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NUMERICAL SOLUTION OF TWO DIMENSIONAL REACTION-DIFFUSION EQUATION USING OPERATIONAL MATRIX METHOD BASED ON GENOCCHI POLYNOMIAL – PART II: ERROR BOUND AND STABILITY ANALYSIS

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Abstract. In this article, an operational matrix method with Genocchi polynomials is applied to solve a two dimensional space-time fractional order nonlinear reaction-diffusion equation. Applying collocation method and using the said matrix, fractional order non-linear partial differential equation is reduced to a system of algebraic equations, which have been solved using Newton iteration method. The salient features of the article are finding the stability analysis and error bound of the proposed scheme and pictorial presentations of the numerical solution of the concerned equation for different particular cases to show the effect of reaction term on the solution profile and also the change of its behaviour when the system approaches from standard order to fractional order. The accuracy of our proposed method is validated through the error analysis of the obtained numerical results with the existing analytical results for two particular cases of the concerned fractional order nonlinear model.

Key words: fractional PDE, diffusion equation, operational matrix, Genocchi polynomial, collocation method.

1. INTRODUCTION

The derivative and integral are the traditional mathematics and we understand the properties and their applications as well as how extensively the scientific professionals and engineers are using it for the advancement of modern technology. Fractional Calculus is a field of mathematics that is derived from the traditional definitions of calculus and its operators have been also derived from the differential and integral operators by taking the exponents with real value. A literature survey on various methods used for solving fractional differential equation and physical interpretation of diffusion equation is discussed in part-I [1]. Some definition on fractional order derivative and Genocchi polynomials with properties of Kronecker product is also given in part-I. To find the solution of the concerned two-dimensional nonlinear fractional order model, the fractional order operational matrix of derivative has been used which is already discussed in part-I [1].

2. ERROR BOUND AND STABILITY ANALYSIS

In this section, we have found the upper bound by means of Genocchi polynomial [2-5] on the error which are expecting in our approximation. Let us consider the following space

$$\prod_M = \text{Span}\,\{G_1(x), G_2(x), ...G_M(x), G_1(y), G_2(y), ...G_M(y), G_1(t), G_2(t), ...G_M(t)\}\,.$$

Let $\tilde{u}(x,y,t) \in \prod_M$ be the best approximation of $u(x,y,t)$. Then using the definition of best approximation, we get

$$\| u(x,y,t)-\tilde{u}(x,y,t)\|_\infty \le \| u(x,y,t)-w(x,y,t)\|_\infty, \quad \forall w(x,y,t)\in \prod_M .$$

The above inequality will remain true if $w(x,y,t)$ represents the interpolating polynomial for $u(x,y,t)$ at points (x_i, y_j, t_k) where x_i, y_j, t_k for $0\le i,j,k\le M$ are respectively roots of $G_{M+1}(x), G_{M+1}(y)$ and $G_{M+1}(t)$. Then by the procedure given in article [6], we get

$$u(x,y,t)-w(x,y,t)=\frac{1}{(M+1)!}\frac{\partial^{M+1}u(\mu,y,t)}{\partial x^{M+1}}\prod_{i=0}^{M}(x-x_i)+\frac{1}{(M+1)!}\frac{\partial^{M+1}u(x,\zeta,t)}{\partial y^{M+1}}\prod_{i=0}^{M}(y-y_j)+$$

$$+\frac{1}{(M+1)!}\frac{\partial^{M+1}u(x,y,\eta)}{\partial t^{M+1}}\prod_{i=0}^{M}(t-t_k)-\frac{\partial^{3M+3}u(\mu',\zeta',\eta')}{\partial x^{M+1}\partial y^{M+1}\partial t^{M+1}}\times\frac{\prod_{i=0}^{M}(x-x_i)\prod_{j=0}^{M}(y-y_j)\prod_{k=0}^{M}(t-t_k)}{(M+1)!(M+1)!(M+1)!},$$

where $\mu',\zeta',\eta',\mu,\zeta,\eta\in[0,1]$. Now by the properties of norm we have

$$\left\| u(x,y,t)-w(x,y,t)\right\|_\infty \le \frac{1}{(M+1)!}\max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(\mu,y,t)}{\partial x^{M+1}}\right|\left\|\prod_{i=0}^{M}(x-x_i)\right\|+$$

$$+\frac{1}{(M+1)!}\max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,\zeta,t)}{\partial y^{M+1}}\right|\left\|\prod_{i=0}^{M}(y-y_j)\right\|+\frac{1}{(M+1)!}\max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,y,\eta)}{\partial t^{M+1}}\right|\left\|\prod_{i=0}^{M}(t-t_k)\right\|+$$

$$+\max_{x,y,t\in[0,1]}\left|\frac{\partial^{3M+3}u(\mu',\zeta',\eta')}{\partial x^{M+1}\partial y^{M+1}\partial t^{M+1}}\right|\times\frac{\left\|\prod_{i=0}^{M}(x-x_i)\right\|\,\left\|\prod_{j=0}^{M}(y-y_j)\right\|\,\left\|\prod_{k=0}^{M}(t-t_k)\right\|}{(M+1)!(M+1)!(M+1)!}.$$

Since $u(x,y,t)$ is a continuous differential function in the interval $[0,1]$ then there exist constants A_1, A_2, A_3 and A_4 such that

$$\max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(\mu,y,t)}{\partial x^{M+1}}\right|\le A_1,\qquad \max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,\zeta,t)}{\partial y^{M+1}}\right|\le A_2,$$

$$\max_{x,y,t\in[0,1]}\left|\frac{\partial^{M+1}u(x,y,\eta)}{\partial t^{M+1}}\right|\le A_3,\qquad \max_{x,y,t\in[0,1]}\left|\frac{\partial^{3M+3}u(\mu',\zeta',\eta')}{\partial x^{M+1}\partial y^{M+1}\partial t^{M+1}}\right|\le A_4.$$

Now to minimize the factor we proceed as

$$\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{M}(x-x_i)\right|=\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\frac{G_{M+1}(x)}{M+1}\right|,$$

where $(M+1)$ is the leading coefficient of Genocchi polynomial of order $(M+1)$ and Genocchi polynomial satisfies

$$\max_{x\in[0,1]}\left|G_{M+1}(x)\right|\le\frac{4e^{\pi}\pi^{-M-1}\left(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi)\right)}{-2+2^{M+1}}.$$

From above inequalities we obtain

$$\left\|u(x,y,t)-\tilde{u}(x,y,t)\right\|_\infty\le\frac{4e^{\pi}\pi^{-M-1}\left(-2^{M+1}\Gamma(2+M,\pi)+e^{\pi}\Gamma(2+M,2\pi)\right)}{-2+2^{M+1}}\times\left(\frac{A_1+A_2+A_3}{(M+1)!}+\frac{A_4}{\left((M+1)!\right)^3}\right).$$

So, an upper bound for approximate solution of the absolute errors is obtained, which shows that approximation $\tilde{u}(x,y,t)$ converges to the exact solution $u(x,y,t)$ and validates the stability of proposed scheme.

3. SOLUTION OF THE PROBLEM

In this section, we apply operational matrix method based on Genocchi polynomials of fractional order derivatives to obtain the solution of the following two dimensional non-linear time-space fractional order PDE given by

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \kappa_1(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial x^{\beta}} + \kappa_2(x,y)\frac{\partial^{\gamma} u(x,y,t)}{\partial y^{\gamma}} + \kappa_3 u(1-u) + f(x,y,t), \tag{1}$$

where $0<\alpha\le 1,\ 1\le\beta\le 2,\ 1\le\gamma\le 2$, $\kappa_1(x,y)$ is the longitudinal diffusion coefficient, and $\kappa_2(x,y)$ is the transverse diffusion coefficient, $\kappa_3(x,y)$ is the reaction coefficient and $f(x,y,t)$ is the force term.

The initial and boundary conditions are considered as

$$u(x,y,0) = f_1(x,y), \tag{2}$$

$$u(0,y,t) = f_2(y,t), \tag{3}$$

$$u(x,0,t) = f_3(x,t), \tag{4}$$

$$\frac{\partial u(1,y,t)}{\partial x} = f_4(x,y,t), \tag{5}$$

$$\frac{\partial u(x,1,t)}{\partial y} = f_5(x,y,t), \tag{6}$$

where $0\le x\le 1$, $0\le y\le 1$ and $0\le t\le 1$. If $\kappa_3 = 0$, then system is known as conservative system and $\kappa_3 \ne 0$ implies that it is the non-conservative. $\kappa_3 < 0$ is known as sink term and $\kappa_3 > 0$ is known as source term.

Let us shall approximate $u(x,y,t)$ by Genocchi polynomial as

$$u(x,y,t) = \sum_{l=1}^{M}\sum_{m=1}^{M}\sum_{n=1}^{M} c_{lmn} G_m(t) G_l(x) G_n(y), \tag{7}$$

where c_{lmn} are unknown coefficients for $l = 1,2,\dots M$, $m = 1,2,\dots M$ and $n = 1,2,\dots M$.

Equation (7) is rewritten as

$$u(x,y,t) = \psi^{\mathrm{T}}(t).V.\big(\psi(x)\otimes\psi(y)\big), \tag{8}$$

where $V = [c_{lmn}]_{M\times M^2}$ is an $M\times M^2$ matrix of unknowns, $(\cdot)$ denotes the inner product [1], and $\psi(t) = \big(G_1(t).....G_M(t)\big)^{\mathrm{T}}$ is a column vector. If $\psi(x,t)$ is column vector formed by Genocchi polynomial then

$$\frac{\partial^{\beta}}{\partial x^{\beta}}\big(\psi(x,t)\big) = \frac{\partial^{\beta}}{\partial x^{\beta}}\big(\psi(x)\otimes\psi(t)\big) = \frac{\partial^{\beta}\psi(x)}{\partial x^{\beta}}\otimes\psi(t) = P^{\beta}\psi(x)\otimes I\,\psi(t) = \big(P^{\beta}\otimes I\big)\big(\psi(x)\otimes\psi(t)\big).$$

Similarly,

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\big(\psi(x,t)\big) = \frac{\partial^{\alpha}}{\partial t^{\alpha}}\big(\psi(x)\otimes\psi(t)\big) = \psi(x)\otimes\frac{\partial^{\alpha}\psi(t)}{\partial t^{\alpha}} = I\,\psi(x)\otimes P^{\alpha}\,\psi(t) = (I\otimes P^{\alpha})\big(\psi(x)\otimes\psi(t)\big).$$

Now operating fractional derivative of order α on equation (8), we get

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \big(P^{\alpha}\phi(t)\big)^{\mathrm{T}}.C.\big(\psi(x)\otimes\psi(y)\big). \tag{9}$$

Similarly,

$$\frac{\partial^{\beta}u(x,y,t)}{\partial x^{\beta}}=\phi^{\mathrm{T}}(t).C.(P^{\beta}\otimes I)\big(\psi(x)\otimes\psi(y)\big), \tag{10}$$

$$\frac{\partial^{\gamma}u(x,y,t)}{\partial y^{\gamma}}=\phi^{\mathrm{T}}(t).C.(I\otimes P^{\gamma})\big(\psi(x)\otimes\psi(y)\big). \tag{11}$$

Now equations (2)-(6) with the aid of the equation (8) give rise to

$$\phi^{\mathrm{T}}(0).C.\big(\psi(x)\otimes\psi(y)\big)=f_1(x,y), \tag{12}$$

$$\phi^{\mathrm{T}}(t).C.\big(\psi(0)\otimes\psi(y)\big)=f_2(y,t), \tag{13}$$

$$\phi^{\mathrm{T}}(t).C.\big(\psi(x)\otimes\psi(0)\big)=f_3(x,t), \tag{14}$$

$$\phi^{\mathrm{T}}(t).C.(P^{1}\otimes I)\big(\psi(1)\otimes\psi(y)\big)=f_4(x,y,t), \tag{15}$$

$$\phi^{\mathrm{T}}(t).C.(I\otimes P^{1})\big(\psi(x)\otimes\psi(1)\big)=f_5(x,y,t). \tag{16}$$

After putting the values of $u(x,y,t)$, $\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$, $\frac{\partial^{\beta}u}{\partial x^{\beta}}$ and $\frac{\partial^{\gamma}u}{\partial y^{\gamma}}$ from equations (8)-(11), we collocate equations (1), (2)–(6) at points $x_i=\frac{i}{M}$, $y_j=\frac{j}{M}$ and $t_k=\frac{k}{M}$ for $i,j,k=1,2,\ldots,M$.

After collocating we get non linear system of algebraic equations. By solving that system of equations and finding V, the numerical solution of our proposed model (1) can be obtained.

4. RESULTS AND DISCUSSION

In this section, a drive has been taken to validate the effectiveness of our proposed method first through applying it on two spatial fractional order problems $(\alpha=1)$ and compared those obtained results with the existing analytical results for different particular cases.

If we consider $\beta=\gamma=1.5$, $\kappa_3=0$, our proposed model (1) is reduced to

$$\frac{\partial u(x,t)}{\partial t}=\kappa_1(x,y)\frac{\partial^{1.5}u(x,t)}{\partial x^{1.5}}+\kappa_2(x,y)\frac{\partial^{1.5}u(x,t)}{\partial y^{1.5}}+f(x,y,t). \tag{17}$$

Taking, $\kappa_1(x,y)=\frac{(3-2x)\Gamma(3-\alpha)}{2}$, $\kappa_2(x,y)=\frac{(4-y)\Gamma(3-\beta)}{6}$ and the forcing function as $f(x,y,t)=e^{-t}\left(x^2(-y^{3/2}+y-4)y^{3/2}+\surd x(2x-3)y^3\right)$ and considering the initial and boundary conditions

$$u(x,y,0)=x^2y^3,\ u(0,y,t)=0,\ u(x,0,t)=0,\ \frac{\partial u(1,y,t)}{\partial x}=2e^{-t}y^3,\ \frac{\partial u(x,1,t)}{\partial y}=2e^{-t}x^2,$$

the exact solution of the above problem will be found as $u(x,y,t)=e^{-t}x^2y^3$.

The absolute error is calculated between the exact solution and the approximate solution using our proposed method for $M=4$, which is displayed in Table 1. The results clearly predict that our numerical results are in complete agreement with the existing results. The similarity nature for both the solutions can also be found from Fig. 1. The variations of absolute error for different values of M are depicted in Table 2.

Table 1

Variations of absolute error for different time and space for first case taking $M = 4$

$x\downarrow$ $t\rightarrow$	0.2	0.4	0.6	0.8	1
.2	1×10^{-5}	2.6×10^{-5}	4.5×10^{-5}	5.9×10^{-5}	6.5×10^{-5}
.4	2.2×10^{-5}	5.4×10^{-5}	8.6×10^{-5}	1.1×10^{-4}	1.2×10^{-4}
.6	3.4×10^{-5}	7.7×10^{-5}	1.1×10^{-4}	1.5×10^{-4}	1.6×10^{-4}
.8	4.3×10^{-5}	9.2×10^{-5}	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}
1	4.5×10^{-5}	9.4×10^{-5}	1.3×10^{-4}	1.7×10^{-4}	1.8×10^{-4}

We consider another particular case of the proposed model (1) as $\beta=1.8, \gamma=1.6$, $\kappa_3=0$ so that the model is reduced to a non-linear two dimensional fractional order differential equation as

$$\frac{\partial u(x,t)}{\partial t}=\kappa_1(x,y)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}}+\kappa_2(x,y)\frac{\partial^{1.6}u(x,t)}{\partial y^{1.6}}+f(x,y,t)\,. \tag{18}$$

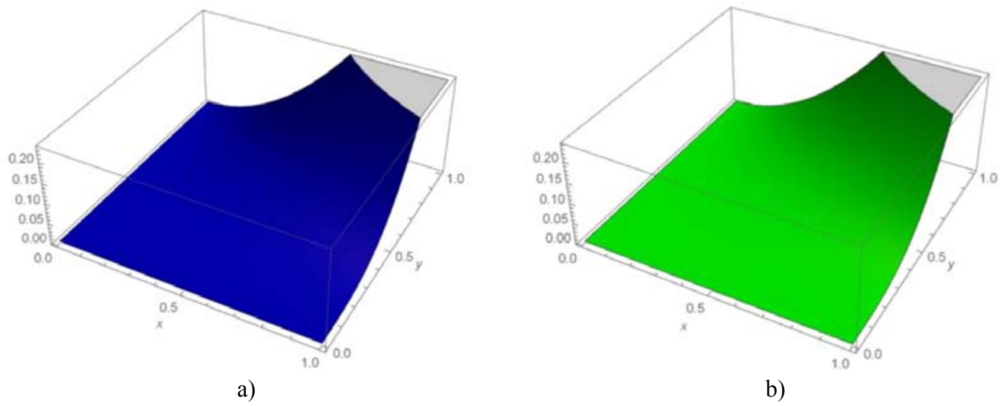
Taking $\kappa_1(x,y)=\dfrac{\Gamma(2.2)x^{2.8}y}{6}$, $\kappa_2(x,y)=\dfrac{2xy^{2.6}}{\Gamma 4.6}$ and the forcing function as

$$f(x,y,t)=e^{-t}\left(x^2(-y^{3/2}+y-4)y^{2.6}+\sqrt{x}(2x-3)y^3\right).$$

Table 2

Variations of absolute error for different values of M at time $t = 0.2$

$x\downarrow$ $M\rightarrow$	4	6	8
0.2	1×10^{-5}	7.3×10^{-6}	8.2×10^{-9}
0.4	2.2×10^{-5}	6.4×10^{-6}	2.4×10^{-8}
0.6	3.4×10^{-5}	2.7×10^{-6}	4.7×10^{-8}
0.8	4.3×10^{-5}	4.6×10^{-6}	1.2×10^{-8}
1	4.5×10^{-5}	1.1×10^{-6}	3.6×10^{-8}



a) b)

Fig. 1 – Plots of $u(x,y,t)$ vs. x and y for $M = 4$ in case of numerical and exact solution respectively for $t = 0.5$.

Considering the initial and boundary conditions as

$$u(x,y,0)=x^3y^{3.6},\ \ u(0,y,t)=0,\ \ u(x,0,t)=0,\ \ \frac{\partial u(1,y,t)}{\partial x}=3e^{-t}y^{3.6},\ \ \frac{\partial u(x,1,t)}{\partial y}=3.6e^{-t}x^3,$$

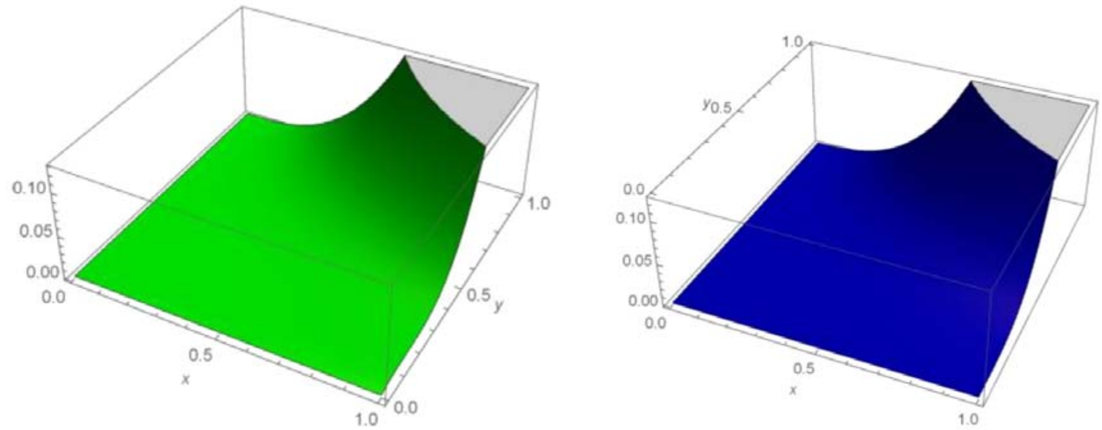
the equation (18) has the exact solution given by $u(x,y,t)=e^{-t}x^2y^{3.6}$.

The absolute error for different values of x and y are shown in the Table 3 for $M=4$. The Fig. 2 clearly shows the similarity of the results of the exact solution and our proposed solution.

Table 3

Variations of absolute error for different time and space for second case taking $M = 4$

$x\downarrow\ t\rightarrow$	0.2	0.4	0.6	0.8	1
.2	3.9×10^{-5}	3.7×10^{-4}	1.2×10^{-3}	2.8×10^{-3}	5.5×10^{-3}
.4	3.6×10^{-4}	3×10^{-4}	1×10^{-3}	2.3×10^{-3}	4.5×10^{-3}
.6	3.9×10^{-5}	2.3×10^{-5}	7.6×10^{-4}	1.8×10^{-3}	3.5×10^{-3}
.8	4.2×10^{-5}	1.6×10^{-4}	5.3×10^{-4}	1.3×10^{-3}	2.7×10^{-3}
1	4.2×10^{-5}	1.1×10^{-4}	13.8×10^{-4}	1.0×10^{-3}	2.1×10^{-3}



a) b)

Fig. 2 – Plots of $u(x,y,t)$ vs. x and y for $M = 4$ in case of numerical and exact solution for $t = 0.5$.

After the confirmation of accuracy and efficiency of the method, it is applied to find numerical solutions of the considered non linear two dimensional space time fractional order non-conservative system (1) under the following initial and boundary conditions as

$$u(x,y,0)=x\,, \tag{19}$$

$$u(0,y,t)=0\,, \tag{20}$$

$$u(x,0,t)=xt\,, \tag{21}$$

$$\frac{\partial u(1,y,t)}{\partial x}=t\,, \tag{22}$$

$$\frac{\partial u(x,1,t)}{\partial y}=0\,. \tag{23}$$

The numerical results for different particular cases are depicted through Figs. 3–6 at $t=0.5$. It is seen from Fig. 3 (left) that for non-linear time fractional reaction diffusion equation ($\beta=\gamma=2$, $\kappa_3=+1$), the sub-diffusion phenomena of solute concentration occur and overshoots of sub diffusion decrease with the increase in α. It is seen from Fig. 3 (right) that similar sub-diffusions are found for non-linear spatial fractional reaction-diffusion equation ($\alpha=1,\gamma=2\,,\kappa_3=1$). Here initially solute concentration decreases as β increases. It is seen from Fig. 4 (left) that solute concentration increases as γ increases. The effect of reaction term on the solution profile for standard order reaction-diffusion equation $(\alpha=1,\beta=\gamma=2)$ is shown through Fig. 4 (left) without the presence of force term. It is clear from the figures that overshoots of the sub-diffusions of solute concentration decrease for the case of sink term ($\kappa_3=-1$) as compared to source term ($\kappa_3=+1$).

4. CONCLUSION

In this article three important goals have been achieved. First one, the use of collocation method based on Genocchi polynomial to solve the two dimensional non linear reaction-diffusion equation in presence of the force term. Second one is the pictorial representations of the nature of overshoots during subdiffusion as the system approaches from standard order to fractional order. Third one is the exhibitions of decrease of solute concentration due to the presence of sink term for standard order as well as fractional order.

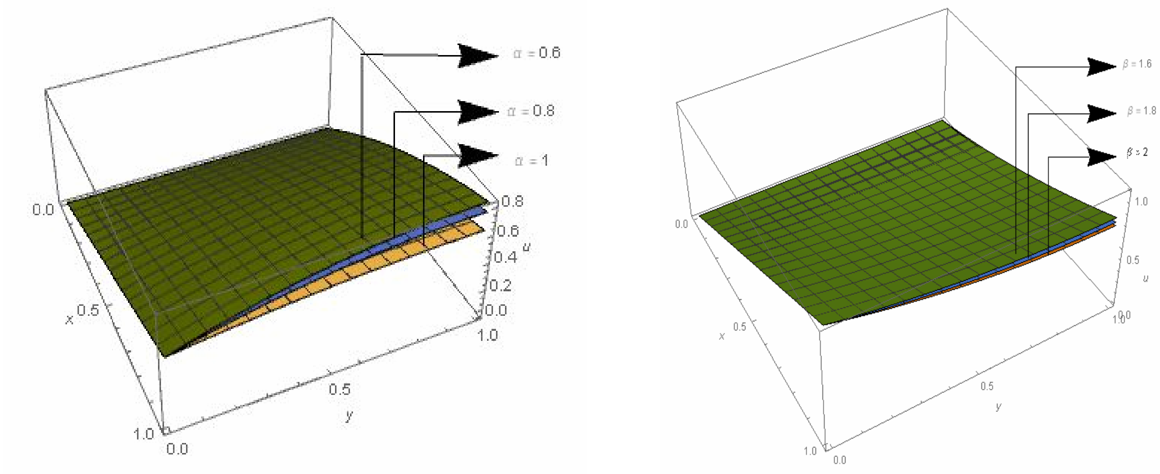


Fig. 3 – Plots of $u(x, y, t)$ vs. x and y for $t = 0.5$, for various α when $\kappa_3 = 1$, $\beta = \gamma = 2$ (left), and for various β when $\kappa_3 = 1$, $\alpha = 1, \gamma = 2$ (right).

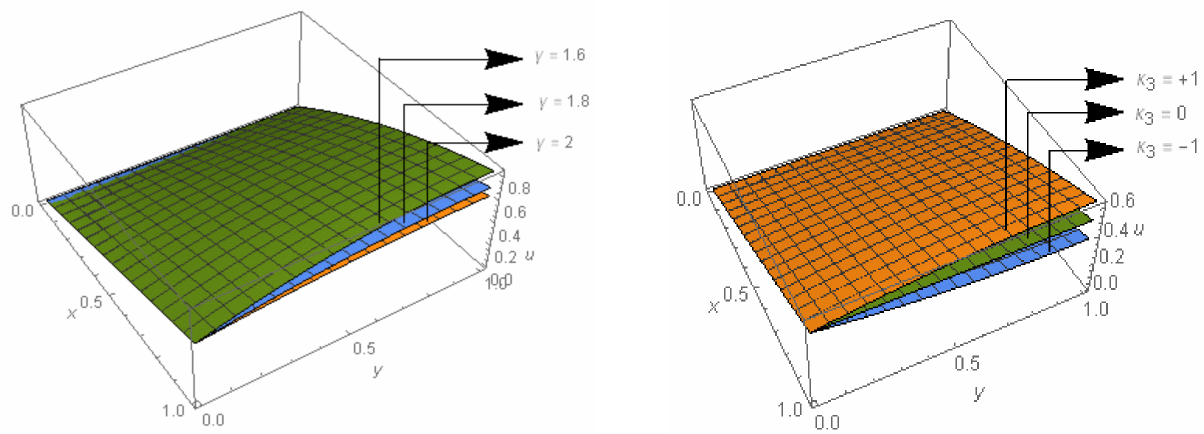


Fig. 4 – Plots of $u(x, y, t)$ vs. x and y for $t = 0.5$, for various γ when $\kappa_3 = 1$, $\beta = 2, \alpha = 1$ (left), and for various β when $\beta = 2$, $\alpha = 1, \gamma = 2$ (right).

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