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QUANTUM VERSION OF ACCARDI AND BOZEJKO'S UNIVERSAL CONVOLUTION

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Abstract. In their famous work [1], L. Accardi and M. Bożejko have introduced a surprising convolution among the real probability laws with finite moments of any order, by proving this is universal in the sense that any symmetric such probability law is infinitely divisible with respect to it, and a central limit law. We present a quick generalization of their construction to the so-called Jacobi-Szegő distributions introduced by M. Anshelevich and J.D. Williams [6] in the operator-valued non-commutative frame considered by D.-V.Voiculescu [20] or R. Speicher [16] for the free probability theory [19].

Key words: non-crossing partitions, complete positivity, algebraic Jacobi-Szegő parameters, operator-valued quantum probability space or non-commutative distribution, (pre-) Hilbert C*-module.

1. INTRODUCTION

If μ is a probability law on the real line $\mathbb{R}$ with finite moments of every order, then, it is well-known (see, e.g., [5, 8, 17]), μ is associated to two scalar sequences $\{\lambda_n, \alpha_n; n \geq 0\}$ such that $\lambda_n, \alpha_n \in \mathbb{R}$, and $\alpha_n \geq 0$; namely, the Jacobi-Szegő parameters corresponding to μ.

A remarkable concept of convolution among the real probability laws with all moments is defined in [1], in terms of the involved Jacobi-Szegő parameters, and the authors prove this convolution is universal, in the sense that every symmetric such probability law is infinitely divisible and a central limit law with respect to this convolution. Their study emphasized a famous quantum decomposition of any classical random variable with all moments as a sum of creation, annihilation and conservation operators on an one-mode interacting Fock space. The interacting Fock spaces appeared from the stochastic limit of quantum theory applied to the non-relativistic quantum electrodynamics without dipole approximation [3] and generalize some well-known second quantization functors (Boson, Fermion, q -deformed) [3, 4]. For this kind of Fock spaces, the quanta in the n-particle subspace are not independent: they interact in a highly nonlinear way; and the vacuum distribution of the field operators is (not Gaussian, as classically, but) a nonlinear deformation of Wigner semi-circular laws; see [3]. The relevance of these concepts and results is reflected by a diversity of applications in several fields, involving apparently faraway topics,- as quantum stochastic independences, spectral analysis of large graphs or asymptotic representation theory-, illustrated, e.g., by the monograph [10] in theoretical and mathematical physics, or the Springer Brief [14] and the rich literature referenced therein.

Moreover, Accardi and Bożejko observed the so-called (generalized) Gaussianization phenomenon; the fact that, e.g., every moment of a symmetric probability law as above may be expressed in terms of its Jacobi-Szegő parameters involving only non-crossing pair partitions (and singletons, in the non-symmetric case), hence it looks as the moment of a (generalized) Gauss law. See, e.g., [1, Corollary 5.1], for details; and Th. 3.1 (iii) in section 3 below.

A first result of Gaussianization (in Accardi and Bożejko's sense) has been presented in [7]. See also [13], for other results in the symmetric case.

M. Anshelevich and J.D. Williams [6] have extended the notion of real probability law with finite moments of all orders to the operator-valued non-commutative setting considered by D.-V. Voiculescu in [20] (or R. Speicher in [16]) for the free probability theory [19], and ipso facto conferring much depth to the whole theory. In this frame, with an arbitrary algebra B, the Jacobi-Szegő parameters are sequences $\lambda_n \in B$, and linear maps $\alpha_n : B \to B, n \geq 0$; in particular, λ_n, α_n being Hermitian, and, respectively, completely positive, when B is involutive algebra; see, for details, the same section 3.

In the present Note, we generalize the Accardi-Bożejko universal convolution to the operator-valued quantum distributions introduced in [6] and quickly derive (in section 4, Corollaries 4.6-4.7) that any symmetric such distribution is infinitely divisible and a quantum central limit law with respect to this general convolution.

2. PRELIMINARIES

We recall some well-known general notions as in, e.g., [1, 4, 6, 10, 16] or [11], and send to [15, 12] for more information about complete-positivity or the involved (pre-)Hilbert C*-modules.

If S is a finite totally ordered set, we denote by $NC_{1,2}(S)$ the non-crossing partitions of S for which every block has at most two elements; calling blocks the non-void subsets defining a partition (in general). We call pair a block having only two elements; and singleton a block having a single element. For $k, l \in S$, denote by $k \sim_\pi l$ the fact that k and l belong to the same block of a partition π. Remind that a partition π is called non-crossing if there are not $k_1 < l_1 < k_2 < l_2$ in S such that $k_1 \sim_\pi k_2 \nsim_\pi l_1 \sim_\pi l_2$. When π is non-crossing, and V is a block of π, say V is inner, if there exists another block W of π containing V (i.e., there are $k, l \in W$, such that $k < v < l$, for all $v \in V$). For a block V in $\pi \in NC_{1,2}(S)$, the depth in π of V is the number, denoted $d(V, \pi)$, equal to the number of all the blocks of π containing V. If S is a disjoint union of non-empty subsets S_i, and $\pi \in NC_{1,2}(S)$ such that $\pi = \cup \pi_i$, with some $\pi_i \in NC_{1,2}(S_i)$, we write $\pi = \coprod \pi_i$. When S is a set with n elements, abbreviate $NC_{1,2}(S)$ by $NC_{1,2}(n)$.

We consider an involutive algebra as being a (complex) associative algebra with an involution $*$ (i.e. a conjugate linear anti-automorphism). If A is an involutive algebra, an element $a \in A$ is Hermitian if $a = a^*$. The cone A_+ of positive elements in A consists of finite sums $\sum a_i^* a_i$, with $a_i \in A$. Denote by $M_n(A)$ the involutive algebra of $n \times n$ matrices $[a_{ij}]$ with entries from A. When A is C*-algebra, then $A_+ = \{a^* a;\, a \in A\}$; and $M_n(A)$ becomes C*-algebra.

If $Q : A \to B$ is a linear map between involutive algebras, this is Hermitian if $Q(a)^* = Q(a^*)$ for all $a \in A$; it is positive if $Q(A_+) \subset B_+$; and it is completely positive, if all maps $Q_n : M_n(A) \to M_n(B)$ given by $Q_n([a_{ij}]) = [Q(a_{ij})]$, if $[a_{ij}] \in M_n(A)$, are positive. If A and B are involutive algebras, A being unital, any positive linear map $Q : A \to B$ is Hermitian. When $B \subset A$ is an inclusion of (involutive) algebras, a (positive) conditional expectation of A onto B is a (positive) B-B-bimodule map, which is a projection on B. If $B \subset A$ is an inclusion of involutive algebras, but B is C*-algebra, then any positive conditional expectation of A onto B is completely positive.

If B is an involutive algebra, a semi-inner product B-module M is a (right) B-module endowed with a sesquilinear map $<\cdot,\cdot>: M \times M \to B$ linear in its second variable such that

$$<x, yb> = <x, y> b\,;\; <x, y>^* = <y, x>;\; <x, x> \in B_+;\text{ for any } x, y \in M \text{ and } b \in B.$$

We denote by $L(M)$ the set of the adjointable operators on M.

When B is C*-algebra, we remind that a semi-inner product B-module M is a (pre-)Hilbert module if the semi-norm $x \mapsto \| <x, x> \|^{1/2}$ is (in-)complete norm on M.

If $B \subset A$ is an inclusion of algebras, and ϕ is a conditional expectation of A onto B, we consider (A,ϕ,B) as quantum or non-commutative B-probability space, and the elements of A as B-valued quantum random variables, according, e.g., to [19, 20]; see also [16]; and send to these references for more information. For operator-valued distributions in non-commutative context we send to [20] (see also [16]).

We recall the general frame from [20] and include the type of convergence for distributions used by us in the final section.

Let B be a unital algebra (over the complex field ℂ). Let $B < X >$ be the algebra freely generated by B and an indeterminate X. Denote $\Sigma_B := \{\nu : B < X > \to B;\ \nu \text{ conditional expectation onto } B\}$. We regard Σ_B as the set of all possible distributions of random variables in a B-probability space (A,ϕ,B), as above. When B is an involutive algebra, we endow $B < X >$ with the natural involution such that $X^* = X$, and regard $\Sigma_B^+ := \{\nu \in \Sigma_B;\ \nu \text{ positive}\}$ as the set of positive distributions. For $\nu \in \Sigma_B$, quantities as $\nu(bXb_1X...b_jXc)$, with $b,c,b_k \in B$, are interpreted as moments of ν.

If (A,ϕ,B) is such a B-probability space, and $a \in A$ is a random variable, the distribution of a with respect to ϕ is $\phi_a := \phi \circ \tau_a$, where $\tau_a : B < X > \to A$ is the unique homomorphism such that $\tau_a | B = id_B$, and $\tau_a(X) = a$. Thus, quantities as $\phi(bab_1a...b_jac)$, with $b,c,b_k \in B$, are called moments of a, with respect to ϕ.

We may consider in section 4, Corollary 4.7, any Hausdorff topology on Σ_B. When B is a Hausdorff topological algebra, we may use, in particular, the weak convergence for distributions, as in [20, 16]. If $\mu_N, \mu \in \Sigma_B$, for every non-negative integer N, we denote $\lim_{N\to\infty} \mu_N = \mu$, if μ is the limit of $(\mu_N)_N$ in Σ_B.

3. OPERATOR-VALUED QUANTUM DISTRIBUTIONS CORRESPONDING TO ALGEBRAIC JACOBI-SZEGO PARAMETERS

Let μ be a probability law on ℝ with finite moments of all orders. Then remind, by famous classical theorems [5, 8, 17], μ is associated to two scalar sequences $\{\lambda_n, \alpha_n; n \geq 0\}$, where $\lambda_n, \alpha_n \in$ ℝ, and $\alpha_n \geq 0$. These Jacobi-Szegő parameters appear, for example, in the well-known three-term recurrence relations involving the monic orthogonal polynomials with respect to μ.

Moreover, the moment generating function of μ admits a continued fraction expansion of Jacobi-Stieltjes type

$$\sum_{n=0}^{\infty} \mu[x^n] z^n = \cfrac{1}{1 - \lambda_0 z - \cfrac{\alpha_0 z^2}{1 - \lambda_1 z - \cfrac{\alpha_1 z^2}{\dots}}}.$$

The moments of μ are obtained from the Jacobi-Szegő parameters by sums over non-crossing partitions or Motzkin paths; e.g.,

$$\mu[x^n] = \sum_{\pi \in NC_{1,2}(n)} \prod_{\substack{V\in\pi \\ |V|=1}} \lambda_{d(V,\pi)} \cdot \prod_{\substack{V\in\pi \\ |V|=2}} \alpha_{d(V,\pi)},$$

where $NC_{1,2}(n)$ and $d(V,\pi)$ have the meaning before, $|V|$ being the cardinality of the block V (see, e.g., [1, 7, 9, 10, 18] and also [11], for details).

Remind μ may be considered as a (positive) linear functional on the (involutive) algebra $\mathbb{C}[X]$ of polynomials over the complex numbers.

We recall the next basic theorem (see, e.g., [1, 5–8, 17]; and also [9–11, 14, 18] for more information).

THEOREM 3.1. *Consider two scalar sequences* $\{\lambda_n, \alpha_n; n \geq 0\}$, *where* $\lambda_n, \alpha_n \in \mathbb{R}$, *and* $\alpha_n \geq 0$, *and a probability law* μ *on* $\mathbb{R}$ *with finite moments of all orders. The following statements are equivalent.*

(i) *The monic orthogonal polynomials corresponding to* μ *satisfy the three-terms recurrence relations*

$$p_{n+1}(X) = (X - \lambda_n) p_n(X) - \alpha_{n-1} p_{n-1}(X); \quad p_{-1} = 0,\ p_0 = 1.$$

(ii) μ *is the distribution of the Jacobi-Stieltjes type tridiagonal matrix*

$$\begin{pmatrix} \lambda_0 & \alpha_0 & 0 & 0 & \cdots \\ 1 & \lambda_1 & \alpha_1 & 0 & \cdots \\ 0 & 1 & \lambda_2 & \alpha_2 & \cdots \\ 0 & 0 & 1 & \lambda_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with respect to the vector state corresponding to the vacuum $e := (1,0,0,0,...)$.

(iii) *For every* n

$$\mu[x^n] = \sum_{\pi \in NC_{1,2}(n)} \prod_{\substack{V \in \pi \\ |V|=1}} \lambda_{d(V,\pi)} \cdot \prod_{\substack{V \in \pi \\ |V|=2}} \alpha_{d(V,\pi)}.$$

Moreover, μ *has all its moments of odd order null if and only if all* $\lambda_n = 0$; *and* μ *has finite support if and only if some* $\alpha_n = 0$; *in any of* (i)–(iii). □

An operator-valued non-commutative version of the previous fact is given by the following two statements from [6, Prop. 3.1 and 3.3] (see also [11], for some more general versions); involving *algebraic* Jacobi-Szegő parameters. In [6], Prop. 3.1 is stated for a unital C*-algebra B, but is true more generally.

PROPOSITION 3.2. *Let* B *be a unital involutive algebra, let* $\lambda_n \in B$, *and let* $\alpha_n : B \to B$ *be (generally, non-unital) linear maps*; $n \geq 0$. *On the vector space* $B < X >$ *define the* B*-valued sesquilinear map*

$$< b_0 X b_1 X \cdots b_k X b, c_0 X c_1 X \cdots c_l X c >_\alpha := \delta_{kl}\, b^* \alpha_0 (b_k^* \alpha_1 (... \alpha_k (b_0^* c_0) c_1 ...) c_k) c,$$

and $< b, c >_\alpha := b^* c$, if $b_i, c_j, b, c \in B$.

When all α_n *are completely positive, this sesquilinear map is a semi-inner product. Let* E *be the* B-B*-bimodule* $B < X >$ *endowed with* $< \cdot, \cdot >_\alpha$, *but omit* α *in this notation, when no confusion is possible. On* E *define the operators* ($b, b_j \in B$) :

$l^*(b_0 X b_1 X ... b_j X b) := X b_0 X b_1 X ... b_j X b$; $l(b_0 X b_1 X ... b_j X b) := \alpha_j(b_0) b_1 X ... b_j X b$, $l(b) := 0$;

$P(b_0 X b_1 X ... b_j X b) := \lambda_{j+1} b_0 X b_1 X ... b_j X b$, $P(b) := \lambda_o b$; *and* $x := l + l^* + P \in L(E)$, *when all* α_n *and all* $\lambda_n \in B$ *are Hermitian.*

Thus, in this case, x *is symmetric; in the sense that* $< xu, v > = < u, xv >$, *for all* $u, v \in E$.

Therefore the distribution $\mu := \varphi_x$ *of* x, *where* $\varphi := \omega_e$ *is the conditional expectation (with respect to the vacuum* $e := 1_B$) *given by* $\omega_e(T) := < e, Te >$, *if* $T \in L(E)$, *is a positive* B-B*-bimodule map, when all* λ_n *are Hermitian, and all* α_n *are completely positive.* □

We denote by $\mu = J(\lambda_n, \alpha_n; n \geq 0)$ the B-valued distribution given by the previous statement and call this *the Jacobi-Szegő distribution with Jacobi parameters* $\{\lambda_n, \alpha_n; n \geq 0\}$, preserving the terminology from [6].

We can even *define* these Jacobi-Szegő distributions as in the next statement, assuming only B is a unital algebra.

The moments of the distribution $\mu = J(\lambda_n, \alpha_n; n \geq 0)$ in Σ_B, as before, are described in the following way (see Prop. 3.3 in [6], for details).

PROPOSITION 3.3. *Let* B *be a unital involutive algebra, let* $\lambda_n \in B$, *and let* $\alpha_n : B \to B$ *be (generally, non-unital) linear maps*; $n \geq 0$. *Then the Jacobi-Szegő distribution* $\mu = J(\lambda_n, \alpha_n; n \geq 0) \in \Sigma_B$ *may be given by*

$$\mu\,(Xb_1X...b_jX) = \sum_{\pi \in NC_{1,2}(j+1)} k_\pi(X, b_1X, ..., b_jX),\ \textit{for all}\ j,\ \textit{and all}\ b_1, ..., b_j \in B;$$

the quantities $k_\pi(X, b_1X, ..., b_jX) \in B$ *being defined, by the parameters* $\{\lambda_n, \alpha_n; n \geq 0\}$, *in the way indicated in* [6, Prop. 3.3], *which involves the depth in* π *of the blocks belonging to* π. □

Namely, these $k_\pi(X, b_1X, ..., b_jX) \in B$, for $\pi \in NC_{1,2}(j+1)$, can be described as follows.

1) If π has only one block, then: $k_\pi(X, bX) := \alpha_0(b)$, when π reduces to a pair; $k_\pi(X) := \lambda_0$, when π reduces to a singleton;

2) If $\pi = \sigma \coprod \rho$, with $\sigma \in NC_{1,2}(\{0, ..., s\})$ and $\rho \in NC_{1,2}(\{s+1, ..., j\})$, then

$$k_\pi(X, b_1X, ..., b_jX) := k_\sigma(X, b_1X, ..., b_sX) \cdot b_{s+1} \cdot k_\rho(X, b_{s+2}X, ..., b_jX);$$

3) If π consists of the block $(1, j+1)$ and the subpartition $\sigma := \pi \cap \{2, ..., j\}$, then

$k_\pi(X, b_1X, ..., b_jX) := \alpha_0(b_1 \cdot l_\sigma(X, b_2X, ..., b_{j-1}X) \cdot b_j)$; and $l_\sigma(\cdot, ..., \cdot)$ have the following sense.

More generally, for a subpartition σ of $\pi \in NC_{1,2}(j+1)$, therefore with $\sigma \in NC_{1,2}(S)$, and $S \subset \{1, ..., j+1\}$, the quantities $l_\sigma(X, b_2X, ..., b_sX) \in B$ can be described as below.

1) If σ has only one block, then that is an inner block (also denoted) σ of π, thus the depth in π of that block $d(\sigma, \pi) \geq 1$, and $l_\sigma(X, bX) := \alpha_{d(\sigma,\pi)}(b)$, when σ is a pair; but $l_\sigma(X) := \lambda_{d(\sigma,\pi)}$, when σ is a singleton;

2) If $\sigma = \rho \coprod \tau$, with $\rho \in NC_{1,2}(k)$ and $\tau \in NC_{1,2}(\{k+1, ..., s\})$, then

$$l_\sigma(X, b_2X, ..., b_sX) := l_\rho(X, b_2X, ..., b_kX) \cdot b_{k+1} \cdot l_\tau(X, b_{k+2}X, ..., b_sX);$$

3) If σ consists of the block $(1, s)$ and the subpartition $\tau := \sigma \cap \{2, ..., s-1\}$, $s \geq 3$, then $l_\sigma(X, b_2X, ..., b_sX) := \alpha_{d_\pi(s)}(b_2 \cdot l_\tau(X, b_3X, ..., b_{s-1}X) \cdot b_s)$, denoting by $d_\pi(s)$ the depth of the inner block $(1, s)$ in π. □

Denote by $\Sigma_{J;B}$ the set of the Jacobi-Szegő distributions in Σ_B, defined as in the previous statement. When B is involutive algebra, denote by $\Sigma^+_{J;B}$ the set of all $J(\lambda_n, \alpha_n; n \geq 0)$ in $\Sigma_{J;B}$ such that all λ_n are Hermitian, and all α_n are completely positive. When all $\lambda_n = 0$, we denote $J(0, \alpha_n; n \geq 0)$ by $J(\alpha_n; n \geq 0)$ and call it the symmetric Jacobi-Szegő distribution with Jacobi parameters $\{\alpha_n; n \geq 0\}$. We send to [6, section 3] for concrete examples of distributions in $\Sigma_{J;B}$, including B-valued versions of Bernoulli, arcsine, semi-circular, free Poisson, free binomial, and, more generally, free Meixner distributions.

4. ACCARDI-BOZEJKO UNIVERSAL CONVOLUTION FOR OPERATOR-VALUED QUANTUM DISTRIBUTIONS

Let B be a unital (non-necessary involutive) algebra in the sequel.

Let $\mu_i = J(\lambda^i_n, \alpha^i_n; n \geq 0)$ be two Jacobi-Szegő distributions in Σ_B.

We define their quantum (*universal*) convolution as follows, extending Accardi and Bożejko's definition in [1, section 6] from the case when B is the complex field $\mathbb{C}$.

Definition 4.1. The (universal) convolution of μ_1 and μ_2 is the unique Jacobi-Szegő distribution, denoted $\mu_1 \times \mu_2$, in Σ_B with the Jacobi parameters $\left\{\lambda_n^1 + \lambda_n^2, \alpha_n^1 + \alpha_n^2; n \geq 0\right\}$, in the natural sense. □

We need the next notion of dilation of a general distribution in Σ_B (see, e.g., [16, Def. 4.2.1]).

Definition 4.2. For $t \in \mathbb{R}$, the dilation $D_t\mu \in \Sigma_B$ of the distribution $\mu \in \Sigma_B$ is given for any monomial $p = Xb_1X...b_jX \in B\langle X\rangle$ by $(D_t\mu)(p) = t^{j+1}\mu(p)$. □

The dilation of an operator-valued Jacobi-Szegő distribution has the form below, in agreement to the scalar Jacobi parameters case (see, e.g., [1, 10]).

PROPOSITION 4.3. *The dilation* $D_t\mu$ *of the Jacobi-Szegő distribution* $\mu = J(\lambda_n, \alpha_n; n \geq 0)$ *in* Σ_B, $t \in \mathbb{R}$, *is the distribution in* $\Sigma_{J;B}$ *with the Jacobi parameters* $\left\{t\lambda_n, t^2\alpha_n; n \geq 0\right\}$, *in the natural sense*.

Proof. Denote $\bar{\mu} := J(t\lambda_n, t^2\alpha_n; n \geq 0)$. It remains to check that $(D_t\mu)(p) = \bar{\mu}(p)$, for any monomial $p = Xb_1X...b_jX \in B\langle X\rangle$. Let k_π, l_σ, and $\bar{k}_\pi, \bar{l}_\sigma$ be the quantities corresponding to μ and $\bar{\mu}$, described (in section 3) in terms of $\left\{\lambda_n, \alpha_n; n \geq 0\right\}$, and $\left\{t\lambda_n, t^2\alpha_n; n \geq 0\right\}$, respectively. In view of Def. 4.2 , we may conclude due to the next properties, by the very definition of a Jacobi-Szegő distribution as in Prop. 3.3; namely:

i) $t^{j+1}k_\pi(X, b_1X, ..., b_jX) = \bar{k}_\pi(X, b_1, ..., b_jX)$, for all $j \geq 0$, all $\pi \in NC_{1,2}(j+1)$, and all $b_1, ..., b_j \in B$;

ii) $t^s l_\sigma(X, b_2X, ..., b_sX) = \bar{l}_\sigma(X, b_2X, ..., b_sX)$, for all $j \geq 0$, all subpartition $\sigma \in NC_{1,2}(S)$, with $S = \left\{1, ..., s\right\}$, of $\pi \in NC_{1,2}(j+1)$, and all $b_2, ..., b_s \in B$.

Indeed, firstly remark these three details:

1) If π reduces to a pair, then $k_\pi(X, bX) := \alpha_0(b)$ and $\bar{k}_\pi(X, bX) := t^2\alpha_0(b)$; but, if π reduces to a singleton, then $k_\pi(X) := \lambda_0$ and $\bar{k}_\pi(X) := t\lambda_0$;

2) If $\pi = \sigma \coprod \rho$, with $\sigma \in NC_{1,2}(\{0, ..., s\})$ and $\rho \in NC_{1,2}(\{s+1, ..., j\})$, $0 \leq s < j$, then

$$k_\pi(X, b_1X, ..., b_jX) := k_\sigma(X, b_1X, ..., b_sX) \cdot b_{s+1} \cdot k_\rho(X, b_{s+2}X, ..., b_jX) \text{ and}$$
$$\bar{k}_\pi(X, b_1X, ..., b_jX) := \bar{k}_\sigma(X, b_1X, ..., b_sX) \cdot b_{s+1} \cdot \bar{k}_\rho(X, b_{s+2}X, ..., b_jX).$$

3) If π consists of the block $(1, j+1)$ and the subpartition $\sigma := \pi \cap \left\{2, ..., j\right\}$, $j \geq 2$, then

$$k_\pi(X, b_1X, ..., b_jX) := \alpha_0(b_1 \cdot l_\sigma(X, b_2X, ..., b_{j-1}X) \cdot b_j), \text{ and similarly}$$
$$\bar{k}_\pi(X, b_1X, ..., b_jX) := t^2\alpha_0(b_1 \cdot \bar{l}_\sigma(X, b_2X, ..., b_{j-1}X) \cdot b_j).$$

Therefore, the property i) results by a natural induction, via the property ii); for $s < j$.

Secondly, remark the other three details:

1) If σ reduces to a pair, then $l_\sigma(X, bX) := \alpha_{d(\sigma,\pi)}(b)$ and $\bar{l}_\sigma(X, bX) := t^2\alpha_{d(\sigma,\pi)}(b)$; but, if σ reduces to a singleton, then $l_\sigma(X) := \lambda_{d(\sigma,\pi)}$ and $\bar{l}_\sigma(X) := t\lambda_{d(\sigma,\pi)}$; denoting by $d(\sigma, \pi)$ the depth of the block σ in π.

2) If $\sigma = \rho \coprod \tau$, with $\rho \in NC_{1,2}(\{1, ..., k\})$ and $\tau \in NC_{1,2}(\{k+1, ..., s\})$, $1 \leq k < s$, then

$$l_\sigma(X, b_2X, ..., b_sX) := l_\rho(X, b_2X, ..., b_kX) \cdot b_{k+1} \cdot l_\tau(X, b_{k+2}X, ..., b_sX) \text{ and}$$
$$\bar{l}_\sigma(X, b_2X, ..., b_sX) := \bar{l}_\rho(X, b_2X, ..., b_kX) \cdot b_{k+1} \cdot \bar{l}_\tau(X, b_{k+2}X, ..., b_sX).$$

3) If σ consists of the block $(1, s)$ and the subpartition $\tau := \sigma \cap \left\{2, ..., s-1\right\}$, $s \geq 3$, then $l_\sigma(X, b_2X, ..., b_sX) := \alpha_{d_\pi(s)}(b_2 \cdot l_\tau(X, b_3X, ..., b_{s-1}X) \cdot b_s)$, and also $\bar{l}_\sigma(X, b_2X, ..., b_sX) := t^2\, \alpha_{d_\pi(s)}(b_2 \cdot \bar{l}_\tau(X, b_3X, ..., b_{s-1}X) \cdot b_s)$; where $d_\pi(s)$ is the depth of the inner block $(1, s)$ in π.

In consequence, the property ii) follows by a natural induction, too. □

The analogue of the infinite-divisibility with respect to the above convolution is clear. Infinite-divisibility being relevant in the frame of positive distributions, we require B be involutive algebra in the next

Definition 4.4. Let B be an involutive algebra. Then $\mu \in \Sigma_B^+$ is infinitely-divisibile with respect to the above convolution, if, for every positive integer n, there exists $\mu_{1/n} \in \Sigma_{J;B}^+$ such that $\mu = \underbrace{(\mu_{1/n} \times \cdots \times \mu_{1/n})}_{n \text{ times}}$. □

The next statement generalizes [1, Th.6.1] from the scalar-valued case $B = \mathbb{C}$. Then we infer the announced facts.

THEOREM 4.5. *The quantum (universal) convolution introduced before has these properties:*
i) *commutativity and associativity;*
ii) *positivity: if* B *is an involutive algebra, and* $\mu, \nu \in \Sigma_{J;B}^+$, *then* $\mu \times \nu \in \Sigma_{J;B}^+$;
iii) $D_t(\mu \times \nu) = (D_t\mu) \times (D_t\nu)$, *for all* $t \in \mathbb{R}$, *and* $\mu, \nu \in \Sigma_{J;B}$;
iv) *for any symmetric distribution* $\mu = J(\alpha_n; n \ge 0) \in \Sigma_{J;B}$, *and any positive integer* N, *it results*

$$\underbrace{(D_{1/\sqrt{N}}\mu) \times \cdots \times (D_{1/\sqrt{N}}\mu)}_{N \text{ times}} = \mu .$$

Proof. As in [1], remark i) and (by Prop. 4.3) iii) are obvious. The positivity property ii) is immediate via Prop. 3.2. Denote by $S_N(\mu)$ the left hand side of the conclusion in iv). Therefore, since $S_N(\mu) = D_{1/\sqrt{N}}\underbrace{(\mu \times \cdots \times \mu)}_{N \text{ times}}$, by i) and iii), and Def. 4.1 implies $\underbrace{\mu \times \cdots \times \mu}_{N \text{ times}} = J(N\lambda_n, N\alpha_n; n \ge 0)$, for any $\mu = J(\lambda_n, \alpha_n; n \ge 0)$ in $\Sigma_{J;B}$, we get that $S_N(\mu) = \mu$, via Prop. 4.3, when μ is symmetric. □

Consequently, by Prop. 3.2, the quantum *universal* convolution '$\times$' introduced before confirmes its name.

COROLLARY 4.6. *Let B be a unital involutive algebra. Then any symmetric distribution* $\mu \in \Sigma_{J;B}^+$ *is infinitely divisible with respect to the quantum convolution* '$\times$', *defined above*. □

Moreover, any such symmetric distribution can be realized as a quantum central limit for the involved convolution. Results of this kind are called universal central limit theorems in [2] (this remarkable work including a constructive approach, even for the non-symmetric case). See [7] for a first such theorem, [13] for similar statements; and also [11] for some operator-valued versions.

COROLLARY 4.7. *Let B be a unital Hausdorff topological algebra, and consider any Hausdorff topology on* Σ_B. *Let* $\mu \in \Sigma_{J;B}$ *be any symmetric distribution*. *Then* $\lim_{N\to\infty} D_{1/\sqrt{N}}\underbrace{(\mu \times \cdots \times \mu)}_{N \text{ times}} = \mu$; *i.e.,* μ *is a (quantum) central limit law with respect to the quantum universal convolution* '$\times$'. □

We send to [11] for a more general version of these facts, and not only, in multivariate frame.

Some results from [11] have been presented at the 19th and 20th Conference of Probability and Statistics Society of Romania in 2016, respectively 2017.

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