

CONGRUENCE IDENTITIES INVOLVING SUMS OF ODD DIVISORS FUNCTION

Mircea MERCA

University of Craiova, Department of Mathematics
A. I. Cuza 13, Craiova 200585, Romania

Corresponding author: Mircea MERCA, E-mail: mircea.merca@profinfo.edu.ro

Abstract. In this paper, inspired by a classical connections between partitions and divisors, we investigate some congruence identities involving sums of the odd divisor function $\sigma_{odd}(n)$ which is defined by $\sigma_{odd}(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d$. In this context, we conjectured that the congruence

$$\sum_{k=-\infty}^{\infty} \sigma_{odd}(n - k(3k - 1)/2) \equiv \begin{cases} n \pmod{m}, & \text{if } n = j(3j - 1)/2, j \in \mathbb{Z}, \\ 0 \pmod{m}, & \text{otherwise.} \end{cases}$$

is valid for any positive integer n if and only if $m \in \{2, 3, 6\}$.

Key words: theta series, partitions, divisors.

1. INTRODUCTION

The object of our investigations is the divisor function $\sigma_{odd}(n)$ which is defined as the sum of the odd positive divisors of n , i.e.,

$$\sigma_{odd}(n) := \sum_{\substack{d|n \\ d \text{ odd}}} d.$$

Throughout this paper, we consider $\sigma_{odd}(n) = 0$ for $n \leq 0$. Recall that the function $\sigma_{odd}(n)$ is the coefficient of q^n in the following Lambert series expansion

$$\sigma_{odd}(n) = [q^n] \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}, \quad |q| < 1. \quad (1)$$

On the other hand, the function $\sigma_{odd}(n)$ appears naturally as the coefficients of a modular form. It is related to the eta η -Dedekind function and Eisenstein series $E_{2,2}$.

Recall [2, Chap. 3] that the Dedekind eta function $\eta(\tau)$ is given by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (2)$$

where $q = e^{2\pi i \tau}$ and $\text{Im}(\tau) > 0$. It is well known that the η -function is a modular form of weight $1/2$ and level 1 for a certain character of order 24 of the metaplectic double cover of the modular group. The eta quotient $\eta(\tau)/\eta(2\tau)$ is equal to

$$\frac{\eta(\tau)}{\eta(2\tau)} = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}). \quad (3)$$

By the logarithmic derivative of this formula, we get the Eisenstein series

$$E_{2,2}(\tau) = -\frac{1}{24} + q \sum_{n=1}^{\infty} \frac{-(2n-1)q^{2n-2}}{q^{2n-1}} = -\frac{1}{24} - \sum_{n=1}^{\infty} \sigma_{\text{odd}}(n)q^n. \quad (4)$$

It is known that $E_{2,2}(\tau)$ is a modular form for the congruence subgroup $\Gamma_0(2)$ [5, pp. 18-19].

In [4, Chap. 3, Section 3.3], the odd divisor function $\sigma_{\text{odd}}(n)$ is related to the topic of *sums of four squares*. More details about arithmetic properties of $\sigma_{\text{odd}}(n)$ can be found in [3].

A partition of a positive integer n is a sequence of positive integers whose sum is n . The order of the summands is unimportant when writing the partitions of n , but for consistency, a partition of n will be written with the summands in a nonincreasing order [1]. The Euler partition function $p(n)$ gives the number of ways of writing the nonnegative integer n as a sum of positive integers, where the order of addends is not considered significant. For example, the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1 and 1 + 1 + 1 + 1 + 1. Thus, $p(5) = 7$. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}$$

and the expansion starts as

$$\frac{1}{(q; q)_{\infty}} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + \dots$$

Here and throughout this paper, we use the following customary q -series notation:

$$(a; q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$

$$(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n.$$

Because the infinite product $(a; q)_{\infty}$ diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a; q)_{\infty}$ appears in a formula, we shall assume $|q| < 1$.

The divisors of numbers have been studied from the point of view of partitions of integers for a long time. It is well known that Euler's partition function $p(n)$ and the sum of divisors function

$$\sigma(n) := \sum_{d|n} d$$

satisfy common recursive relations with only $p(0)$ different from $\sigma(0)$:

$$\sum_{k=-\infty}^{\infty} (-1)^k p(n - P_5(k)) = \delta_{0,n}, \quad \text{with } p(0) = 1$$

and

$$\sum_{k=-\infty}^{\infty} (-1)^k \sigma(n - P_5(k)) = 0, \quad \text{with } \sigma(0) \text{ replaced by } n, \quad (5)$$

where $\delta_{i,j}$ is the Kronecker delta and

$$P_m(n) := \left(\frac{m}{2} - 1\right)n^2 - \left(\frac{m}{2} - 2\right)n$$

is the n th generalized m -gonal number. It is clear that the divisors functions $\sigma(n)$ and $\sigma_{\text{odd}}(n)$ have the same parity, i.e.,

$$\sigma(n) \equiv \sigma_{\text{odd}}(n) \pmod{2}.$$

By identity (5), we easily deduce the following parity results.

THEOREM 1. For $n \geq 0$,

$$\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - P_5(k)) \equiv 1 \pmod{2}$$

if and only if n is an odd generalized pentagonal number.

It is well known that $\sigma_{\text{odd}}(n)$ is odd if and only if n is a square or a twice square. Thus we deduce the following parity result.

COROLLARY 1. Let n be a positive integer. The number of representations of n as the sum of a generalized pentagonal number and a square or a twice square is odd if and only if n is an odd generalized pentagonal number.

The first generalized pentagonal numbers are

$$0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \dots$$

As we can see, 51 is an odd generalized pentagonal number that can be represented as a sum of a generalized pentagonal number and a square or twice square in five different ways:

$$51 = 1 + 2 \cdot 5^2 = 2 + 7^2 = 15 + 6^2 = 26 + 5^2 = 35 + 4^2.$$

In this article, we investigate the positive integers m for which the following congruence identities are valid for any positive integer n :

$$\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - P_m(k)) \equiv \begin{cases} n \pmod{2}, & \text{if } n = P_m(j), j \in \mathbb{Z}, \\ 0 \pmod{2}, & \text{otherwise,} \end{cases} \quad (6)$$

$$\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - P_5(k)) \equiv \begin{cases} n \pmod{m}, & n = P_5(j), j \in \mathbb{Z}, \\ 0 \pmod{m}, & \text{otherwise,} \end{cases} \quad (7)$$

$$\sum_{k=-\infty}^{\infty} (-1)^{P_3(-k)} \sigma_{\text{odd}}(n - P_5(k)) \equiv \begin{cases} (-1)^{P_3(-j)} \cdot n \pmod{m}, & \text{if } n = P_5(j), j \in \mathbb{Z}, \\ 0 \pmod{m}, & \text{otherwise.} \end{cases} \quad (8)$$

It is clear that the case $m = 5$ of (6) is the case $m = 2$ of (7) and (8). Note that generalized 3-gonal numbers are triangular numbers and generalized 4-gonal numbers are squares of integers. It is known that generalized hexagonal numbers are identical with triangular numbers. We have the following equivalent form of the congruence (6): *The number of representation of n as the sum of a generalized m -gonal number and a square or a twice square is odd if and only if n is an odd generalized m -gonal number.*

There is a substantial amount of numerical evidence to conjecture the following assertions.

CONJECTURE 1. The congruence (6) is valid for any positive integer n if and only if $m \in \{5, 6\}$.

CONJECTURE 2. The congruence (7) is valid for any positive integer n if and only if $m \in \{2, 3, 6\}$.

CONJECTURE 3. The congruence (8) is valid for any positive integer n if and only if $m \in \{2, 4\}$.

In Section 2, we prove one implication of Conjecture 1, i.e., if $m \in \{5, 6\}$, then (6) holds for any positive integer n . We remark that, the case $m = 6$ of this implication reads as follows.

THEOREM 2. For $n \geq 0$,

$$\sum_{k=0}^{\infty} \sigma_{\text{odd}}(n - P_3(k)) \equiv 1 \pmod{2}$$

if and only if n is an odd triangular number.

The other implication has been verified for all integers m with $m < 100000$. In Section 3, we prove one implication of Conjecture 2, i.e., if $m \in \{2, 3, 6\}$, then (7) holds for any positive integer n . In Section 4, we prove one implication of Conjecture 3, i.e., if $m \in \{2, 4\}$, then (8) holds for any positive integer n .

2. PROOFS OF THEOREMS 1 AND 2

As usual, we denote by $Q(n)$ the number of integer partitions of n into odd parts. For example, $Q(7) = 5$ because the five partitions of 7 odd parts are 7, $5 + 1 + 1$, $3 + 3 + 1$, $3 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1 + 1$. We remark that the generating function of $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q; q^2)_{\infty}} = (-q; q)_{\infty}.$$

The logarithmic differentiation of the generating function for $Q(n)$ gives:

$$\frac{d}{dq} \frac{1}{(q; q^2)_{\infty}} = -\frac{1}{(q; q^2)_{\infty}^2} \frac{d}{dq} (q; q^2)_{\infty} = \frac{1}{(q; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-2}}{1-q^{2n-1}}.$$

On the other hand, we have

$$\frac{d}{dq} \frac{1}{(q; q^2)_{\infty}} = \frac{d}{dq} \sum_{n=0}^{\infty} Q(n)q^n = \sum_{n=1}^{\infty} nQ(n)q^{n-1}.$$

Thus we deduce that

$$\sum_{n=1}^{\infty} nQ(n)q^n = (-q; q)_{\infty} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} = (-q; q)_{\infty} \sum_{n=1}^{\infty} \sigma_{\text{odd}}(n)q^n. \quad (9)$$

The following theta identity is often attributed to Gauss [1, p.23, eqs. (2.2.13)]:

$$1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}. \quad (10)$$

By (9) and (10), we obtain

$$\begin{aligned} nQ(n) + 2 \sum_{k=1}^{\infty} (-1)^k (n-k^2)Q(n-k^2) &= [q^n] \left(\frac{(q; q)_{\infty}}{(-q; q)_{\infty}} \cdot (-q; q)_{\infty} \sum_{n=1}^{\infty} \sigma_{\text{odd}}(n)q^n \right) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \sigma_{\text{odd}}(n - k(3k-1)/2), \end{aligned}$$

where we have invoked Euler's pentagonal number theorem

$$(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \quad (11)$$

In this way, we deduce that

$$nQ(n) \quad \text{and} \quad \sum_{k=-\infty}^{\infty} (-1)^k \sigma_{\text{odd}}(n - k(3k-1)/2)$$

have the same parity. According to [6, Corollary 4.7], $Q(n)$ is odd if and only if n is a generalized pentagonal

number. This concludes the proof of Theorem 1.

In order to prove Theorem 2, we consider another theta series identity of Gauss: [1, p.23, eqs. (2.2.13)]

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \quad (12)$$

Considering this identity, we can write

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^k Q(n - k(3k - 1)) &= [q^n] \left((q^2; q^2)_\infty \cdot \frac{1}{(q; q^2)_\infty} \right) \\ &= [q^n] \sum_{n=0}^{\infty} q^{n(n+1)/2} \\ &= \begin{cases} 1, & \text{if } n \text{ is a triangular number,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we deduce that

$$n \sum_{k=-\infty}^{\infty} Q(n - k(3k - 1)) \equiv 1 \pmod{2} \quad (13)$$

if and only if n is an odd triangular number.

On the other hand, taking into account (9), we can write

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^k (n - k(3k - 1)) Q(n - k(3k - 1)) &= [q^n] \left((q^2; q^2)_\infty \sum_{n=1}^{\infty} n Q(n) q^n \right) \\ &= [q^n] \left(\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{n=1}^{\infty} \sigma_{\text{odd}}(n) q^n \right) \\ &= \sum_{k=0}^{\infty} \sigma_{\text{odd}}(n - k(k + 1)/2). \end{aligned}$$

By this identity, taking into account that $k(3k - 1)$ is even, we deduce that

$$\sum_{k=0}^{\infty} \sigma_{\text{odd}}(n - k(k + 1)/2) \equiv n \sum_{k=-\infty}^{\infty} Q(n - k(3k - 1)) \pmod{2}.$$

The proof of Theorem 2 follows easily considering (13).

3. CONGRUENCES MODULO 2, 3 AND 6

The congruence provided by Theorem 1 motivates us to look for other similar results involving the divisor function σ_{odd} and generalized pentagonal numbers. We experimentally found that the coefficient of q^n in the series

$$\begin{aligned} (-q; q)_\infty \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}} & \quad (14) \\ &= q + 2q^2 + 6q^3 + 6q^4 + 11q^5 + 12q^6 + 19q^7 + 18q^8 + 24q^9 + 30q^{10} + 36q^{11} + 36q^{12} \\ & \quad + 36q^{13} + 48q^{14} + 57q^{15} + 60q^{16} + 60q^{17} + 66q^{18} + 72q^{19} + 84q^{20} + 84q^{21} + 106q^{22} + \dots \end{aligned}$$

is congruent to 0 modulo 6 if and only if n is not a generalized pentagonal number or n is a generalized pentagonal number congruent to 0 modulo 6. For $0 < r < 6$ we notice that the coefficient of q^n in (14) is congruent to r modulo 6 if and only if n is a generalized pentagonal number congruent to r modulo 6. Considering the Jacobi triple product identity

$$(q; q)_\infty (z; q)_\infty (q/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2}, \quad |q| < 1, z \neq 0,$$

we deduce that

$$(-q; q)_\infty \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty} = (-q; q^3)_\infty (-q^2; q^3)_\infty (q^3; q^3)_\infty = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}.$$

Thus we can state the following result.

THEOREM 3. For $n > 0$, $m \in \{2, 3, 6\}$,

$$\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - P_5(k)) \equiv \begin{cases} n \pmod{m}, & \text{if } n = P_5(j), j \in \mathbb{Z}, \\ 0 \pmod{m}, & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 2. We have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - k(3k-1)/2) &= [q^n] \left(\left(\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} \right) \left(\sum_{n=1}^{\infty} \sigma_{\text{odd}}(n) q^n \right) \right) \\ &= [q^n] \left((-q; q)_\infty \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \right) \\ &= nQ(n) + 2 \sum_{k=1}^{\infty} (-1)^k (n-3k^2) Q(n-3k^2) \\ &= n \left(Q(n) + 2 \sum_{k=1}^{\infty} (-1)^k Q(n-3k^2) \right) - 6 \sum_{k=1}^{\infty} (-1)^k k^2 Q(n-3k^2) \end{aligned}$$

and

$$\begin{aligned} Q(n) + 2 \sum_{k=1}^{\infty} (-1)^k Q(n-3k^2) &= [q^n] \left((-q; q)_\infty \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \right) \\ &= [q^n] \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} \\ &= \begin{cases} 1, & \text{if } n \text{ is a generalized pentagonal number,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For $m \in \{2, 3, 6\}$, we deduce that

$$\sum_{k=-\infty}^{\infty} \sigma_{\text{odd}}(n - k(3k-1)/2) \equiv \begin{cases} n \pmod{m}, & \text{if } n \text{ is a generalized pentagonal number,} \\ 0 \pmod{m}, & \text{otherwise.} \end{cases}$$

This concludes the proof. □

4. CONGRUENCES MODULO 2 AND 4

In this section, we prove the following congruence identity.

THEOREM 4. For $n > 0$, $m \in \{2, 4\}$,

$$\sum_{k=-\infty}^{\infty} (-1)^{P_3(-k)} \sigma_{\text{odd}}(n - P_5(k)) \equiv \begin{cases} (-1)^{P_3(-j)} \cdot n \pmod{m}, & \text{if } n = P_5(j), j \in \mathbb{Z}. \\ 0 \pmod{m}, & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem is quite similar to the proof of Theorem 3. Considering Euler's pentagonal number theorem (11), we deduce that

$$(-q; q)_{\infty} \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} = (-q; q^2)_{\infty} (q^2; q^2)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(3n-1)/2}.$$

We can write

$$\begin{aligned} Q(n) + 2 \sum_{k=1}^{\infty} (-1)^k Q(n - 2k^2) &= [q^n] \left((-q; q)_{\infty} \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \right) \\ &= \begin{cases} (-1)^{j(j-1)/2}, & \text{if } n = j(3j-1)/2, j \in \mathbb{Z}, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \sigma_{\text{odd}}(n - k(3k-1)/2) &= [q^n] \left(\left(\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} q^{n(3n-1)/2} \right) \left(\sum_{n=1}^{\infty} \sigma_{\text{odd}}(n) q^n \right) \right) \\ &= [q^n] \left((-q; q)_{\infty} \frac{(q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \right) \\ &= nQ(n) + 2 \sum_{k=1}^{\infty} (-1)^k (n - 2k^2) Q(n - 2k^2) \\ &= n \left(Q(n) + 2 \sum_{k=1}^{\infty} (-1)^k Q(n - 2k^2) \right) - 4 \sum_{k=1}^{\infty} (-1)^k k^2 Q(n - 2k^2). \end{aligned}$$

The proof follows easily. □

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Received March 16, 2021