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ON GENERALISED TRIPLETS OF HILBERT SPACES

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Abstract. We compare the concept of triplet of closely embedded Hilbert spaces with that of generalised triplet of Hilbert spaces in the sense of Berezanskii by showing when they coincide, when they are different, and when starting from one of them one can naturally produce the other one that essentially or fully coincides.

Key words: Generalised triplet of Hilbert spaces, closed embedding, triplet of closely embedded Hilbert spaces, rigged Hilbert spaces.

1. INTRODUCTION

There are two basic paradigms of mathematical models in quantum physics: one due to J. von Neumann based on Hilbert spaces and their linear operators and the other due to P.A.M. Dirac based on the bra-ket duality. The two paradigms have been connected by L. Schwartz's theory of distributions [7] and the rigged Hilbert space method, originated by I.M. Gelfand and his school [5] that turned out to be a powerful tool in analysis, partial differential equations, and mathematical physics.

An important more rigorous formalisation of the construction of rigged Hilbert spaces was done by Yu.M. Berezanskii [1] through a scale of Hilbert spaces and where the main step is taken by the so-called *triplet of Hilbert spaces*. More precisely, a triplet of Hilbert spaces, denoted $\mathscr{H}_+ \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{H}_-$, means that: $\mathscr{H}_+$, $\mathscr{H}$, and $\mathscr{H}_-$ are Hilbert spaces, the embeddings are continuous (bounded linear operators), the space $\mathscr{H}_+$ is dense in $\mathscr{H}$, the space $\mathscr{H}$ is dense in $\mathscr{H}_-$, and the space $\mathscr{H}_-$ is the dual of $\mathscr{H}_+$ with respect to $\mathscr{H}$, that is, $\|\varphi\|_- = \sup\{|\langle h, \varphi\rangle_{\mathscr{H}} \mid \|h\|_+ \leq 1\}$, for all $\varphi \in \mathscr{H}$. Extending these triplets on both sides, one may get a scale of Hilbert spaces that yields, by an inductive and, respectively, projective limit method, a *rigged Hilbert space* $\mathscr{S} \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{S}'$.

In this note we compare two generalisations of the concept of triplet of Hilbert spaces, one obtained in [4] and called triplet of closely embedded Hilbert spaces and the other due to Yu.M. Berezanskii [1] and called generalised triplet of Hilbert spaces. In this respect, the two concepts share some common traits, one of the most interesting being the symmetry, see Proposition 5.3 in [4] and Corollary 3.1. Also, in Example 3.1 and Example 3.2 we show that for the case of two of the toy models, weighted L^2 spaces and Dirichlet type spaces on the unit polydisc, that we have used in order to derive the axiomatisation of triplets of closely embedded Hilbert spaces, these two concepts coincide.

On the other hand, the two generalisations of triplets of Hilbert spaces are rather different in nature, when considered in the abstract sense. The main results of this note are Theorem 3.1 , Theorem 3.2, and Theorem 3.3, that show when they coincide, when they are different, and when starting from one of them one can naturally produce the other one that essentially or fully coincides.

2. TRIPLETS OF CLOSELY EMBEDDED HILBERT SPACES

A Hilbert space $\mathscr{H}_+$ is called *closely embedded* in the Hilbert space $\mathscr{H}$ if:

(ceh1) There exists a linear manifold $\mathscr{D} \subseteq \mathscr{H}_+ \cap \mathscr{H}$ that is dense in $\mathscr{H}_+$.

(ceh2) The embedding operator j_+ with domain $\mathscr{D}$ is closed, as an operator $\mathscr{H}_+ \to \mathscr{H}$.

More precisely, axiom (ceh1) means that on $\mathscr{D}$ the algebraic structures of $\mathscr{H}_+$ and $\mathscr{H}$ agree, while the meaning of the axiom (ceh2) is that the embedding j_+, defined by $j_+x = x$ for all $x \in \mathscr{D} \subseteq \mathscr{H}_+$, is a closed operator when considered as an operator from $\mathscr{H}_+$ to $\mathscr{H}$. In case $\mathscr{H}_+ \subseteq \mathscr{H}$ and the embedding operator $j_+ \colon \mathscr{H}_+ \to \mathscr{H}$ is continuous, one says that $\mathscr{H}_+$ is *continuously embedded* in $\mathscr{H}$. The operator $A = j_+ j_+^*$ is called the *kernel operator* of the closely embedded Hilbert space $\mathscr{H}_+$ with respect to $\mathscr{H}$.

Given a linear operator T defined on a linear submanifold of $\mathscr{H}$ and valued in $\mathscr{G}$, for two Hilbert spaces $\mathscr{H}$ and $\mathscr{G}$, such that its null space $\mathrm{Null}(T)$ is a closed subspace of $\mathscr{H}$, on the linear manifold $\mathrm{Dom}(T) \ominus \mathrm{Null}(T)$ we consider the norm

$$|x|_T := \|Tx\|_{\mathscr{G}}, \quad x \in \mathrm{Dom}(T) \ominus \mathrm{Null}(T), \tag{2.1}$$

and let $\mathscr{D}(T)$ be the Hilbert space completion of the pre-Hilbert space $\mathrm{Dom}(T) \ominus \mathrm{Null}(T)$, with respect to the norm $|\cdot|_T$ associated to the inner product $(\cdot,\cdot)_T$,

$$(x,y)_T = \langle Tx, Ty\rangle_{\mathscr{G}}, \quad x,y \in \mathrm{Dom}(T) \ominus \mathrm{Null}(T). \tag{2.2}$$

We consider the operator i_T, as an operator defined in $\mathscr{D}(T)$ and valued in $\mathscr{H}$, as follows:

$$i_T x := x, \quad x \in \mathrm{Dom}(i_T) = \mathrm{Dom}(T) \ominus \mathrm{Null}(T). \tag{2.3}$$

The operator i_T is closed if and only if T is a closed operator, cf. Lemma 3.1 in [4]. In addition, the construction of $\mathscr{D}(T)$ is actually a renorming process, more precisely, the operator Ti_T admits a unique isometric extension $\widehat{T} \colon \mathscr{D}(T) \to \mathscr{G}$, cf. Proposition 3.2 in [4].

Let now T be a linear operator acting from a Hilbert space $\mathscr{G}$ to another Hilbert space $\mathscr{H}$ and such that its null space $\mathrm{Null}(T)$ is closed. A pre-Hilbert space structure on $\mathrm{Ran}(T)$ is introduced by the positive definite inner product $\langle\cdot,\cdot\rangle_T$ defined by

$$\langle u,v\rangle_T = \langle x,y\rangle_{\mathscr{G}}, \tag{2.4}$$

for all $u = Tx$, $v = Ty$, $x,y \in \mathrm{Dom}(T) \ominus \mathrm{Null}(T)$. Let $\mathscr{R}(T)$ be the completion of the pre-Hilbert space $\mathrm{Ran}(T)$ with respect to the corresponding norm $\|\cdot\|_T$, where $\|u\|_T^2 = \langle u,u\rangle_T$, for $u \in \mathrm{Ran}(T)$. The inner product and the norm on $\mathscr{R}(T)$ are denoted by $\langle\cdot,\cdot\rangle_T$ and, respectively, $\|\cdot\|_T$ throughout. Consider the *embedding operator* $j_T \colon \mathrm{Dom}(j_T)(\subseteq \mathscr{R}(T)) \to \mathscr{H}$ with domain $\mathrm{Dom}(j_T) = \mathrm{Ran}(T)$ defined by

$$j_T u = u, \quad u \in \mathrm{Dom}(j_T) = \mathrm{Ran}(T). \tag{2.5}$$

By definition, $(\mathscr{H}_+;\mathscr{H}_0;\mathscr{H}_-)$ is called a *triplet of closely embedded Hilbert spaces* if:

(th1) $\mathscr{H}_+$ is a Hilbert space closely embedded in the Hilbert space $\mathscr{H}_0$, with the closed embedding denoted by j_+, and such that $\mathrm{Ran}(j_+)$ is dense in $\mathscr{H}_0$.

(th2) $\mathscr{H}_0$ is closely embedded in the Hilbert space $\mathscr{H}_-$, with the closed embedding denoted by j_-, and such that $\mathrm{Ran}(j_-)$ is dense in $\mathscr{H}_-$.

(th3) $\mathrm{Dom}(j_+^*) \subseteq \mathrm{Dom}(j_-)$ and for every vector $y \in \mathrm{Dom}(j_-) \subseteq \mathscr{H}_0$ we have

$$\|y\|_- = \sup\Big\{ \frac{|\langle x,y\rangle_{\mathscr{H}_0}|}{\|x\|_+} \mid x \in \mathrm{Dom}(j_+),\ x \neq 0 \Big\}. \tag{2.6}$$

By axiom (th3), it follows that actually the inclusion in (2.6) is an equality

$$\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-). \tag{2.7}$$

Given three Hilbert spaces $\mathscr{H}_+$, $\mathscr{H}_0$, and $\mathscr{H}_-$, $(\mathscr{H}_+;\mathscr{H}_0;\mathscr{H}_-)$ makes a triplet of closely embedded Hilbert spaces if and only the axioms (th1), (th2), and

(th3)′ $\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-)$ and $\|j_- y\|_- = \|j_+^* y\|_+$, for all $y \in \operatorname{Dom}(j_-)$.

hold.

The concept of a triplet of closely embedded Hilbert spaces was obtained in [4] as a consequence of a model, starting from a positive selfadjoint operator H in a Hilbert space $\mathscr{H}$ with trivial kernel, and a factorisation $H = T^*T$, with T a closed operator densely defined in $\mathscr{H}$ having trivial kernel and dense range in the Hilbert space $\mathscr{G}$, and based on the spaces of type $\mathscr{D}(T)$ and $\mathscr{R}(T)$. Under these assumptions, $(\mathscr{D}(T);\mathscr{H};\mathscr{R}(T^*))$ is a triplet of closely embedded Hilbert spaces with some additional remarkable properties, see Theorem 4.1 in [4]. On the other hand, the properties of the triplet $(\mathscr{D}(T);\mathscr{H};\mathscr{R}(T^*))$ can be proven for any other triplet of closely embedded Hilbert spaces, see Theorem 5.1 in [4]. The fact that there is a rather general model for triplets of closely embedded Hilbert spaces can be used to prove certain existence and uniqueness results, see Theorem 5.2 in [4]. Another consequence of the existence of the model is a certain ”left-right” symmetry, see Proposition 5.3 in [4].

3. MAIN RESULTS

3.1. Generalised Triplets in the Sense of Berezanskii

As defined in [1] at page 57, $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ is called a *generalised triplet* of Hilbert spaces if:

(gt1) $\mathscr{D} = \mathscr{H} \cap \mathscr{H}_0 \cap \mathscr{H}'$ is a linear subspace dense in each of the Hilbert spaces $\mathscr{H}$, $\mathscr{H}_0$, $\mathscr{H}'$.

(gt2) The sesquilinear form $b(\varphi,\psi) = \langle \varphi,\psi\rangle_{\mathscr{H}_0}$, $\varphi,\psi \in \mathscr{D}$, has the property

$$|b(\varphi,u)| \leq \|\varphi\|_{\mathscr{H}'} \|u\|_{\mathscr{H}}, \quad \varphi,u \in \mathscr{D},$$

and hence it can be uniquely extended to a continuous sesquilinear form $\mathscr{H}' \times \mathscr{H} \ni (\varphi,v) \mapsto b(\varphi,v) \in \mathbb{C}$.

(gt3) For each $u \in \mathscr{H}$ there exists a unique vector $\varphi_u \in \mathscr{H}'$ such that $\langle u,v\rangle_{\mathscr{H}} = b(\varphi_u,v)$, for all $v \in \mathscr{H}$.

It is preferable to reformulate this definition in operator theoretical terms.

LEMMA 3.1 ([1], page 58). *Let $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}'$ be Hilbert spaces. Then $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ is a generalised triplet of Hilbert spaces if and only if* (gt1) *holds and there exists $B\colon \mathscr{H}' \to \mathscr{H}$, a contractive and boundedly invertible operator, such that*

$$\langle \varphi,u\rangle_{\mathscr{H}_0} = \langle B\varphi,u\rangle_{\mathscr{H}}, \quad \varphi,u \in \mathscr{D}. \tag{3.1}$$

In addition, the operator B is uniquely determined, subject to these properties.

As a consequence, it can be shown that the concept of generalised triplet of Hilbert spaces has a property of symmetry similar to that of the concept of triplet of closely embedded Hilbert spaces.

COROLLARY 3.1 ([1], Theorem 1.2.10). *If $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ is a generalised triplet of Hilbert spaces, then the same is true for $(\mathscr{H}';\mathscr{H}_0;\mathscr{H})$.*

3.2. Starting with a Triplet of Closely Embedded Hilbert Space

We first investigate the possibility of making a generalised triplet from a triplet of closely embedded Hilbert spaces.

THEOREM 3.1. *Let $(\mathscr{H}_+;\mathscr{H}_0;\mathscr{H}_-)$ be a triplet of closely embedded Hilbert spaces and let $\mathscr{D}=\mathscr{H}_+\cap\mathscr{H}_0\cap\mathscr{H}_-$. Then, $(\mathscr{H}_+;\mathscr{H}_0;\mathscr{H}_-)$ is a generalised triplet of Hilbert spaces if and only if one, hence all, of the following mutually equivalent conditions holds:*

(a) *The sesquilinear form $(\mathscr{D};\|\cdot\|_-)\times(\mathscr{D};\|\cdot\|_+)\ni(\varphi,u)\mapsto\langle\varphi,u\rangle_{\mathscr{H}_0}\in\mathbb{C}$ is separately continuous.*

(b) *The sesquilinear form $(\mathscr{D};\|\cdot\|_-)\times(\mathscr{D}:\|\cdot\|_+)\ni(\varphi,u)\mapsto\langle\varphi,u\rangle_{\mathscr{H}_0}\in\mathbb{C}$ is jointly continuous.*

(c) $|\langle\varphi,u\rangle_{\mathscr{H}_0}|\leq\|\varphi\|_{\mathscr{H}_-}\|u\|_{\mathscr{H}_+}$ *for all $\varphi,u\in\mathscr{D}$.*

Proof. Let $(\mathscr{H}_+;\mathscr{H}_0;\mathscr{H}_-)$ be a triplet of closely embedded Hilbert spaces and let $\mathscr{D}=\mathscr{H}_+\cap\mathscr{H}_0\cap\mathscr{H}_-$. In order to prove that the axiom (gt1) holds, we first prove that $\mathscr{D}$ is dense in each of $\mathscr{H}_+$ and $\mathscr{H}_0$. To see this, we first observe that

$$\begin{aligned}\operatorname{Dom}(j_+^*j_+)&=\{u\in\operatorname{Dom}(j_+)\mid j_+u\in\operatorname{Dom}(j_+^*)\}\\&=\operatorname{Dom}(j_+)\cap\operatorname{Dom}(j_+^*),\ \text{since } j_+u=u \text{ for all } u\in\operatorname{Dom}(j_+)\\&=\operatorname{Dom}(j_+)\cap\operatorname{Dom}(j_-),\ \text{since } \operatorname{Dom}(j_+^*)=\operatorname{Dom}(j_-), \text{ see (2.7)},\\&=\operatorname{Ran}(j_+)\cap\operatorname{Ran}(j_-),\ \text{since } j_+ \text{ and } j_- \text{ are identity operators on their domains.}\end{aligned}$$

Thus, $\operatorname{Dom}(j_+^*j_+)$ is a subspace of each of the spaces $\mathscr{H}_+$, $\mathscr{H}_0$, and $\mathscr{H}_-$, hence, in particular, it is a subspace of $\mathscr{D}$. On the other hand, since $\operatorname{Dom}(j_+^*j_+)$ is a core for j_+, for any $u\in\operatorname{Dom}(j_+)$ there exists a sequence $(u_n)_n$ of vectors in $\operatorname{Dom}(j_+^*j_+)$ such that $\|u_n-u\|_{\mathscr{H}_+}\to 0$ and $\|u_n-u\|_{\mathscr{H}_0}\to 0$ as $n\to\infty$, hence, since $\operatorname{Dom}(j_+)$ is dense in $\mathscr{H}_+$ and $\operatorname{Ran}(j_+)$ is dense in $\mathscr{H}_0$, it follows that $\operatorname{Dom}(j_+^*j_+)$ in dense in each of $\mathscr{H}_+$ and $\mathscr{H}_0$. In particular, $\mathscr{D}$ is dense in each of $\mathscr{H}_+$ and $\mathscr{H}_0$.

In order to finish proving that the axiom (gt1) holds, it remains to prove that $\mathscr{D}$ is dense in $\mathscr{H}_-$ as well. To this end, by the symmetry property as in [4], Proposition 5.3, $(\mathscr{H}_-;\mathscr{H}_0;\mathscr{H}_+)$ is a triplet of closely embedded Hilbert spaces as well and, hence, using the fact just proven that $\mathscr{D}$, whose definition does not depend on the order in which we consider the spaces, is dense in the leftmost component of the triplet, it follows that $\mathscr{D}$ is dense in $\mathscr{H}_-$ as well.

In order to complete the proof, by Lemma 3.1 it is sufficient to prove that there exists a contractive and boundedly invertible operator $B\colon\mathscr{H}_-\to\mathscr{H}_+$ such that (3.1) holds. Indeed, for arbitrary $\varphi\in\operatorname{Dom}(j_+^*)=\operatorname{Dom}(j_-)$ and $u\in\operatorname{Dom}(j_+)$ we have

$$\langle\varphi,u\rangle_{\mathscr{H}_0}=\langle\varphi,j_+u\rangle_{\mathscr{H}_0}=\langle j_+^*\varphi,u\rangle_{\mathscr{H}_+}=\langle V\varphi,u\rangle_{\mathscr{H}_+},\tag{3.2}$$

where $V=j_+^*$ but considered as an operator defined in $\mathscr{H}_-$ and valued in $\mathscr{H}_+$. By [4], Theorem 5.1, there exists a unique unitary operator $\widetilde{V}\colon\mathscr{H}_-\to\mathscr{H}_+$ that extends the operator V. Letting $B=\widetilde{V}$, since B is unitary if follows that it is contractive and boundedly invertible. In particular, (3.2) can be rewritten as

$$\langle\varphi,u\rangle_{\mathscr{H}_0}=\langle B\varphi,u\rangle_{\mathscr{H}_+},\quad\varphi\in\operatorname{Dom}(j_-),\ u\in\operatorname{Dom}(j_+).\tag{3.3}$$

We assume now that the condition (a) holds and prove that (3.3) holds for all $\varphi,u\in\mathscr{D}$, that is, (3.1). To this end, fix $\varphi\in\operatorname{Dom}(j_-)$ for the moment and consider an arbitrary vector $u\in\mathscr{D}$. As proven before, $\operatorname{Dom}(j_+^*j_+)$ lies in $\mathscr{D}$ and is dense with respect to the norm $\|\cdot\|_{\mathscr{H}_+}$, hence, there exists a sequence $(u_n)_n$ of vectors in $\operatorname{Dom}(j_+^*j_+)$ such that $\|u_n-u\|_{\mathscr{H}_+}\to 0$ as $n\to\infty$. By condition (a), $\langle\varphi,u_n\rangle_{\mathscr{H}_0}\to\langle\varphi,u\rangle_{\mathscr{H}_0}$ as $n\to\infty$, and by (3.3) we have

$$\langle\varphi,u_n\rangle_{\mathscr{H}_0}=\langle B\varphi,u_n\rangle_{\mathscr{H}_+},\quad n\in\mathbb{N},$$

hence, we can pass to the limit as $n \to \infty$ to obtain that (3.3) holds for all $u \in \mathscr{D}$ and all $\varphi \in \mathrm{Dom}(j_-)$. Next, a similar reasoning, with fixing $u \in \mathscr{D}$ and approximating $\varphi \in \mathscr{D}$ accordingly, shows that (3.3) holds for all $u, \varphi \in \mathscr{D}$.

It is clear that (c)⇒(b)⇒(a). Observe that we have just proven before that (a)⇒(c), hence the conditions (a), (b), and (c) are mutually equivalent.

The converse implication is clear. □

***Example* 3.1**. *Weighted L^2 Spaces.* Let $(X;\mathfrak{A})$ be a measurable space on which we consider a σ-finite measure μ. A function w defined on X is called a *weight* with respect to the measure space $(X;\mathfrak{A};\mu)$ if it is measurable and $0 < w(x) < \infty$, for μ-almost all $x \in X$. Note that $\mathscr{W}(X;\mu)$, the collection of weights with respect to $(X;\mathfrak{A};\mu)$, is a multiplicative unital group. For an arbitrary $w \in \mathscr{W}(X;\mu)$, consider the measure ν whose Radon-Nikodym derivative with respect to μ is w, denoted $\mathrm{d}\nu = w\,\mathrm{d}\mu$, that is, for any $E \in \mathfrak{A}$ we have $\nu(E) = \int_E w\,\mathrm{d}\mu$. It is easy to seee, e.g. see [3], that ν is always σ-finite.

In [3], Theorem 2.1, it is proven that $(L^2_w(X;\mu); L^2(X;\mu); L^2_{w^{-1}}(X;\mu))$ is a triplet of closely embedded Hilbert spaces, provided that w is a weight on the σ-finite measure space $(X;\mathfrak{A};\mu)$. More precisely, the closed embeddings $j_\pm$ of $L^2_w(X;\mu)$ in $L^2(X;\mu)$ and of $L^2(X;\mu)$ in $L^2_{w^{-1}}(X;\mu)$ have maximal domains $L^2_w(X;\mu) \cap L^2(X;\mu)$ and, respectively, $L^2(X;\mu) \cap L^2_{w^{-1}}(X;\mu)$. It is a routine exercise to check that, if $\varphi, u \in L^2(X;\mu) \cap L^2_w(X;\mu) \cap L^2_{w^{-1}}(X,\mu)$ then

$$|\langle \varphi, u \rangle_{L^2(X;\mu)}| \leq \|\varphi\|_{L^2_w(X;\mu)} \|u\|_{L^2_{w^{-1}}(X;\mu)},$$

hence, $(L^2_w(X;\mu); L^2(X;\mu); L^2_{w^{-1}}(X;\mu))$ is a generalised triplet of Hilbert spaces as well, by Theorem 3.1. Observe that, in the proof of Theorem 2.1 in [3], it was directly proven that $L^2_w(X;\mu) \cap L^2(X;\mu) \cap L^2_{w^{-1}}(X;\mu)$ is dense in each of the spaces $L^2_w(X;\mu)$, $L^2(X;\mu)$, and $L^2_{w^{-1}}(X;\mu)$.

***Example* 3.2**. *Dirichlet Type Spaces.* For a fixed natural number N consider the unit polydisc $\mathbb{D}^N = \mathbb{D} \times \cdots \times \mathbb{D}$, the direct product of N copies of the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. We consider $H(\mathbb{D}^N)$ the algebra of functions holomorphic in the polydisc, that is, the collection of all functions $f \colon \mathbb{D}^N \to \mathbb{C}$ that are holomorphic in each variable, equivalently, there exists $(a_k)_{k \in \mathbb{Z}^N_+}$ with the property that

$$f(z) = \sum_{k \in \mathbb{Z}^N_+} a_k z^k, \quad z \in \mathbb{D}^N, \tag{3.4}$$

where the series converges absolutely and uniformly on any compact subset in $\mathbb{D}^N$. Here and in the sequel, for any multi-index $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N_+$ and any $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ we let $z^k = z_1^{k_1} \cdots z_N^{k_N}$.

Let $\alpha \in \mathbb{R}^N$ be fixed. The *Dirichlet type space* $\mathscr{D}_\alpha$, see [8] and [6], is defined as the space of all functions $f \in H(\mathbb{D}^N)$ with representation (3.4) subject to the condition

$$\sum_{k \in \mathbb{Z}^N_+} (k+1)^\alpha |a_k|^2 < \infty,$$

where, $(k+1)^\alpha = (k_1+1)^{\alpha_1} \cdots (k_N+1)^{\alpha_N}$. The linear space $\mathscr{D}_\alpha$ is naturally organized as a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\alpha$

$$\langle f, g \rangle_\alpha = \sum_{k \in \mathbb{Z}^N_+} (k+1)^\alpha a_k \overline{b_k},$$

where f has representation (3.4) and similarly $g(z) = \sum_{k \in \mathbb{Z}^N_+} b_k z^k$, for all $z \in \mathbb{D}^N$, and norm $\|\cdot\|_\alpha$ defined by

$$\|f\|^2_\alpha = \sum_{k \in \mathbb{Z}^N_+} (k+1)^\alpha |a_k|^2.$$

It is proven in [3], Theorem 3.1, that, if $\alpha, \beta \in \mathbb{R}^N$ are arbitrary multi-indices, then $(\mathscr{D}_\beta; \mathscr{D}_\alpha; \mathscr{D}_{2\alpha-\beta})$ is a triplet of closely embedded Hilbert spaces. It is a simple exercise to check that

$$|\langle f,g\rangle_\alpha| \leq \|f\|_{2\alpha-\beta}\|g\|_\beta,$$

whenever $f,g\in\mathscr{D}_\beta\cap\mathscr{D}_\alpha\cap\mathscr{D}_{2\alpha-\beta}$, hence, by Theorem 3.1 $(\mathscr{D}_\beta;\mathscr{D}_\alpha;\mathscr{D}_{2\alpha-\beta})$ is a generalised triplet of Hilbert spaces as well. Note that, in this particular case, $\mathscr{D}_\beta\cap\mathscr{D}_\alpha\cap\mathscr{D}_{2\alpha-\beta}$ contains $\mathscr{P}_N$, the linear space of polynomial functions in N complex variables, that is dense in each of the Dirichlet type spaces $\mathscr{D}_\beta$, $\mathscr{D}_\alpha$, and $\mathscr{D}_{2\alpha-\beta}$.

3.3. Starting with a Generalised Triplet

We now consider a generalised triplet of Hilbert spaces $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$. First, we investigate the possibility of making a triplet of closely embedded Hilbert spaces out of it, in a natural fashion, and in such a way that it "essentially" coincides with it. Let $\mathscr{D}=\mathscr{H}\cap\mathscr{H}_0\cap\mathscr{H}'$ be the linear subspace that is dense in each of $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}'$ and, in view of Lemma 3.1, consider the contractive linear operator $B\colon\mathscr{H}'\to\mathscr{H}$ that is boundedly invertible and such that (3.1) holds.

Let $j_{+,0}\colon \operatorname{Dom}(j_{+,0})(\subseteq\mathscr{H})\to\mathscr{H}_0$ be the linear operator with domain $\operatorname{Dom}(j_{+,0}):=\mathscr{D}$, considered as a subspace of $\mathscr{H}$, the embedding of $\mathscr{D}$ in $\mathscr{H}_0$, that is, $j_{+,0}u=u$ for all $u\in\mathscr{D}$. We observe that, for any $u,\varphi\in\mathscr{D}$, we have

$$\langle\varphi,j_{+,0}u\rangle_{\mathscr{H}_0}=\langle\varphi,u\rangle_{\mathscr{H}_0}=\langle B\varphi,u\rangle_{\mathscr{H}}, \tag{3.5}$$

hence $\mathscr{D}\subseteq\operatorname{Dom}(j_{+,0}^*)$ and $B|\mathscr{D}=j_{+,0}^*|\mathscr{D}$. In particular, $j_{+,0}^*$ is defined on a subspace dense in $\mathscr{H}_0$, hence $j_{+,0}$ is closable. Let $T\in\mathscr{C}(\mathscr{H}_0,\mathscr{H})$ be the closure of the operator $j_{+,0}$. Thus, $\mathscr{D}\subseteq\operatorname{Dom}(T)$, $Tu=u$ for all $u\in\mathscr{D}$, and $j_{+,0}^*=T^*$, in particular, $T^*|\mathscr{D}=B|\mathscr{D}$. Therefore,

$$\operatorname{Null}(T)=\mathscr{H}\ominus\overline{\operatorname{Ran}(T^*)}\subseteq\mathscr{H}\ominus\overline{B\mathscr{D}}=\mathscr{H}\ominus\mathscr{H}=0,$$

where we have taken into account that B is boundedly invertible and $\mathscr{D}$ is dense in $\mathscr{H}'$. This shows that T is one-to-one. Since $T\mathscr{D}\supseteq j_{+,0}\mathscr{D}=\mathscr{D}$, which is dense in $\mathscr{H}_0$ as well, it follows that T has dense range in $\mathscr{H}_0$.

We can now consider the triplet of closely embedded Hilbert spaces $(\mathscr{R}(T);\mathscr{H}_0;\mathscr{D}(T^*))$, where j_T is the closed embedding of $\mathscr{R}(T)$ in $\mathscr{H}_0$ and $i_{T^*}^{-1}$ is the closed embedding of $\mathscr{H}_0$ in $\mathscr{D}(T^*)$. In the following we show that the subspace $\mathscr{D}$ is densely contained in $\mathscr{R}(T)$, that on $\mathscr{D}$ the inner product of $\mathscr{H}$ coincides with that of $\mathscr{R}(T)$, and that on $\mathscr{D}$ the topological structure of $\mathscr{H}'$ coincides with that of $\mathscr{D}(T^*)$.

Since $\mathscr{D}$ is a subspace of $\operatorname{Dom}(T)$ and T acts on $\mathscr{D}$ like the identity operator, it follows that $\mathscr{D}$ is a subspace of $\operatorname{Ran}(T)$, hence a subspace of $\mathscr{R}(T)$. In addition, taking into account that T is the closure of the embedding operator $j_{+,0}$, for any vector $x\in\operatorname{Dom}(T)$ there exists a sequence $(x_n)_n$ of vectors in $\mathscr{D}$ such that $\|x-x_n\|_{\mathscr{H}}\to0$ and $\|j_{+,0}x_n-Tx\|_{\mathscr{H}_0}\to0$, as $n\to\infty$. Hence

$$\|x_n-Tx\|_T=\|x_n-x\|_{\mathscr{H}}\to0,\ \text{as } n\to\infty,$$

which shows that $\mathscr{D}$ is dense in $\operatorname{Ran}(T)$ with respect to the norm $\|\cdot\|_T$, see (2.4). Since $\operatorname{Ran}(T)$ is dense in $\mathscr{R}(T)$ with respect to the norm $\|\cdot\|_T$, it follows that $\mathscr{D}$ is dense in $\mathscr{R}(T)$. On the other hand, for any vector $x\in\mathscr{D}$ we have

$$\|x\|_T=\|Tx\|_T=\|x\|_{\mathscr{H}},$$

that is, on $\mathscr{D}$ the norms of the two Hilbert spaces $\mathscr{H}$ and $\mathscr{R}(T)$ coincide.

On the other hand, as a consequence of (3.5) and taking into account that $T^*=j_{+,0}^*$, it follows that $\mathscr{D}\subseteq\operatorname{Dom}(T^*)$ and $T^*|\mathscr{D}=B|\mathscr{D}$. Thus, $\mathscr{D}$ is a subspace of $\mathscr{D}(T^*)$ and

$$|x|_{T^*}=\|T^*x\|_{\mathscr{H}}=\|Bx\|_{\mathscr{H}},\quad x\in\mathscr{D}. \tag{3.6}$$

Taking into account that $B\colon\mathscr{H}'\to\mathscr{H}$ is bounded and boundedly invertible, this implies that on $\mathscr{D}$ the norms $|\cdot|_{T^*}$ and $\|\cdot\|_{\mathscr{H}'}$ are equivalent.

We have proven the following

THEOREM 3.2. *Let $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ be a generalised triplet of Hilbert spaces, let $\mathscr{D}=\mathscr{H}\cap\mathscr{H}_0\cap\mathscr{H}'$ be the linear subspace which is dense in each of $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}'$, and let $B\colon \mathscr{H}'\to\mathscr{H}$ denote the contractive linear operator that is boundedly invertible and such that* (3.1) *holds.*

(1) *Let $j_{+,0}$ denote the linear operator with domain* $\operatorname{Dom}(j_{+,0})=\mathscr{D}$*, considered as a subspace of $\mathscr{H}$, and with codomain in $\mathscr{H}_0$, defined by $j_{+,0}u=u$ for all $u\in\mathscr{D}$. Then $j_{+,0}$ is closable.*

(2) *Let T denote the closure of $j_{+,0}$. Then $T\in\mathscr{C}(\mathscr{H},\mathscr{H}_0)$, is one-to-one, has dense range, and $T^*|\mathscr{D}=B|\mathscr{D}$.*

(3) *The triplet of closely embedded Hilbert spaces $(\mathscr{R}(T);\mathscr{H}_0;\mathscr{D}(T^*))$, where j_T is the closed embedding of $\mathscr{R}(T)$ in $\mathscr{H}_0$ and $i_{T^*}^{-1}$ is the closed embedding of $\mathscr{H}_0$ in $\mathscr{D}(T^*)$, has the following properties:*

(i) *$\mathscr{D}$ is densely contained in $\mathscr{R}(T)$ and on $\mathscr{D}$ the inner product of $\mathscr{H}$ coincides with that of $\mathscr{R}(T)$;*

(ii) *$\mathscr{D}$ is a subspace of $\mathscr{D}(T^*)$ and on $\mathscr{D}$ the norms $\|\cdot\|_{\mathscr{H}'}$ and $|\cdot|_{T^*}$ are equivalent.*

The triplet of closely embedded Hilbert spaces $(\mathscr{R}(T);\mathscr{H}_0;\mathscr{D}(T^*))$ constructed out of the generalised triplet of Hilbert spaces $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ as in Theorem 3.2, "essentially" coincides with the triplet of generalised Hilbert spaces $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ on the linear manifold $\mathscr{D}$, that is dense in each of the spaces $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}'$, modulo a norm equivalent with $\|\cdot\|_{\mathscr{H}'}$. If we want the generalised triplet $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ to be a triplet of closely embedded Hilbert spaces itself, this depends on a rather general question of when a closed embedding can be obtained from an unbounded embedding by taking its closure. We record this fact in the following

***Remark* 3.1**. Let $\mathscr{H}$ and $\mathscr{G}$ be two Hilbert spaces such that there exists $\mathscr{D}_0$ a linear manifold of both $\mathscr{H}$ and $\mathscr{G}$ that is dense in $\mathscr{G}$ and let the embedding operator $j_0\colon \mathscr{D}_0\to\mathscr{H}$ be defined by $j_0u=u$ for all $u\in\mathscr{D}_0$. Then, the following assertions are equivalent:

(a) j_0 is closable, as an operator defined in $\mathscr{G}$ and valued in $\mathscr{H}$, and the closure $j=\overline{j_0}$ is a closed embedding of $\mathscr{G}$ in $\mathscr{H}$.

(b) For every sequence (u_n) of vectors in $\mathscr{D}_0$ that is Cauchy with respect to both norms $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{G}}$, there exists $u\in\mathscr{H}\cap\mathscr{G}$ (of course, unique) such that $\|u_n-u\|_{\mathscr{H}}\to 0$ and $\|u_n-u\|_{\mathscr{G}}\to 0$ as $n\to\infty$.

We can now approach the main question of this section referring to characterisations of those generalised triplets of Hilbert spaces that are also triplets of closely embedded Hilbert spaces.

THEOREM 3.3. *Let $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ be a generalised triplet of Hilbert spaces, let $\mathscr{D}=\mathscr{H}\cap\mathscr{H}_0\cap\mathscr{H}'$ be the linear subspace which is dense in each of $\mathscr{H}$, $\mathscr{H}_0$, and $\mathscr{H}'$, and let $B\colon \mathscr{H}'\to\mathscr{H}$ denote the contractive linear operator that is boundedly invertible and such that* (3.1) *holds. Then, $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ is a triplet of closely embedded Hilbert spaces, modulo a renorming of $\mathscr{H}'$ with an equivalent norm, if and only if the following three conditions hold:*

(i) *For any sequence (u_n) of vectors in $\mathscr{D}$ that is Cauchy with respect to both norms $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}_0}$, there exists $u\in\mathscr{H}\cap\mathscr{H}_0$ such that $\|u_n-u\|_{\mathscr{H}}\to 0$ and $\|u_n-u\|_{\mathscr{H}_0}\to 0$ as $n\to\infty$.*

(ii) *For any sequence (φ_n) of vectors in $\mathscr{D}$ that is Cauchy with respect to both norms $\|\cdot\|_{\mathscr{H}'}$ and $\|\cdot\|_{\mathscr{H}_0}$, there exists $\varphi\in\mathscr{H}'\cap\mathscr{H}_0$ such that $\|\varphi_n-\varphi\|_{\mathscr{H}'}\to 0$ and $\|\varphi_n-\varphi\|_{\mathscr{H}_0}\to 0$ as $n\to\infty$.*

(iii) *For every vector $\varphi\in\mathscr{H}_0$ with the property that the linear functional $\mathscr{D}\ni u\mapsto\langle y,\varphi\rangle_{\mathscr{H}_0}$ is bounded with respect to the norm $\|\cdot\|_{\mathscr{H}}$, there exists a sequence $(\varphi_n)_n$ of vectors in $\mathscr{D}$ such that $\|\varphi_n-\varphi\|_{\mathscr{H}_0}\to 0$ and $\|\varphi_n-\varphi\|_{\mathscr{H}'}$, as $n\to\infty$.*

Proof. We first assume that the generalised triplet of Hilbert spaces $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ satisfies all conditions (i)–(iii). Consider the operator $j_{+,0}$ with domain $\operatorname{Dom}(j_{+,0})=\mathscr{D}$, viewed as a subspace of $\mathscr{H}$, as the embedding in $\mathscr{H}_0$, that is, $j_{+,0}u=u$ for all $u\in\mathscr{D}$. By (3.5), $\mathscr{D}\subseteq\operatorname{Dom}(j_{+,0}^*)$ and hence $j_{+,0}$ is closable. By condition (i), see Remark 3.1, it follows that the closure j_+ of $j_{+,0}$ is an embedding, that is, for all $u\in\operatorname{Dom}(j_+)$ we have $j_+u=u$. With notation as in Theorem 3.2, this means that $T=j_+$ is a closed embedding of $\mathscr{H}$ in $\mathscr{H}_0$. In addition, by condition (i) it also follows that

$$\langle\varphi,u\rangle_{\mathscr{H}_0}=\langle B\varphi,u\rangle_{\mathscr{H}},\quad u\in\operatorname{Dom}(j_+),\ \varphi\in\mathscr{D}. \tag{3.7}$$

Further on, by changing the norm $\|\cdot\|_{\mathscr{H}'}$ with an equivalent norm, without loss of generality we can assume that the operator $B\colon \mathscr{H}' \to \mathscr{H}$ is unitary. We consider the embedding operator $i_{-,0} : \mathscr{D}(\subseteq \mathscr{H}') \to \mathscr{H}_0$, with domain $\mathscr{D} \subseteq \mathscr{H}'$ and range in $\mathscr{H}_0$, defined by $i_{-,0}\varphi = \varphi$ for all $\varphi \in \mathscr{D}$. We observe that, for any $u, \varphi \in \mathscr{D}$ we have

$$\langle i_{-,0}\varphi, u\rangle_{\mathscr{H}_0} = \langle \varphi, u\rangle_{\mathscr{H}_0} = \langle B\varphi, u\rangle_{\mathscr{H}} = \langle \varphi, B^* u\rangle_{\mathscr{H}'}, \tag{3.8}$$

hence $\mathscr{D} \subseteq \operatorname{Dom}(i_{-,0}^*)$ and $B^*|\mathscr{D} = i_{-,0}^*$. Therefore, $i_{-,0}^*$ is defined on a subspace dense in $\mathscr{H}_0$ and hence $i_{-,0}$ is closable. From condition (ii) it follows that i_-, the closure of the operator $i_{-,0}$, is an embedding, that is, $\operatorname{Dom}(i_-) \subseteq \mathscr{H}_0 \cap \mathscr{H}'$ and $i_-\varphi = \varphi$ for all $\varphi \in \operatorname{Dom}(i_-)$. Clearly, $\operatorname{Dom}(i_-) = \operatorname{Ran}(i_-)$ is dense in both $\mathscr{H}_0$ and $\mathscr{H}'$, hence we can consider $j_- = i_-^{-1}$, which is a closed embedding of $\mathscr{H}_0$ in $\mathscr{H}'$ with dense range. In addition, by condition (ii) and (3.7) it follows that

$$\langle \varphi, u\rangle_{\mathscr{H}_0} = \langle B\varphi, u\rangle_{\mathscr{H}}, \quad u \in \operatorname{Dom}(j_+),\ \varphi \in \operatorname{Dom}(j_-). \tag{3.9}$$

So far, we have shown that the triplet $(\mathscr{H};\mathscr{H}_0;\mathscr{H}')$ satisfies the axioms (th1) and (th2), with respect to the closed embeddings j_+ and j_- defined as before. Recalling that B is unitary, from (3.9) it follows that, for every $\varphi \in \operatorname{Dom}(j_-)$, we have

$$\sup\Big\{\frac{|\langle \varphi, u\rangle_{\mathscr{H}_0}|}{\|u\|_+} \mid u \in \operatorname{Dom}(j_+),\ u \neq 0\Big\} = \|B\varphi\|_{\mathscr{H}} = \|\varphi\|_{\mathscr{H}'},$$

hence (2.6) holds. It only remains to prove that $\operatorname{Dom}(j_+^*) \subseteq \operatorname{Dom}(j_-)$. To this end, let $\varphi \in \operatorname{Dom}(j_+^*)$, hence, the linear functional $\mathscr{D} \ni u \mapsto \langle j_+ u, \varphi\rangle_{\mathscr{H}_0} = \langle u, \varphi\rangle_{\mathscr{H}_0}$ is continuous with respect to the norm $\|\cdot\|_{\mathscr{H}}$. By condition (iii), there exist a sequence $(\varphi_n)_n$ of vectors in $\mathscr{D}$ such that $\|\varphi_n - \varphi\|_{\mathscr{H}_0} \to 0$ and $\|\varphi_n - \varphi\|_{\mathscr{H}'}$, as $n \to \infty$, which means that $\varphi \in \operatorname{Dom}(j_-)$. □

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