAIM 1. $\delta(H) \geq a+1$.

Proof. Using Lemma 2, $n \geq \frac{(a+b-1)(a+b-2)+1}{b} + \frac{bk}{b-1}$ and $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$, we deduce

$$\begin{aligned}
\delta(G) &> n - \frac{n-1}{\frac{(a+b-1)(n-1)}{bn-a-b-bk}} = \frac{(a-1)n+a+b+bk}{a+b-1} \\
&\geq \frac{(a-1)\left(\frac{(a+b-1)(a+b-2)+1}{b} + \frac{bk}{b-1}\right) + a+b+bk}{a+b-1} \\
&\geq \frac{(a-1)\left(\frac{(a+b-1)(a+b-2)+1}{b} + k\right) + a+b+bk}{a+b-1} \\
&= \frac{(a-1)(a+b-2)}{b} + k + \frac{\frac{a-1}{b} + a+b}{a+b-1} \\
&\geq \frac{(a-1)(a+b-2)}{b} + k + \frac{a+b}{a+b-1}.
\end{aligned}$$

If $a = 1$, then $\delta(G) > k + \frac{1+b}{b} > 1+k$. By the integrity of $\delta(G)$, $\delta(G) \geq 2+k = a+k+1$.

If $a \geq 2$, then $\delta(G) > \frac{(a-1)(a+b-2)}{b} + k + \frac{a+b}{a+b-1} \geq (a-1)+k+1+\frac{1}{a+b-1} > a+k$. Applying the integrity of $\delta(G)$, $\delta(G) \geq a+k+1$.

Hence, we gain that $\delta(G) \geq a+k+1$. Combining this with $H = G - Q$ and $|Q| = k$, we possess that $\delta(H) \geq a+1$. Claim 1 is proved. □

CLAIM 2. $|S| \geq 2$.

Proof. If $|S| = 0$, then it follows from (1), $\varepsilon(S,T) = 0$ and Claim 1 that $-1 = \varepsilon(S,T) - 1 \geq \gamma_H(S,T) = d_H(T) - a|T| \geq (\delta(H) - a)|T| \geq |T| \geq 0$, a contradiction.

If $|S| = 1$, then by (1), $\varepsilon(S,T) \leq 1$ and Claim 1, we find

$$\begin{aligned}
0 &= \varepsilon(S,T) - 1 \geq \gamma_H(S,T) = b|S| + d_{H-S}(T) - a|T| \\
&\geq b|S| + d_H(T) - |T| - a|T| \geq b|S| + \delta(H)|T| - |T| - a|T| \\
&= b|S| + (\delta(H) - a - 1)|T| \geq b|S| = b \geq 2,
\end{aligned}$$

this is a contradiction. Hence, $|S| \geq 2$. We verify Claim 2. □

CLAIM 3. $T \neq \emptyset$.

Proof. Assume that $T = \emptyset$. Using (1), $\varepsilon(S,T) \leq 2$ and Claim 2, we deduce $\varepsilon(S,T) - 1 \geq \gamma_H(S,T) = b|S| \geq |S| \geq 2 \geq \varepsilon(S,T)$, which is a contradiction. Claim 3 is justified. □

Note that $T \neq \emptyset$ by Claim 3. Therefore, we may define $d = \min\{d_{H-S}(t) : t \in T\}$. In light of the definition of T, we derive $0 \leq d \leq a$.

Case 1. $d = 0$.
CLAIM 4. $\frac{bn-a-b-bk}{n-1} \geq 1$.
Proof. Note that $n \geq \frac{(a+b-1)(a+b-2)+1}{b} + \frac{bk}{b-1}$, $a \geq 1$ and $b \geq \max\{a, 2\}$. Thus, we verify that

$$\begin{aligned} & bn-a-b-bk-(n-1) = (b-1)n-a-b-bk+1 \\ \geq\ & (b-1)\Big(\frac{(a+b-1)(a+b-2)+1}{b} + \frac{bk}{b-1}\Big) - a-b-bk+1 \\ =\ & \frac{(b-1)(a+b-1)(a+b-2)}{b} + \frac{b-1}{b} - a - b + 1 \\ \geq\ & (b-1)(a+b-2) - (a+b-2) - \frac{1}{b} \\ =\ & (b-2)(a+b-2) - \frac{1}{b} \geq -\frac{1}{b} > -1. \end{aligned}$$

Note that $bn-a-b-bk-(n-1)$ is an integer. Hence, we gain $bn-a-b-bk-(n-1) \geq 0$, that is,

$$\frac{bn-a-b-bk}{n-1} \geq 1.$$

Claim 4 is demonstrated. □

Let $\beta = |\{t \in T : d_{H-S}(t) = 0\}|$. Obviously, $\beta \geq 1$ by $d = 0$. Writing $W = V(H) \setminus S = V(G) \setminus (S \cup Q)$, we deduce that $W \neq \emptyset$ and $|N_G(W)| \leq n - \beta$. Combining these with the definition of $bind(G)$ and $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$, we gain that

$$n - \beta \geq |N_G(W)| \geq bind(G)|W| > \frac{(a+b-1)(n-1)}{bn-a-b-bk}(n-k-|S|),$$

namely,

$$|S| > n - k - \frac{(n-\beta)(bn-a-b-bk)}{(a+b-1)(n-1)}. \tag{2}$$

Note that $d_{H-S}(T) \geq |T| - \beta$. By (2), $|S| + |T| + |Q| = |S| + |T| + k \leq n$, $\varepsilon(S,T) \leq 2$ and Claim 4, we gain

$$\begin{aligned} \gamma_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \geq b|S| + |T| - \beta - a|T| \\ &= b|S| - (a-1)|T| - \beta \geq b|S| - (a-1)(n-k-|S|) - \beta \\ &= (a+b-1)|S| - (a-1)(n-k) - \beta \\ &> (a+b-1)\Big(n-k-\frac{(n-\beta)(bn-a-b-bk)}{(a+b-1)(n-1)}\Big) - (a-1)(n-k) - \beta \\ &\geq (a+b-1)\Big(n-k-\frac{(n-1)(bn-a-b-bk)}{(a+b-1)(n-1)}\Big) - (a-1)(n-k) - 1 \\ &= a+b-1 \geq b \geq 2 \geq \varepsilon(S,T), \end{aligned}$$

contradicting (1).

Case 2. $1 \leq d \leq a$.
Select $t_1 \in T$ satisfying $d_{H-S}(t_1) = d$. Let $X = (V(H) \setminus S) \setminus N_{H-S}(t_1)$. Visibly, $t_1 \in X$ and $t_1 \notin N_G(X)$. Hence, $X \neq \emptyset$ and $N_G(X) \neq V(G)$. From the definition of $bind(G)$, $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$ and $|V(H)| = |V(G)| - k = n - k$, we derive

$$n - 1 \geq |N_G(X)| \geq bind(G)|X| > \frac{(a+b-1)(n-1)}{bn-a-b-bk} \cdot (n-k-|S|-d),$$

namely,

$$|S| > n-k-d-\frac{bn-a-b-bk}{a+b-1}. \quad (3)$$

It follows from (1), (3), $|S|+|T|+k \le n$ and $\varepsilon(S,T) \le 2$ that

$$\begin{aligned}
1 &\ge \varepsilon(S,T)-1 \ge \gamma_H(S,T) = b|S|+d_{H-S}(T)-a|T| \\
&\ge b|S|+d|T|-a|T| = b|S|-(a-d)|T| \\
&\ge b|S|-(a-d)(n-k-|S|) \\
&= (a+b-d)|S|-(a-d)(n-k) \\
&> (a+b-d)\Big(n-k-d-\frac{bn-a-b-bk}{a+b-1}\Big)-(a-d)(n-k),
\end{aligned}$$

i.e.,

$$1 > (a+b-d)\Big(n-k-d-\frac{bn-a-b-bk}{a+b-1}\Big)-(a-d)(n-k). \quad (4)$$

Subcase 2.1. $d=1$.
By applying (4), we obtain

$$1 > (a+b-1)\Big(n-k-1-\frac{bn-a-b-bk}{a+b-1}\Big)-(a-1)(n-k) = 1,$$

it is a contradiction.

Subcase 2.2. $2 \le d \le a$.
Let $f(d) = (a+b-d)(n-k-d-\frac{bn-a-b-bk}{a+b-1})-(a-d)(n-k)$. Then in view of $n \ge \frac{(a+b-1)(a+b-2)+1}{b}+\frac{bk}{b-1}$ and $2 \le d \le a$, we have

$$\begin{aligned}
f'(d) &= -n+k+d+\frac{bn-a-b-bk}{a+b-1}-a-b+d+n-k \\
&= \frac{bn-a-b-bk}{a+b-1}+2d-a-b \\
&\ge \frac{(a+b-1)(a+b-2)+1+bk-a-b-bk}{a+b-1}+2d-a-b \\
&= 2d-3 \ge 1 > 0.
\end{aligned}$$

Thus, we deduce that $f(d) \ge f(2)$. Using (4) and $n \ge \frac{(a+b-1)(a+b-2)+1}{b}+\frac{bk}{b-1}$, we gain

$$\begin{aligned}
1 &> f(d) \ge f(2) = (a+b-2)\Big(n-k-2-\frac{bn-a-b-bk}{a+b-1}\Big)-(a-2)(n-k) \\
&= \frac{b(n-k)-(a+b-2)^2}{a+b-1} \ge \frac{(a+b-1)(a+b-2)+1-(a+b-2)^2}{a+b-1} = 1,
\end{aligned}$$

a contradiction. Finishing the proof of Theorem 3. □

3. REMARK

We now claim that $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$ in Theorem 3 is sharp, namely, we construct a graph which shows that we cannot replace $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$ by $bind(G) \ge \frac{(a+b-1)(n-1)}{bn-a-b-bk}$ in Theorem 3.

Let a,b,k be nonnegative integers with $2 \le a \le b$ and b being odd, l be any enough large positive integer with $\frac{2(a-1)l+1}{b}$ being an integer, and $G = K_{\frac{2(a-1)l+1}{b}+k} \vee (lK_2)$ be a graph of order n, where $\vee$ means “join”. Let

$H = G - Q$, where $Q \subseteq V(K_{\frac{2(a-1)l+1}{b}+k})$ with $|Q| = k$. Then H is not fractional $[a,b]$-covered by Lemma 1 since for $S = V(K_{\frac{2(a-1)l+1}{b}+k}) \setminus Q$ and $T = V(lK_2)$, we deduce

$$\gamma_H(S,T) = b|S| + d_{H-S}(T) - a|T| = 2(a-1)l + 1 + 2l - a \cdot (2l) = 1 < \varepsilon(S,T),$$

where $\varepsilon(S,T) = 2$ since S is not independent. And so, G is not fractional (a,b,k)-critical covered. Moreover, we admit that

$$bind(G) = \frac{|N_G(V(lK_2) \setminus \{u\})|}{|V(lK_2) \setminus \{u\}|} = \frac{n-1}{2l-1} = \frac{(a+b-1)(n-1)}{(a+b-1)(2l-1)} = \frac{(a+b-1)(n-1)}{bn-a-b-bk},$$

where $u \in V(lK_2)$. Hence, the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-a-b-bk}$ in Theorem 3 is best possible.

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