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PHOTON-LIMITED IMAGE RESTORATION USING A WELL-POSED NONLINEAR FOURTH-ORDER HYPERBOLIC PDE-BASED MODEL

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Abstract. We consider a novel photon-limited image restoration technique in this research paper. The proposed denoising approach uses a hyperbolic nonlinear fourth-order partial differential equation-based model that is treated mathematically here, its well-posedness being demonstrated. Its weak and unique solution is then computed numerically, by solving the proposed PDE model using a finite difference method-based numerical approximation algorithm. The obtained iterative approximation scheme provides an effective Poisson noise removal, our successful restoration experiments being also described.

Key words: photon-limited imaging, quantum noise, hyperbolic fourth-order partial differential equation-based model, finite difference-based method, numerical approximation scheme.

1. INTRODUCTION

The photon image devices capture the images by counting the photon detections at various spatial locations over a certain period of observation. Since this photon emission and detection process is characterized by an inherently discrete nature, the signal-dependent errors of the image acquisition systems generate the so called quantum or shot noise [1].

This type of noise deteriorates seriously the captured images both quantitatively and qualitatively, producing the so called photon-limited images. The quantum noise is modelled as a Poisson process, obeying a Poisson law. For this reason, it is also called Poison noise, being characterized by the next Poisson probability distribution [2]:

$$P(n) = \frac{e^{-\mu}\mu^{n}}{n!},\ n \geq 0\,. \tag{1}$$

Many photon-limited image restoration techniques have been developed in the last decades. Some classic nonlinear filters, such as the 2D median filter [3], can be used for the shot noise removal, but they are not effective enough and may affect the essential image features. So, some much more effective quantum denoising approaches have been introduced. They include the Non-Local Mean – NLM Poisson filter [4], the Poisson Reducing Bilateral Filter – PRBF [5], the Multi Scale Variance Stabilizing Transform (MS-VST) [6], the moving average filter and the wavelet-based filtering methods [7].

The partial differential equation (PDE) – based models have been widely used for image denoising and restoration in the last 35 years, in both variational and non-variational form. While most of them represent powerful additive Gaussian noise removal solutions [8], some PDE-based shot noise filtering algorithms have been also developed.

The variational quantum denoising schemes are based on the total variation (TV) regularization [9,10]. While these models provide an effective Poisson noise filtering and conserve the boundaries, they may generate the undesired image staircasing.

Fourth-order PDE models deal successfully with this unintended staircase effect. So, we consider here a non-variational and nonlinear fourth-order PDE-based quantum noise removal framework that preserves the essential image details and overcomes the undesirable effects.

We have conducted a high amount of research in the variational and diffusion-based image restoration domain in the last 12 years, developing numerous effective white additive Gaussian noise filtering techniques based on parabolic and hyperbolic PDE models of second and fourth order [11-13]. The novel fourth-order differential model proposed in the next section is hyperbolic and also has a well-posed character, the existence of a unique weak solution being treated in the third section.

Then, the proposed model is discretized by applying a numerical approximation algorithm that is consistent to it and is constructed by using the finite difference method [14]. This iterative discretization scheme proposed in the fourth section has been successfully tested on many photon-limited images, the shot denoising experiments being discussed in the fifth section. The conclusions of this work are drawn in the final section.

2. NONLINEAR HYPERBOLIC FOURTH-ORDER PDE-BASED RESTORATION MODEL

A novel nonlinear hyperbolic PDE model for photon-limited image restoration is proposed in this section. It is composed of the following fourth-order partial differential equation and several boundary conditions:

$$\begin{cases} \alpha \dfrac{\partial^2 u}{\partial t^2} + \beta \dfrac{\partial u}{\partial t} + \lambda \nabla^2 \left(\varphi\left(\|\nabla u\| \right) \nabla^2 u \right) + \varepsilon \left(\dfrac{u - u_0}{|u| + \eta} \right) = 0, \\ u(x,y,0) = u_0(x,y), \quad \forall (x,y) \in \Omega \subseteq R^2, \\ u_t(x,y,0) = u_1(x,y), \quad \forall (x,y) \in \Omega, \\ u(x,y,t) = 0, \quad \forall (x,y) \in \partial\Omega, \\ \dfrac{\partial u}{\partial \vec{n}}(x,y,t) = 0, \quad \forall (x,y) \in \partial\Omega, \end{cases} \tag{2}$$

where the coefficients are $\alpha, \beta, \lambda, \varepsilon, \eta \in (0,1]$, $\Omega \subseteq R^2$ represents the image domain and $u_0 \in L^2(\Omega)$ is the observed photon-limited image. Here $\nabla^2 = \Delta$.

We propose the following diffusivity function of this nonlinear partial differential equation-based model:

$$\varphi : [0,\infty) \to [0,\infty),$$
$$\varphi(s) = \sqrt[k-1]{\frac{\delta(x,y,t)}{\left| \gamma s^k + \xi \log 10\left(\delta(x,y,t) \right) \right|}}, \tag{3}$$

where the coefficients are $\xi \in (0.5,1]$, $\gamma \in (0,1]$ and $k \geq 3$, and the conductance parameter $\delta(x,y,t)$ represents a positive function that depends on the coordinates and the statistics of the evolving image. It will be computed by using an algorithm which is described in the following sections.

The function given by (3) is properly modelled for a detail-preserving restoration process [8], being positive, monotonically decreasing and convergent to zero. It is also a Lipschitz function, since its derivative is bounded, because it exists a constant $C > 0$ such that

$$\varphi'(s) \in (0,C) => \varphi \in \mathrm{Lip}(R). \tag{4}$$

The detail-preserving denoising is also assured by the hyperbolic character of the fourth-order PDE model (2). So, the presence of the second-order time derivative in the partial differential equation (2) provides much sharper image boundaries, thus enhancing the essential details.

This nonlinear hyperbolic PDE-based model is non-variational, since it cannot be obtained from the minimization of an energy cost functional, but it is well-posed, admitting a unique variational solution. Its validity is treated mathematically in the next section, where the existence and unicity of that weak solution is demonstrated.

3. A MATHEMATICAL TREATMENT ON THE VALIDITY OF THE PDE MODEL

The well-posedness of the proposed fourth-order hyperbolic PDE restoration model is investigated in this section. Let us set $H = L^2(\Omega)$, $V = H^2(\Omega) \cap H_0^1(\Omega)$ and note that $V \subset H$ with dense and compact embedding. Here $H^2(\Omega)$ and $H_0^1(\Omega)$ represent standard Sobolev spaces. Also, the Lipschitz property of the function φ is important to this mathematical treatment [15].

The function $u : [0,T] \times \Omega \to R$ is said to be a variational solution to the hyperbolic equation (2) if the following conditions hold:

$$\begin{cases} u, u_t \in L^\infty(0,T;V),\ u_{tt} \in L^\infty(0,T;H), \\ u(x,y,0) = u_0(x,y), \\ \int\limits_\Omega \left(\alpha u_{tt}(x,y,t) + \beta u_t(x,y,t)\right) v(x,y,t)\,\mathrm{d}x\mathrm{d}y + \lambda \int\limits_\Omega \varphi\left(\|\nabla u\|\right)(x,y,t)\,\Delta u(x,y,t)\,\Delta v(x,y,t)\,\mathrm{d}x\mathrm{d}y + \\ \qquad + \varepsilon \int\limits_\Omega \dfrac{u(x,y,t) - u_0(x,y)}{|u(x,y,t)| + \eta} v(x,y,t)\,\mathrm{d}x\mathrm{d}y = 0,\ \forall v \in V,\ t \in [0,T]. \end{cases} \tag{5}$$

As regards the equation (2) we have the following existence result:

PROPOSITION 1. Under above assumptions there is a unique variational (weak) solution u.

Proof (sketch). We shall construct the following sequence: $\{u_n\} \subset L^\infty(0,T;V)$, where we have $(u_n)_t \subset L^\infty(0,T;V)$, $(u_n)_{tt} \subset L^\infty(0,T;H)$, which is defined by:

$$\begin{cases} \alpha (u_n)_{tt} + \beta (u_n)_t + \lambda \Delta\left(\varphi\left(\|\nabla u_{n-1}\|\right)\Delta u_n\right) + \varepsilon \dfrac{u_n - u_0}{|u_n| + \eta} = 0, \text{ on } (0,T)\times\Omega, \\ u_n(x,y,t) = 0,\ \dfrac{\partial u_n}{\partial n}(x,y,t) = 0 \text{ on } (0,T)\times\partial\Omega, \\ u_n(x,y,t) = u_0,\ (u_n)_t(x,y,0) = u_1 \text{ on } \Omega. \end{cases} \tag{6}$$

By applying Theorem 1.1 described in [15], it follows that for any u_{n-1} there is a unique weak solution u_n to (6). Then, one can prove the convergence of $\{u_n\}$ to a variational solution u for the PDE-based model (1), by proving just that $\{u_n\}$ and $\{(u_n)_t\}$ are bounded in $L^2(0,T;V)$ and $\{(u_n)_{tt}\}$ is bounded in $L^2(0,T;H)$, and so $\{u_n\}$ and $\{(u_n)_t\}$ are compact in $L^2(0,T;H)$.

Indeed, by multiplying the equation in (6) by u_n, and then integrating on $[0,t]\times\Omega$, we get after some calculation:

$$\alpha \int\limits_\Omega u_n(x,y,t)\,(u_n)_t(x,y,t)\,\mathrm{d}x\mathrm{d}y + \frac{\beta}{2}\int\limits_\Omega u_n^2(x,y,t)\,\mathrm{d}x\mathrm{d}y + \lambda \int\limits_0^t \int\limits_\Omega \varphi\left(\|\nabla u_{n-1}\|\right)|\Delta u_n|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s =$$
$$= \frac{\beta}{2}\int\limits_\Omega |u_0|^2\,\mathrm{d}x\mathrm{d}y + \alpha \int\limits_\Omega u_1 u_0\,\mathrm{d}x\mathrm{d}y + \alpha \int\limits_0^t \int\limits_\Omega (u_n)_t^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s - \varepsilon \int\limits_0^t \int\limits_\Omega \frac{u_n - u_0}{|u_n| + \eta} u_n\,\mathrm{d}x\mathrm{d}y\mathrm{d}s \tag{7}$$

and therefore

$$\beta \int\limits_\Omega u_n^2(x,y,t)\,\mathrm{d}x\mathrm{d}y + \int\limits_0^t \int\limits_\Omega |\Delta u_n|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s \le c \int\limits_0^t \int\limits_\Omega (u_n)_t^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s + c,\ \ \forall t \in [0,T],\ \text{on } (0,T)\times\Omega, \tag{8}$$

where c is a positive constant and independent of n. Next, we multiply (6) by $(u_n)_t$ and then integrate it on $[0,t]\times\Omega$ to get the following estimate:

$$\frac{\alpha}{2}\int_{\Omega}(u_n)_t^2(x,y,t)\,\mathrm{d}x\mathrm{d}y+\beta\int_0^t\int_{\Omega}(u_n)_t^2(x,y,s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}s+\frac{\lambda}{2}\int_0^t\int_{\Omega}\varphi\left(\left\|\nabla u_{n-1}\right\|\right)\left|\Delta u_n\right|_s^2\mathrm{d}x\mathrm{d}y\mathrm{d}s=$$

$$=-\varepsilon\int_0^t\int_{\Omega}\frac{u_n-u_0}{|u_n|+\eta}(u_n)_t\,\mathrm{d}x\mathrm{d}y\mathrm{d}s+\frac{\alpha}{2}\int_{\Omega}u_1^2\,\mathrm{d}x\mathrm{d}y. \tag{9}$$

Finally, multiplying by Δu_n and $\Delta(u_n)_t$ and using (8) and (9), we get:

$$\int_{\Omega}\left((u_n)_t^2+u_n^2\right)\mathrm{d}x\mathrm{d}y+\int_0^t\int_{\Omega}|\Delta u_n|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s+\int_0^t\int_{\Omega}|\nabla(u_n)_t|^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}s\le c,\quad\forall t\in[0,T] \tag{10}$$

as claimed.

4. NUMERICAL APPROXIMATION ALGORITHM

The proposed differential model is solved numerically by applying an effective numerical approximation scheme that is developed using the finite difference method [14]. A grid of space size h and time step Δt is used for this purpose.

Thus, the space coordinates are quantized as $x=ih,\ y=jh,\ i\in\{1,...,I\},\ j\in\{1,...,J\}$ and the time coordinate is quantized as $t=n\Delta t,\ n\in\{1,...,N\}$, where the support image is $[Ih\times Jh]$. The partial differential equation in (1) is approximated by using the finite differences [14]. It can be written as follows:

$$\alpha\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial u}{\partial t}+\varepsilon\left(\frac{u-u_0}{|u|+\eta}\right)=-\lambda\nabla^2\left(\varphi\left(\|\nabla u\|\right)\Delta u\right). \tag{11}$$

The left term of the equation (11) is then discretized, by applying the central differences [14], as following:

$$\alpha\frac{u_{i,j}^{n+\Delta t}+u_{i,j}^{n-\Delta t}-2u_{i,j}^{n}}{\Delta t^2}+\beta\frac{u_{i,j}^{n+\Delta t}-u_{i,j}^{n-\Delta t}}{2\Delta t}+\varepsilon\left(\frac{u_{i,j}^{n}-u_{i,j}^{0}}{\left|u_{i,j}^{n}\right|+\eta}\right)=u_{i,j}^{n+\Delta t}\left(\frac{\alpha}{\Delta t^2}+\frac{\beta}{2\Delta t}\right)+$$

$$+u_{i,j}^{n-\Delta t}\left(\frac{\alpha}{\Delta t^2}-\frac{\beta}{2\Delta t}\right)+u_{i,j}^{n}\left(\frac{\varepsilon}{\left|u_{i,j}^{n}\right|+\eta}-\frac{2\alpha}{\Delta t^2}\right)-u_{i,j}^{0}\frac{\varepsilon}{\left|u_{i,j}^{n}\right|+\eta} \tag{12}$$

that leads to

$$u_{i,j}^{n+1}\left(\frac{2\alpha+\beta}{2}\right)+u_{i,j}^{n-1}\left(\frac{2\alpha-\beta}{2}\right)+u_{i,j}^{n}\left(\frac{\varepsilon}{\left|u_{i,j}^{n}\right|+\eta}-2\alpha\right)-u_{i,j}^{0}\frac{\varepsilon}{\left|u_{i,j}^{n}\right|+\eta} \tag{13}$$

if one considers $\Delta t=1$.

The right term of (11) is then approximated numerically. First, one computes $\varphi_{i,j}=\varphi\left(\left\|\nabla u_{i,j}^{n}\right\|\right)$, where

$$\left\|\nabla u_{i,j}^{n}\right\|\approx\sqrt{\left(\frac{u_{i+h,j}^{n}-u_{i-h,j}^{n}}{2h}\right)^2+\left(\frac{u_{i,j+h}^{n}-u_{i,j-h}^{n}}{2h}\right)^2}. \tag{14}$$

Therefore, from (3) we get:

$$\varphi_{i,j} = \sqrt[k-1]{\frac{\delta(i,j,n)}{\left|\gamma\left\|\nabla u_{i,j}^n\right\|^k + \xi\log10\left(\delta(i,j,n)\right)\right|}}\,, \tag{15}$$

where the conductance parameter is computed by using the next algorithm:

$$\delta(i,j,n) = \left|\nu n + \mu\left(\left\|\nabla u_{i,j}^{n-1}\right\|\right)\right|, \tag{16}$$

where $\nu \in (0,1]$ and μ returns the average of the argument. Thus, one obtains the following discretization for the right term in (7):

$$-\lambda\Delta\varphi^{i,j} = -\lambda\frac{\varphi^{i+1,j} + \varphi^{i-1,j} + \varphi^{i,j+1} + \varphi^{i,j-1} - 4\varphi^{i,j}}{h^2}\,, \tag{17}$$

where

$$\varphi^{i,j} = \varphi_{i,j}\nabla^2 u_{i,j} = \varphi_{i,j}\frac{u_{i+h,j}^n + u_{i-h,j}^n + u_{i,j+h}^n + u_{i,j-h}^n - 4u_{i,j}}{h^2}\,. \tag{18}$$

Therefore, by choosing $h = \Delta t = 1$, we obtain the following iterative explicit numerical approximation algorithm:

$$u_{i,j}^{n+1} = u_{i,j}^n\left(\frac{4\alpha\left(\left|u_{i,j}^n\right| + \eta\right) - 2\varepsilon}{(2\alpha+\beta)\left(\left|u_{i,j}^n\right| + \eta\right)}\right) + u_{i,j}^{n-1}\left(\frac{\beta - 2\alpha}{\beta + 2\alpha}\right) + u_{i,j}^0\frac{2\varepsilon}{(2\alpha+\beta)\left(\left|u_{i,j}^n\right| + \eta\right)} -$$
$$-\frac{2\lambda\left(\varphi^{i+1,j} + \varphi^{i-1,j} + \varphi^{i,j+1} + \varphi^{i,j-1} - 4\varphi^{i,j}\right)}{2\alpha+\beta},\quad \forall i \in \{1,\ldots,I\},\ j \in \{1,\ldots,J\},\ n \in \{1,\ldots,N\}. \tag{19}$$

The finite difference-based numerical approximation scheme given by (19) and corresponding to the proposed nonlinear hyperbolic fourth-order PDE-based model (2) converges fast to its variational solution representing the restored image, u^{N+1}, because the number of the iterations required by an optimal quantum denoising process, N, is quite low. This iterative approximation algorithm has been successfully tested on numerous photon-limited images, the restoration experiments being disscused in the following section.

5. NUMERICAL SIMULATIONS

A lot of numerical experiments have been performed on hundreds of photon-limited images, applying the described filtering algorithm on them. Well-known image collections, such as the volumes of the USC-SIPI database, have been used for our restoration tests.

The proposed nonlinear hyperbolic PDE-based shot denoising approach removes properly this type of noise and overcomes also unintended effects like blurring or staircasing, preserving well the edges, corners and other features. It has a quite low execution time, given its fast-converging numerical approximation algorithm described in the previous section, which reach the optimal restoration after few iterations. However, the number of the required processing steps and, as a result, the running time depend on the size of the photon-limited image and the amount of the Poisson noise that affects it.

Method comparison have been also performed. The quantum denoising framework proposed here outperforms both the conventional filtering algorithms, such as the 2D median filter, and the PDE-based smoothing approaches, such as the total variation (TV) based models adapted for the shot noise, producing better restoration results and also running much faster.

The performance of the developed technique has been assessed using various similarity measures, such as PSNR (Peak Signal to Noise Ratio), SNR (Signal to Noise Ratio) and MSE (Mean-Squared Error) [16]. We have found that the proposed Poisson denoising approach provides better values for these performance metrics. As one can see in the Table 1, the described hyperbolic fourth-order PDE-based Poisson denoising approach achieves higher average PSNR values (in decibels) than other shot filtering techniques.

Table 1

Average PSNRs for several quantum denoising models

Restoration method	**Average PSNR value**
The proposed hyperbolic PDE model	34.3541 (dB)
Median filter	27.9572 (dB)
TV model for Poisson noise	31.4754 (dB)
Bilateral 2D filter	29.8968 (dB)
NLM filter	28.4369 (dB)

A method comparison example is displayed in Fig. 1. Thus, the original [384 × 512] *Peppers* image displayed in Fig. 1a is corrupted by adding a high amount of Poisson noise, its photon-limited version being represented in Fig. 1b. The denoising results produced by several well-known shot filtering techniques are displayed in the next images.

The restoration result obtained by our hyperbolic PDE-based algorithm after $N = 25$ iterations is depicted in Fig. 1c. It is closer to original than the results achieved by the 2D Median, NLM and Bilateral 2D filters and by the variational TV-based model for Poisson noise (after 40 processing steps), which are described in Fig. 1d to Fig. 1g, and it is also produced in a lower running time, comparing to the other image restoration results.

6. CONCLUSIONS

A novel photon-limited image restoration framework has been described in this article. The considered partial differential equation-based technique provides an effective shot denoising and overcomes the undesired effects, preserving the edges and other features.

The nonlinear hyperbolic fourth-order PDE-based denoising model proposed here and its mathematical investigation represent the main contributions of this research. Unlike other PDE-based Poisson noise filtering approaches that represent variational schemes and are based on second-order equations, our restoration model has a non-variational character and a higher order. Its well-posedness is rigorously treated here, the existence and unicity of a variational solution of this nonlinear hyperbolic PDE model being demonstrated.

Another contribution of this work is the iterative finite difference-based numerical approximation algorithm constructed for this model. The proposed numerical discretization scheme of the PDE model converges quite fast to its weak solution.

The iterative discretization scheme has been successfully applied in the Poisson denoising tests performed by us, which illustrate the effectiveness of the described method. Our photon-limited image restoration approach outperforms numerous existing shot noise filtering models, providing much better denoising results, and also executing faster than those conventional and PDE-based quantum noise removal algorithms.

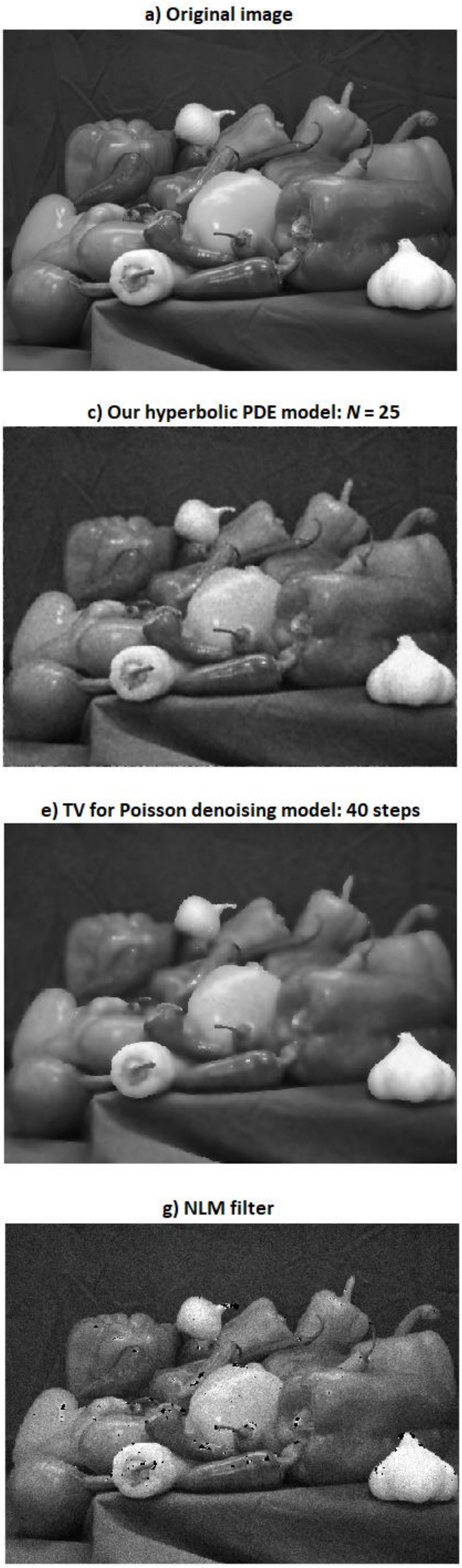

Fig. 1 – A photon-limited image denoised by several filtering models.

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