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STRUCTURE OF FINITE GROUPS WITH SOME QTI-SUBGROUPS

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Abstract. Let G be a group and H be a subgroup of G. H is said to be a TI-subgroup of G if $H \cap H^g \in \{1, H\}$ for every $g \in G$ and H is called a QTI-subgroup of G if $C_G(x) \le N_G(H)$ for any $1 \neq x \in H$. It is easy to see that a TI-subgroup must be a QTI-subgroup, however the converse is not true. It is proved in [10] that if every non-abelian (non-cyclic) subgroup of a group G is a TI-subgroup or a subnormal subgroup, then every non-abelian (non-cyclic) subgroup of G is subnormal in G. In this paper, we generalize the above result and prove that if every non-abelian (non-cyclic) subgroup of a group G is a QTI-subgroup or a subnormal subgroup, then every non-abelian (non-cyclic) subgroup of G is subnormal in G. Furthermore, we give the completely classification of a group G in which every subgroup is a QTI-subgroup or a subnormal subgroup.

Key words: QTI-subgroup, subnormal subgroup, Frobenius group.

1. INTRODUCTION

All groups considered in this paper are finite and G always denotes a finite group.

Let G be a group and H be a subgroup of G. H is said to be a TI-subgroup (or trivial-intersection subgroup) if $H \cap H^g = 1$ or H for every $g \in G$. A number of papers have investigated the influence of TI-subgroups on the structure of a finite group. For example, Walls in [12] described the structure of a group G in which every subgroup is a TI-subgroup. Guo, Li and Flavell in [4] classified the groups all of whose abelian subgroups are TI-subgroups. As generalization, Qian and Tang in [8] introduced the definition of a "QTI-subgroup": A subgroup H of G is called a QTI-subgroup of G if $C_G(x) \le N_G(H)$ for any $1 \neq x \in H$. It is easy to see that a TI-subgroup of G must be a QTI-subgroup of G, however the converse is not true, see Example 1.2 of [8]. Qian and Tang in [8] classified the groups all of whose abelian subgroups are QTI-subgroups. Later, Lu and Guo in [7] investigated the structure of finite groups all of whose second maximal subgroups are QTI-subgroups. More recently, the authors in [2] classified finite groups in which every non-abelian maximal subgroups is a QTI-subgroup.

In [10], the author proved that if every non-abelian (non-cyclic) subgroup of a group G is a TI-subgroup or a subnormal subgroup, then every non-abelian (non-cyclic) subgroup of G is subnormal in G. Since a TI-subgroup of G must be a QTI-subgroup of G, motivated by the above results, we pose the following question:

QUESTION 1.1. *Can we generalize the results about TI-subgroups to QTI-subgroups?*

In this short paper, we answer the above question in several cases and obtain the following two theorems.

THEOREM 1. *Let G be a group such that every non-abelian subgroup of G is a QTI-subgroup or a subnormal subgroup. Then every non-abelian subgroup of G is subnormal in G.*

THEOREM 2. *Let G be a group such that every non-cyclic subgroup of G is a QTI-subgroup or a subnormal subgroup. Then every non-cyclic subgroup of G is subnormal in G.*

Finally, we consider the structure of a group in which every subgroup is a QTI-subgroup or a subnormal subgroup, and we give the completely classification of such groups in the following theorem.

THEOREM 3. *Let G be a group. Then every subgroup of G is a QTI-subgroup or a subnormal subgroup if and only if G is a group of one of the following types:*

(1) *G is nilpotent;*

(2) *$G = N \rtimes M$ is a Frobenius group with the kernel N and a complement M, where N is a minimal normal subgroup of G, N is an elementary abelian p-group for some prime p, M is a cyclic maximal subgroup of G, and the subgroups of G can only be one of the following three cases:*

(I) *K with $K \leq N$;*

(II) *T with $T \leq M^y$ for some $y \in G$;*

(III) *NH with $1 < H \leq M^x$ for some $x \in G$.*

For a group G, $\pi(G)$ is the set of the prime divisors of $|G|$. The prime graph $\Gamma(G)$ of G is defined as follows: the vertex set is $\pi(G)$ and two vertices p, q are joined by an edge if G has an element of order pq. If σ is the vertex set of a connected component of $\Gamma(G)$, then σ is called a prime component of G. According to [13], if $G = N \rtimes M$ is a Frobenius group with the kernel N and a complement M, then $\pi(N)$ and $\pi(M)$ are the prime components of G.

A subgroup H is said to be a CC-subgroup of a group G if $C_G(x) \leq H$ for every $1 \neq x \in H$. By $G = N \rtimes H$ we mean that G is the product of a normal subgroup N and a subgroup H such that $N \cap H = 1$. Other notation and terminology not mentioned are standard, see [9] for instance.

2. PRELIMINARIES

In this section, we will give some lemmas which are useful in the proof of our main results.

LEMMA 2.1 ([7, Lemma 2.1]). *Let G be a group and H be a QTI-subgroup of G. If H is maximal in G, then H is normal in G or H is a Hall subgroup of G.*

LEMMA 2.2 ([9, Theorem 10.5.6(ii)]). *Let G be a Frobenius group with the kernel K and a complement H. Then the Sylow p-subgroups of H are cyclic if $p > 2$ or generalized quaternion if $p = 2$.*

LEMMA 2.3. *Suppose that G is a Frobenius group with the kernel N. If K is a subnormal subgroup of G, then $K \leq N$ or $N < K$.*

Proof. Let $K = K_s \trianglelefteq K_{s-1} \trianglelefteq K_{s-2} \trianglelefteq \cdots \trianglelefteq K_1 \trianglelefteq K_0 = G$ be a composition series from K to G. We will prove the lemma by using induction on s. If $s = 1$, then by Exercise 8.5.7 of [9], we have that $K \leq N$ or $N < K$, and the conclusion is true. Suppose that $K_{s-1} \leq N$ or $N < K_{s-1}$. If $K_{s-1} \leq N$, then $K \leq N$. Suppose that $N < K_{s-1}$. Then N is a Frobenius kernel of the Frobenius group K_{s-1}. Again by Exercise 8.5.7 of [9], we have that $K = K_s \leq N$ or $N < K$. □

LEMMA 2.4 ([6, Theorem 4.1.8(a)]). *Let G be a Frobenius group with Frobenius complement H and Frobenius kernel K. Let U be a subgroup of G such that $U \nsubseteq K$, and let $x \in G$ such that $H^x \cap U \neq 1$. Then either $U \leq H^x$ or U is a Frobenius group with Frobenius complement $H^x \cap U$ and Frobenius kernel $U \cap K$.*

3. PROOFS

In this section, we give the proofs of the main results. We first give the proof of Theorem 1.

PROOF of Theorem 1. Suppose that the theorem is not true and let H be a subgroup of G of the largest order such that H is non-abelian and non-subnormal in G. If $H < N_G(H)$, then $N_G(H)$ is non-abelian and thus is subnormal in G by the choice of H, which gives that H is subnormal in G, a contradiction. Therefore, $H = N_G(H)$. It follows that H is a CC-subgroup of G. If H is non-nilpotent, then by Theorem A of [1], it follows that G is one of the following groups:

(1) $Sz(q), q = 2^{2n+1}, n \geq 1$ and H is solvable;

(2) $PSL(2,2^n), n \geq 2$ and H is solvable;

(3) $PSL(2,q), q \equiv 3(\bmod 4), q > 3$ and H is solvable of odd order $q(q-1)/2$;

(4) A Frobenius group with non-nilpotent complement H.

If $G \cong Sz(q)$ with $q = 2^{2n+1}$ and $n \geq 1$, then there exists a dihedral group K of order $2(q-1)$ such that K is maximal in G by [11]. Then K is non-abelian and K is neither a normal subgroup or a Hall subgroup of G. On the other hand, since K is non-abelian, by hypothesis K is either a QTI-subgroup of G or subnormal in G. If K is a QTI-subgroup of G, then K is normal in G or K is a Hall subgroup of G by Lemma 2.1, which contradicts to the former claim. If K is subnormal in G, then it is normal in G since it is maximal in G, which is also a contradiction.

If $G \cong PSL(2,2^n), n \geq 2$ or $PSL(2,q), q \equiv 3(\bmod 4)$ and $q > 3$, then by Dickson's Theorem ([5, III, Huaptsatz 8.27]), excepting one case: $PSL(2,7)$, G contains two dihedral subgroups D_1 and D_2, both of which are maximal in G. By examining their orders, we have that at least one of D_1 and D_2 is not a Hall subgroup of G, which contradicts to Lemma 2.1. For the case $G \cong PSL(2,7)$, by [3], let M be a maximal subgroup of G with $M \cong S_4$, T be a subgroup of M with $T \cong S_3$ and a be an element of T with order 2. Then $N_G(T) = T$ and $|C_G(a)| = 8$. Therefore, $C_G(a) \nsubseteq N_G(T)$, whence T is not a QTI-subgroup of G. Obviously, T is not subnormal in G, which is a contradiction.

Suppose that G is a Frobenius group with non-nilpotent complement H. Let N be the kernel of G. Since $G/N \cong H$, there is a non-normal maximal subgroup M of G such that $N < M$. Then $N_G(M) = M$. We claim that M is a CC-subgroup of G. In fact, if M is abelian, then as $Z(G) = 1$ and M is maximal in G, we have that $C_G(x) = M$ for every $1 \neq x \in M$, whence M is a CC-subgroup of G. If M is non-abelian, then M is a QTI-subgroup of G. Therefore, $C_G(x) \leq N_G(M) = M$ for every $1 \neq x \in M$, which also shows that M is a CC-subgroup of G. Then M is a Hall subgroup of G since every CC-subgroup must be a Hall subgroup. Since $\pi(G) = \pi(N) \cup \pi(H)$, N is a Hall $\pi(N)$-subgroup of G, H is a Hall $\pi(H)$-subgroup of G and $N < M$, we have that $\pi(M) \cap \pi(H) \neq \emptyset$. Since $\pi(H)$ is a prime component of G and M is a CC-subgroup of G, we have that $\pi(H) \subseteq \pi(M)$. Therefore, $H \leq M$, whence $G = HN \leq M$, which is a contradiction.

Therefore, H is nilpotent, and thus $Z(H) \neq 1$. We claim that H is a self-normalizing TI-subgroup of G. In fact, it suffices to show that $H \cap H^g = 1$ if $H^g \neq H$. For otherwise, let $1 \neq x \in H \cap H^g$. Then $C_G(x) \leq H$ and $C_G(x) \leq H^g$ since H and H^g are QTI-subgroups of G. On the other hand, we have that $\langle Z(H), Z(H^g) \rangle \leq C_G(x)$. It follows that $Z(H) \leq H \cap H^g$. Let $1 \neq y \in Z(H)$. Then $H \leq C_G(y) \leq H^g$ again since H^g is a QTI-subgroup, which is a contradiction. Therefore, the claim is true. It follows that G is a Frobenius group with H a Frobenius complement. Let S be the kernel of G. Since H is non-abelian, we have that H is of even order. Moreover, by Theorem 10.5.6(ii) of [9], H is a direct product of a generalized quaternion 2-group and a cyclic group of odd order. Take $H_1 = \langle h \rangle$ be a subgroup of H of order 2. Since conjugation by h induces an automorphism of S that has order 2 and has no nonidentity fixed points, we have that $s^h = s^{-1}$ for every non-identity s of S. It follows that H_1 normalizes every subgroup of S.

Assume that S is not a subgroup of prime order. Let S_1 be a maximal subgroup of S. Then $S_1 \neq 1$ and $S_1 \rtimes H_1$ is a non-abelian maximal subgroup of $S \rtimes H_1$. Since $S_1 \rtimes H_1$ is non-abelian, by the hypothesis, one has that $S_1 \rtimes H_1$ is subnormal in G or $S_1 \rtimes H_1$ is a QTI-subgroup of G.

(1) Suppose that $S_1 \rtimes H_1$ is subnormal in $S \rtimes H_1$. Then $S_1 \rtimes H_1$ is normal in $S \rtimes H_1$. Since S is the Frobenius kernel of $S \rtimes H_1$, it follows that either $S_1 \rtimes H_1 \leq S$ or $S < S_1 \rtimes H_1$ by Exercise 8.5.7 of [9]. It is obvious that this is impossible.

(2) Suppose that $S_1 \rtimes H_1$ is a non-normal QTI-subgroup of $S \rtimes H_1$. Since $S_1 \rtimes H_1$ is maximal in $S \rtimes H_1$, it follows from Lemma 2.1 that $S_1 \rtimes H_1$ is a Hall subgroup of $S \rtimes H_1$. Since $\pi(S)$ is a prime component of $S \rtimes H_1$, we have that $S \leq S_1 \rtimes H_1$, which gives the contradiction that $S \rtimes H_1 \leq S_1 \rtimes H_1$. Hence S is a cyclic group of order p for some prime p. By N/C-theorem, one has that $H \cong G/S = N_G(S)/C_G(S) \lesssim Aut(S) \cong Z_{p-1}$, which implies that H is cyclic, a contradiction with the fact that H is non-abelian. The final contradiction shows that the theorem is true.

□

Since every TI-subgroup of G must be a QTI-subgroup of G, we immediately have the following corollary.

COROLLARY 1 ([10, Theorem 1]). *Let G be a group such that every non-abelian subgroup of G is a TI-subgroup or a subnormal subgroup. Then every non-abelian subgroup of G is subnormal in G.*

According to Theorem 1, by arguing similarly as in the proof of Corollary 2 in [10], we have the following corollary.

COROLLARY 2 . *Let G be a group such that every non-abelian subgroup of G is a QTI-subgroup or a subnormal subgroup. Then for every prime p dividing $|G|$, G must have either a normal Sylow p-subgroup or else a Sylow p-subgroup is abelian and there exists a normal p-complement.*

Now we give the proof of Theorem 2.

PROOF of Theorem 2. Let G be a group such that every non-cyclic subgroup of G is a QTI-subgroup or a subnormal subgroup. Since every non-abelian subgroup of G must be a non-cyclic subgroup, it follows from Theorem 1 that every non-abelian subgroup of G is subnormal in G.

Suppose that there is at least one non-cyclic subgroup of G which is not subnormal in G and let E be a non-cyclic subgroup which is not subnormal in G with the largest order. Then E is a non-subnormal QTI-subgroup of G. Moreover, $E = N_G(E)$ by the choice of E. We have that E is abelian by the above paragraph. It follows that $E = Z(E) \neq 1$. Similarly arguing as in paragraph 5 of the proof of Theorem 1, we have that E is a TI-subgroup of G. Therefore, G is a Frobenius group with E being a Frobenius complement. As E is abelian, it follows from Lemma 2.2 that E is cyclic, which is a contradiction.

□

Finally, we give the proof of Theorem 3.

PROOF of Theorem 3. For the necessity. Assume that G is non-nilpotent. Then there exists a maximal subgroup M of G such that M is not normal in G. Obviously, M is not subnormal in G. By the hypothesis, M is a QTI-subgroup of G and $M = N_G(M)$. It follows that M is a CC-subgroup of G. By arguing similarly as in the proof of Theorem 1, we have that M is nilpotent and G is a Frobenius group with M as a complement. Let N be the Frobenius kernel of G. Then $G = N \rtimes M$. Furthermore, N is a minimal normal subgroup of G since M is maximal in G. Therefore, N is an elementary abelian p-group for some prime p as N is nilpotent (or by Exercise 6, Chapter 4 of [6]). As every non-abelian subgroup of G is a QTI-subgroup or a subnormal subgroup, it follows from Theorem 1 that every non-abelian subgroup of G is subnormal in G. Therefore, M is abelian, and thus M is cyclic according to Lemma 2.2. In particular, G is solvable.

Let T be a subgroup of G such that $T \nsubseteq N$. Then $\pi(T) \nsubseteq \pi(N)$ as N is a Hall subgroup of G. As $\pi(G) = \pi(N) \cup \pi(M)$, we have that $\pi(T) \cap \pi(M) \neq \emptyset$. Let $\pi = \pi(T) \cap \pi(M)$. As G is solvable and M is a Hall subgroup of G, we have that a Hall π-subgroup of T is contained in some conjugate of M, saying M^x for example. Therefore, $T \cap M^x \neq 1$. Then by Lemma 2.4, either $T \leq M^x$ or $T = (T \cap N) \rtimes (T \cap M^x)$. For the case $T = (T \cap N) \rtimes (T \cap M^x)$, if $T \cap N \neq 1$, then T is non-abelian. By Theorem 1, T must be subnormal in G. Then by Lemma 2.3, we have that $T \leq N$ or $N < T$. Since $T \cap M^x \neq 1$, it can only be that $N < T$, whence $T = N \rtimes (T \cap M^x)$. Therefore, G is a group of the structure (2) in this theorem.

For the sufficiency. If G is nilpotent, then there is nothing to be proved. Suppose that $G = N \rtimes M$ is a group of the type (2) in the theorem. Let K be a subgroup of G. If $K \leq N$, then K is subnormal in G. If $K \leq M^y$ for some $y \in G$, then $M^y \leq N_G(K)$ as M^y is cyclic. Noticing that M^y is maximal in G and Lemma 2.3, we have that $N_G(K) = M^y$. As $Z(G) = 1$, we have that $C_G(x) = M^y = N_G(K)$ for every $1 \neq x \in K$, whence K is a QTI-subgroup of G. If $K = NH$ with $1 < H \leq M^x$, then $(NH)^G = (NH)^{NM^x} = NH^{M^xN} = NH^N = NH$, and thus NH is normal in G. Therefore, every subgroup of G is a QTI-subgroup or a subnormal subgroup.

□

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