THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 22, Number 4/2021, pp. 307–315

ON A NONLINEAR EVOLUTION MODEL IN AN ORDERED BANACH SPACE WITH NORM ADDITIVE ON THE POSITIVE CONE

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Abstract. Previous results on the existence of solutions of a nonlinear evolution equation formulated in an AL-space, by abstracting common properties of collisional kinetic models, are extended to the setting of a partially ordered Banach space with norm additive on the positive cone, which is not necessarily a Banach lattice. An application is sketched in the case of a simple nonlinear von Neumann equation in the space of trace class self-adjoint operators on a separable Hilbert space.

Key words: nonlinear evolution equation, ordered Banach space, abstract state space, positive semigroup, collisional kinetic equation, Povzner inequality, trace class operators, nonlinear von Neumann equation.

1. INTRODUCTION

Let X be a real separable Banach space, (partially) ordered by the order relation $\leq$, for which the positive cone $X_+ := \{g : g \in X; 0 \leq g\}$ is closed and generating (i.e., $X = X_+ - X_+$), and the norm is additive on X_+, i.e.,

$$\|g+h\| = \|g\| + \|h\| \quad \forall g,h \in X_+. \tag{1}$$

Following [1, p. 30] (see also, [2], and [3]), X will be called abstract state space. A first example of such a space is an AL-space (abstract Lebesgue space), i.e. a Banach lattice whose norm is additive on its positive cone. However, an abstract state space is not necessarily a Banach lattice, e.g., the space of trace class self-adjoint operators on some separable Hilbert space, with the trace norm, and the canonical order of the bounded self-adjoint operators[1].

In X, consider the Cauchy problem

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = Af(t) + Q^+(t,f(t)) - Q^-(t,f(t)), \quad f(0) = f_0 \in X_+, \quad t \geq 0, \tag{2}$$

for f defined from $\mathbb{R}_+ := [0,\infty)$ to X_+. Here, A is the infinitesimal generator of a C_0-group of positive linear isometries $\{U^t\}_{t\in\mathbb{R}}$ on X_+ (case $A = 0$ is not excluded), and $Q^\pm$ are (nonlinear) mappings from $\mathbb{R}_+ \times \mathscr{D}$ to X_+, for some $\mathscr{D} \subset X_+$ dense in X_+, such that:

• $\mathbb{R}_+ \ni t \mapsto Q^\pm(t,g(t)) \in X_+$ are (Lebesgue) measurable for every measurable $g : \mathbb{R}_+ \mapsto X_+$ which satisfies $g(t) \in \mathscr{D}$ almost everywhere (a.e.) on $\mathbb{R}_+$.

• For almost all (a.a.) $t \geq 0$, the positive mappings $\mathscr{D} \ni g \mapsto Q^\pm(t,u) \in X_+$ are isotone and o-closed, and their common domain $\mathscr{D}$ is p-saturated (see the Appendix for some known definitions and facts).

The existence of solutions to problem (2) was investigated in [4], under additional conditions (and developing ideas of [5]), in the case of an abstract model generalizing several collisional kinetic equations with

[1] For other examples of abstract state spaces, the reader is referred to [1, pp. 30–31].

common monotonicity properties, and compatible, in some sense [4], with the so-called Povzner inequality [6]. Paper [4] also included applications to examples of the so-called classical kinetic theory (Boltzmann equation, Smoluchowskis coagulation equation, a Povzner-like model with dissipative collisions).

However, the results of [4] were obtained by assuming that X is an AL-space, and may not be directly applied to problems involving ordered Banach spaces that are not Banach lattices, as the aforementioned space of trace class self-adjoint operators, which may be encountered in quantum kinetic modeling.

The present note shows briefly how the main result of [4] can be re-obtained in the more general setting introduced in the beginning of this section, without imposing that X should be a (Banach) lattice. For a more detailed exposure, the reader is referred to [7] (where a "Corrigendum" to an easily correctable error in [4]) was also included, independently of the main content of [7]). Differently from [7], the present work is limited to accounting for the main new ideas behind the aforementioned generalization of the results of [4], and also contains a simple application that extends a result of [8].

2. MAIN RESULT

To state the main result of this paper, we first complete the setting detailed in the previous section with the rest of the assumptions that define the model introduced in [4]:

Assumption (A). There exists a linear operator $\Lambda : \mathscr{D}(\Lambda) \subset X \mapsto X$ such that $(-\Lambda)$ is the infinitesimal generator of a positive C_0-semigroup on X, and $\mathscr{D}_+(\Lambda) \subset \mathscr{D}$, $Q^{\pm}(t, \mathscr{D}_+(\Lambda^k)) \subset \mathscr{D}_+(\Lambda^{k-1})$, $t \geq 0$ a.e., $k = 2,3$ (where we have used the notation $\mathscr{D}_+(\Lambda^k) := \mathscr{D}(\Lambda^k) \cap X_+$, $k = 1,2...$).

Assumption (A_0). There is a number $\lambda_0 > 0$ such that

$$\lambda_0 g \leq \Lambda g, \quad \forall g \in \mathscr{D}_+(\Lambda). \tag{3}$$

Assumption (A_1). There exists a positive, non-decreasing, convex function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$, such that for a.a. $t \geq 0$,

$$0 \leq Q^-(t,g) \leq a(\|\Lambda g\|)\Lambda g, \quad \forall g \in \mathscr{D}_+(\Lambda), \tag{4}$$

and the mapping $\mathscr{D}_+(\Lambda) \ni g \mapsto a(\|\Lambda g\|)\Lambda g - Q^-(t,g) \in X$ is *isotone*.

Assumption (A_2). For a.a. $t \geq 0$,

$$\Delta(t,g) := \left\|\Lambda Q^-(t,g)\right\| - \left\|\Lambda Q^+(t,g)\right\| \geq 0, \quad \forall g \in \mathscr{D}_+(\Lambda^2), \tag{5}$$

and the map $\mathscr{D}_+(\Lambda^2) \ni g \mapsto \Delta(t,g) \in \mathbb{R}_+$ is *isotone*.

Assumption (A_3). There exists a positive non-decreasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that for a.a. $t \geq 0$,

$$\left\|\Lambda^2 Q^+(t,g)\right\| \leq \left\|\Lambda^2 Q^-(t,g)\right\| + \rho(\|\Lambda g\|)\left\|\Lambda^2 g\right\|, \quad \forall g \in \mathscr{D}_+(\Lambda^3). \tag{6}$$

The next remark collects useful immediate consequences of the above assumptions:

Remark 1. (a) Function a is locally Lipschitz continuous on every compact sub-interval of R_+, its derivative is a.e. defined, positive and non-decreasing on $\mathbb{R}_+$, and $a(0) = a(0+)$;

(b) For each $k = 1,2,...$, the linear operator Λ^k is positive, closed, and densely defined (see the Appendix);

(c) For a.a. $t \geq 0$, one has $\Delta(t,g) \leq \|\Lambda Q^-(t,g)\| \leq a(\|\Lambda g\|)\|\Lambda^2 g\|$ and $\|Q^{\pm}(t,g)\| \leq \lambda_0^{-1}\|\Lambda Q^-(t,g)\| \leq a(\|\Lambda g\|)\lambda_0^{-1}\left\|\Lambda^2 g\right\|$, $\forall g \in \mathscr{D}_+(\Lambda^2)$;

(d) $Q^{\pm}(t,0) = 0$ and $\Delta(t,0) = 0$ a.e. on $\mathbb{R}_+$.

Inequality (5) is abstracting common conservation/dissipation properties of several collisional kinetic equations (see [4]). The above model assumptions show some control on $Q^{\pm}(t,f)$, in terms of $\Lambda^k f$. Inequality (6) is an abstract correspondent to the Povzner inequality [5,6] (see also [4]).

In this paper, $L^1(\mathbb{R}_+;X_+)$ ($L^1_{loc}(\mathbb{R}_+;X_+)$) denotes the space of equivalent classes of Lebesgue measurable functions from $\mathbb{R}_+$ to X_+ which are Bochner integrable (locally Bochner integrable) on $\mathbb{R}_+$. Also, $C(\mathbb{R}_+;X_+)$ stands for the space of continuous functions from $\mathbb{R}_+$ to X_+. In addition, $L^1_{k,loc}(\mathbb{R}_+;X_+)$ denotes the space of the measurable mappings $g:\mathbb{R}_+\mapsto\mathscr{D}(\Lambda^k)$, with the property $\Lambda^k g\in L^1_{loc}(\mathbb{R}_+;X_+)$, $k=1,2,\ldots$. We also put $L^1_{0,loc}(\mathbb{R}_+;X_+)=L^1_{loc}(\mathbb{R}_+;X_+)$.

The next theorem shows that the main result of [4] remains valid in our present setting. Let $\mathscr{D}_+(\Lambda^\infty):=\cap_{n\geq 1}\mathscr{D}_+(\Lambda^k)$.

THEOREM 1. *Suppose that $Q^+(t,\mathscr{D}_+(\Lambda^\infty))\subset\mathscr{D}_+(\Lambda^\infty)$, $t\geq 0$ a.e., and $\Lambda^kQ^+(\cdot,\mathscr{D}_+(\Lambda^\infty))\subset L^1_{loc}(\mathbb{R}_+;X_+)$, $k=1,2,\ldots$. Let $f_0\in\mathscr{D}_+(\Lambda^2)$, in (2). Then:*

(a) *If $A=0$, then problem (2) has a unique, positive strong solution f on $\mathbb{R}_+$, such that $f(t)\in\mathscr{D}_+(\Lambda^2)$ for all $t\geq 0$, and $\|\Lambda^2 f\|$ is locally bounded on $\mathbb{R}_+$. Moreover, $\Lambda f\in C(\mathbb{R}_+;X_+)$. Furthermore, f satisfies*

$$\|\Lambda f(t)\|+\int_0^t\Delta(s,f(s))ds=\|\Lambda f_0\|,\quad\forall t\geq 0,\tag{7}$$

and

$$\left\|\Lambda^2 f(t)\right\|\leq\left\|\Lambda^2 f_0\right\|\exp(\rho(\|\Lambda f_0\|)t),\quad\forall t\geq 0;\tag{8}$$

(b) *If $A\neq 0$ and, for each $t>0$, $U^t\mathscr{D}(\Lambda)=\mathscr{D}(\Lambda)$ and $U^t\Lambda=\Lambda U^t$ on $\mathscr{D}(\Lambda)$, then problem (2) has a unique, positive mild solution f on $\mathbb{R}_+$, with the same properties as in (a).*

In applications, equality (7) can be interpreted as a dissipation-conservation relation, while (8) is related to some so-called “moment” estimates [4]. Expression (7) and inequality (8) are, in some sense, integral formulations of (5) and (6), respectively.

In the following, we sketch the proof of the above theorem. The proof is close to the central argument of [4], with two main differences to be briefly pointed out in the next subsection[2].

Here we notice that, by means of Remark 1(d) and (7), it can be seen that if $f_0=0$ in problem (2), then $f(t)\equiv 0$ is the only solution of (2) with the properties stated in Theorem 1. Moreover, (A_1), (A_2), and (7) imply that $f(t)\equiv f_0$ is the unique solution of (2), in the case when $0\neq f_0\in\mathscr{D}_+(\Lambda^2)$ and $a(\|\Lambda f_0\|)=0$.

Therefore, for the rest of this section, we suppose that in problem (2), f_0 satisfies $f_0\neq 0$ and $a(\|\Lambda f_0\|)\neq 0$.

2.1. Sketch of the proof of Theorem 1(a)

In this subsection, we suppose that the conditions of Theorem 1(a) hold.

As in [4], in the case of (2) with $A=0$, we introduce the auxiliary problem

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t}+a(\|\Lambda f_0\|)\Lambda f(t)=B(t,f,f),\quad f(0)=f_0\in X_+,\quad t\geq 0,\tag{9}$$

where a is given by (A_1), and B is formally defined a.e. on $t\in R_+$, by

$$B(t,g,h):=Q^+(t,g(t))-Q^-(t,g(t))+a\left(\|\Lambda g(t)\|+\int_0^t\Delta(s,h(s))\mathrm{d}s\right)\Lambda g(t),\tag{10}$$

for, say, g, $h\in L^1_{2,loc}(\mathbb{R}_+;X_+)$.

Also, consider the integral form of (9) with B replaced by (10)

$$\begin{aligned}f(t)&=f_0+\int_0^t[Q^+(s,f(s))-Q^-(s,f(s))]\mathrm{d}s\\&+\int_0^t\left[a\left(\|\Lambda f(s)\|+\int_0^s\Delta(\tau,f(\tau))d\tau\right)-a(\|\Lambda f_0\|)\right]\Lambda f(s)\mathrm{d}s\quad\forall t\geq 0,\end{aligned}\tag{11}$$

[2] For more details, the reader is referred to [7].

Remark 2. Due to the properties of Λ, $Q^{\pm}$, Δ and a, we have that expression (10) defines a mapping $L^1_{2,loc}(\mathbb{R}_+;X_+) \times L^1_{2,loc}(\mathbb{R}_+;X_+) \ni (g,h) \mapsto B(\cdot,g,h) \in L^1_{1,loc}(\mathbb{R}_+;X_+)$ which is *isotone*, in the sense that if $(g_i,h_i) \in L^1_{2,loc}(\mathbb{R}_+;X_+) \times L^1_{2,loc}(\mathbb{R}_+;X_+)$, $i=1,2$, and $g_1(t) \leq g_2(t)$, $h_1(t) \leq h_2(t)$ a.e. on $\mathbb{R}_+$, then $B(t,g_1,h_1) \leq$ $\leq B(t,g_2,h_2)$ a.e. on $\mathbb{R}_+$.

The next proposition shows that Theorem 1(a) can be proved by investigating (9) instead of (2).

PROPOSITION 1. *Let $\mathbb{R}_+ \ni t \mapsto f(t) \in \mathscr{D}_+(\Lambda^2)$ such that $\|\Lambda^2 f\|$ is locally bounded on $\mathbb{R}_+$.*
(a) *If f is a strong solution to (2) with $A=0$, then $\Lambda f \in C(\mathbb{R}_+;X_+)$, and f satisfies (7);*
(b) *f is a strong solution to (2) with $A=0$ iff it is a strong solution to (9).*

Proof. To prove (a), observe that the properties of f, Λf, $\Lambda Q^{\pm}(\cdot,f)$, as well as Remark 1(b), enable us to apply Λ to the integral form of (2) with $A=0$, and use (31) (see the Appendix) with $\Gamma=\Lambda$. Thus, considering the resulting expression, it remains to invoke its continuity properties, and to evaluate its norm, by taking into account assumption (A_2), and applying (1) together with (30).

The direct statement in (b) follows by introducing (7) in (9).

To prove the converse statement in (b), put $\psi(f)(t) := \|\Lambda f_0\| - \|\Lambda f(t)\| - \int_0^t \Delta(s,f(s))ds$. One needs only show that if f is a solution to (9), then $\psi(f)(t)=0$ for all $t \geq 0$. To this end, one applies Λ to (11) and makes use of (31), with $\Gamma=\Lambda$. Then, the resulting expression is handled conveniently, by applying (1), (30), and using Remark 1(a), the positivity of $\Delta(t,f(t))$, as well as the fact that $\|\Lambda^2 f\|$ is locally bounded on $\mathbb{R}_+$. One ultimately obtains a classical Gronwall inequality for $|\psi(f)(t)|$, with vanishing initial (non-integral) term in the right hand side (r.h.s.) of the inequality, which concludes the proof. □

By virtue of Proposition 1(b), the positive strong solutions of problem (2) can be found among the positive solutions of the mild form of (9)

$$f(t) = V^t f_0 + \int_0^t V^{t-s} B(s,f,f)\mathrm{d}s, \quad t \geq 0 \tag{12}$$

(the integral being in the sense of Bochner), where $\{V^t\}_{t\geq 0}$ is the positive C_0-semigroup on X with infinitesimal generator $L := -a(\|\Lambda f_0\|)\Lambda$.

Proceeding as in the proof of [4, Theorem 3.1], we appeal to the isotonicity of B to demonstrate the existence part of Theorem 1(a) by a monotone iteration scheme. One obtains a norm bounded, increasing sequence in X_+, which is finally shown to converge to a solution of (12). The convergence can be proved by appealing to the (strong) Levi property of X_+ (see the Appendix). Although such a construction was introduced in [4], in the context of an AL-space, it actually remains valid in the more general setting of our paper, due to the following result that extends [4, Lemma 2.1] to the case when X is an abstract state space.

LEMMA 1. *(see [7, Lemma 1] for a slightly more general formulation and proof)*
(a) *For each $g \in X_+$, one has:*

$$0 \leq V^t g \leq \exp(-\lambda_0 a(\|\Lambda f_0\|)t) g \leq g, \quad \forall t \geq 0; \tag{13}$$

(b) *For each $g \in X_+$, there exists an increasing sequence $\{g_n\} \subset \mathscr{D}_+(\Lambda^\infty)$, such that $g_n \nearrow g$ as $n \to \infty$;*

(c) *Let p be a positive integer. If $\{g_n\} \subset \mathscr{D}_+(\Lambda^p)$ is increasing and $\{\Lambda^p g_n\}$ is norm bounded, then there exists $g \in \mathscr{D}_+(\Lambda^p)$ such that $\Lambda^k g_n \nearrow \Lambda^k g$ for all $k=0,1,..,p$;*

(d) *$\mathscr{D}_+(\Lambda^k)$ is p-saturated, $\forall$ $k=1,2,...,\infty$.*

Proof. The proof of the above lemma applies general properties of positive C_0-semigroups, the construction behind the argument of ([9, Theorem 10.3.4]), adapted to positive C_0-semigroups (see (33) in the Appendix), and the Levi's property of X_+. □

Remark 3. Lemma 1(d) and (4) imply that $Q^-(t,\mathscr{D}_+(\Lambda^k)) \subset \mathscr{D}_+(\Lambda^{k-1})$, a.e. on $\mathbb{R}_+$, $k = 1,2,.....\infty$. In particular, Q^- satisfies the inclusion conditions on $Q^\pm(t,\mathscr{D}_+(\Lambda^k))$ imposed in the beginning of this section.[3]

Now by Lemma 1(b), we can choose an increasing sequence $\mathscr{D}_+(\Lambda^\infty) \ni f_{0,n} \nearrow f_0$, as $n \to \infty$, where the first term of the sequence is $f_{0,1} = 0$. Then our approximating sequence is formally given by

$$\begin{aligned} &f_1(t) = 0, \quad f_2(t) = V^t f_{0,2}, \\ &f_n(t) = V^t f_{0,n} + S(t, f_{n-1}, f_{n-2}), \quad t \geq 0; \quad n = 3,4,.... \end{aligned} \tag{14}$$

where

$$S(t,g,h) := \int_0^t V^{t-s} B(s,g,h)\mathrm{d}s, \quad t \geq 0. \tag{15}$$

Here it should be emphasized that (14) is a diagonalization, in some sense, of the iteration scheme considered in [4], and leads to a more general, but simpler analysis than in [4]) (see also [7]).

The next lemmas give a rigorous meaning to (14), and show that it defines a norm bounded increasing sequence of elements in X_+, with useful integrability, and regularity properties.

Let $\mathscr{M}_\infty$ be the family of those $g \in C(\mathbb{R}_+;X_+)$ with the property that, $\forall\, 0 < T < \infty$, there is $g_T \in \mathscr{D}_+(\Lambda^\infty)$, which may depend only on g and T, such that $g(t) \leq g_T$ on $[0,T]$.

LEMMA 2. *For each $n = 1,2,3,...$, one has:*
(a) $f_n \in \mathscr{M}_\infty$. *In particular,* $f_n \in L^1_{k,loc}(\mathbb{R}_+;X_+)$, $k = 0,1,2,3$*;*
(b) f_n *is a.e. differentiable on* $(0,\infty)$*;* $n = 1,2,3,...,$

Proof. The proof of the lemma relies on rather technical arguments [7, Lemmas 3 - 5].

Basically, to demonstrate (a), the key point is to show that S given in (15) satisfies the key inclusion $S(\cdot,\mathscr{M}_\infty,\mathscr{M}_\infty) \subset \mathscr{M}_\infty$ which is then used to obtain inductively from (14) that $f_n \in \mathscr{M}_\infty$ for all $n = 1,2,....$ Finally, it is sufficient to observe that $\mathscr{M}_\infty \subset \in L^1_{k,loc}(\mathbb{R}_+;X_+)$, $k = 0,1,2,....$

To prove (b), one uses (a), the properties of V^t, and (14), to apply a standard argument [10, Ch.4, § 4.2]. □

Remark 4. $Q^\pm(\cdot, f_n(\cdot)) \in L^1_{k,loc}(\mathbb{R}_+;X_+)$ for $k = 0,1,2$.

LEMMA 3. *The sequence $f_n(t)$ is positive and increasing for all $t \geq 0$.*

Proof. Te proof is achieved by a straightforward induction that applies the isotonicity of B, the positivity of V^t, as well as the positivity and monotonicity of $\{f_{0.n}\}$. □

LEMMA 4. *For each $n = 2,3,....$,*

$$f_n(t) + \int_0^t Q^-(s, f_{n-1}(s))\mathrm{d}s \leq f_{0,n} + \int_0^t Q^+(s, f_{n-1}(s))\mathrm{d}s, \quad \forall t \geq 0, \tag{16}$$

$$\|\Lambda f_n(t)\| + \int_0^t \Delta(s, f_{n-1}(s))\mathrm{d}s \leq \|\Lambda f_{0,n}\| \leq \|\Lambda f_0\|, \quad \forall t \geq 0. \tag{17}$$

Proof. Inequalities (16) and (17) are proved inductively by taking advantage of the properties stated in Lemmas 3 and 2, and invoking general known facts (see the Appendix). □

LEMMA 5.

$$\left\|\Lambda^2 f_n(t)\right\| \leq \left\|\Lambda^2 f_0\right\| \exp(\rho(\|\Lambda f_0\|)t), \quad \forall t \geq 0, \quad n = 1,2,.... \tag{18}$$

Proof. To obtain (18), one applies Λ^2 to (16), and uses (31) with $\Gamma = \Lambda^2$. From the resulting expression, one estimates $||\Lambda^2 f_n(t)||$, by applying the monotonicity of the norm, (1), and (30). Finally, by (6) and using that (17) implies $\rho((\|\Lambda f_{n-1}(s)\|) \leq \rho((\|\Lambda f_0\|)$, one obtains a Gronwall inequality yielding (18). □

[3] However, we kept those conditions, in order to have a priori well-defined statements in Assumptions (A_2) and (A_3).

The above results enable us to complete the proof of the existence part in Theorem 1(a). Indeed, by Lemmas 3, 5 and 1(c), $\exists\, f : \mathbb{R}_+ \mapsto \mathscr{D}_+(\Lambda^2)$ measurable, such that, $\forall t \geq 0$,

$$\Lambda^k f_n(t) \nearrow \Lambda^k f(t), \quad as \quad n \to \infty; \quad k = 0, 1, 2. \tag{19}$$

Besides, the monotonicity properties of $Q^{\pm}$ and Δ imply that, for a.a. $t \geq 0$, the sequences $\{\Lambda^k Q^{\pm}(t, f_n(t))\}_{n\geq 1}$, $k = 0, 1$, and $\{\Delta(t, f_n(t))\}_{n\geq 1}$ are increasing. Also, they are bounded, because of Remark 1(c) and (19). Therefore, Levi's property implies that they are convergent. But Λ is closed and $Q^{\pm}(t, \cdot)$ are o-closed a.e. on $\mathbb{R}_+$. Consequently, $\Lambda Q^{\pm}(t, f_n(t)) \nearrow \Lambda Q^{\pm}(t, f(t))$, $k = 0, 1$, and $\Delta(t, f_n(t)) \nearrow \Delta(t, f(t))$ as $n \to \infty$, a.e. on $\mathbb{R}_+$. Taking the limit in (18), we find that f satisfies (8), hence $||\Lambda^2 f||$ is locally bounded on $\mathbb{R}_+$. In particular, $f \in L^1_{k,loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, 2$. Thus by Remark 1(c), we have $Q^{\pm}(\cdot, f(\cdot))$, $\Lambda Q^{\pm}(\cdot, f(\cdot))$, $\Delta(\cdot, f(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$.

On the other hand, due to Lemma 2(b), one can differentiate (14). Then re-arranging conveniently the terms of the resulting expression and integrating again, we get for $n \geq 3$,

$$\begin{aligned} f_n(t) = f_{0,n} &+ \int_0^t [Q^+(s, f_{n-1}(s)) - Q^-(s, f_{n-1}(s))]\mathrm{d}s \\ &+ \int_0^t \left[a\left(\|\Lambda f_{n-1}(s)\| + \int_0^s \Delta(\tau, f_{n-2}(\tau)) d\tau \right) \Lambda f_{n-1}(s) - \right. \\ &\left. - a(\|\Lambda f_0\|)\Lambda f_n(s) \right] \mathrm{d}s, \end{aligned} \tag{20}$$

The above considerations and the fact that a is non-decreasing and continuous enable us to apply conveniently the dominated convergence theorem in (20). It follows that f is solution to (11). Since f satisfies (8), Proposition 1(b) concludes the existence part of Theorem 1(a).

To demonstrate the uniqueness of the solution, we follow [4], by adapting an uniqueness argument of [5]. To put it briefly, if, besides the above constructed solution f, problem (2) with $A = 0$, has another solution F with the properties stated in Theorem 1(a), then $0 \leq f(t) \leq F(t)$ for all $t \geq 0$. Thus, if $\exists\, t_* > 0$ such that $F(t_*) \neq f(t_*)$, then $\|\Lambda f(t_*)\| < \|\Lambda F(t_*)\|$. Since $\Delta(t, \cdot)$ is isotone for a.a. $t \geq 0$, we get $\|\Lambda f_0\| = \|\Lambda f(t_*)\| + \int_0^{t_*} \Delta(s, f(s))\mathrm{d}s <$ $< \|\Lambda F(t_*)\| + \int_0^{t_*} \Delta(s, F(s))\mathrm{d}s$, in contradiction with the fact that both f and F satisfy (7) for all $t \geq 0$. □

2.2. Sketch of the proof of Theorem 1(b)

The proof is as for [4, Corollary 3.1]. Indeed, since a mild solution to Eq. (2) is a $C(\mathbb{R}_+; X_+)$ solution to

$$f(t) = U^t f_0 + \int_0^t U^{t-s}[Q^+(s, f(s)) - Q^-(s, f(s))]\mathrm{d}s, \quad t \geq 0, \tag{21}$$

then by $F(t) := U^{-t} f(t)$ and $Q^{\pm}_U(t, F) := U^{-t} Q^{\pm}(t, U^t F)$, problem (21) is reduced to (2) with $A = 0$, and $Q^{\pm}_U$ instead of $Q^{\pm}$. Then we need only check that Theorem 1(a) applies (with $Q^{\pm}_U$ instead of $Q^{\pm}$).

3. EXAMPLE: SIMPLE NONLINEAR VON NEUMANN EQUATION

In this section, we show how Theorem 1 can be directly applied to a simple generalization of the model considered in [8].

Let $\mathscr{T} = \mathscr{T}(\mathscr{H})$ be the abstract state space of the trace class self-adjoint operators in some separable Hilbert space $\mathscr{H}$, endowed with the trace norm $\|F\|_{tr} := Tr(|F|)$ and the natural order $\leq$ induced by scalar product $(\cdot, \cdot)$ of $\mathscr{H}$ (i.e., $F \leq G$ iff $(f, Ff) \leq (f, Gf)$, $\forall\, f \in \mathscr{H}$). By $\mathscr{T}_+$ we denote the positive cone in $\mathscr{T}$.

Let H be a self-adjoint operator and $\{U^t\}_{t\in\mathbb{R}}$ the C_0-group of positive isometries on $\mathscr{T}$, defined by $U^t F := \exp(-\mathrm{i}Ht) F \exp(\mathrm{i}Ht)$, where $\mathrm{i} = \sqrt{-1}$. The infinitesimal generator A of $\{U^t\}_{t\in\mathbb{R}}$ can be written as $AF := -i[H, F] = i(FH - HF)$, $F \in \mathscr{D}(A)$, where $[\cdot, \cdot]$ is the usual notation for the commutator.

Consider the problem in $\mathscr{T}$

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -i[H,F(t)] + Q(F(t)), \quad F(0) = F_0 \in \mathscr{T}_+, \tag{22}$$

where Q is (possibly) a nonlinear mapping in $\mathscr{T}$.

Equations of the form (22), supplemented with the "conservation" condition

$$||F(t)||_{tr} = ||F_0||_{tr}, \quad t \geq 0, \tag{23}$$

are encountered in quantum mechanical problems modeling the evolution of the so-called quantum density operator, where they are know as nonlinear von Neumann equations (see, e.g., [11,12].

Here, we suppose that H has a purely discrete spectrum[4], and $\{e_n\}_{n\mathbb{N}}$ is the orthonormal basis associated to its eigenvectors.

For some strictly increasing sequence $0 < \lambda_0 < \lambda_1 < \dots < \lambda_n \dots \nearrow \infty$, as $n \to \infty$, let $\{V^t\}_{t\geq 0}$ be the C_0-semigroup on $\mathscr{T}$, defined by

$$(e_n, V^t F e_m) := \exp[-(1+\lambda_n \delta_{n,m})t] F_{n,m}, \tag{24}$$

where $F_{n,m} := (e_n, F e_m)$. Thus, denoting by $(-\Lambda)$ the infinitesimal generator of $\{V^t\}_{t\geq 0}$, we have

$$(e_n, \Lambda F e_m) := (1+\lambda_n \delta_{n,m}) F_{n,m}, \tag{25}$$

hence $\Lambda \geq I$ (where I is the identity operator in $\mathscr{T}$). Obviously, U^t leaves $\mathscr{D}_+(\Lambda^k) := \mathscr{D}(\Lambda^k) \cap X_+$ invariant, and $U^t \Lambda^k = \Lambda^k U^t$ on $\mathscr{D}_+(\Lambda^k)$, $k = 1,2,\dots$

Further, we make precise Q. To this end, let $\{q_n^{\pm}\}_{n\in\mathbb{N}}$ be a family of o-closed, isotone mappings from $D_+(\Lambda)$ to $\mathbb{R}_+$ such that $\sum_{n\in\mathbb{N}}(1+\lambda_n)^{k-1} q_n^{\pm}(F) < \infty$, $\forall\, F \in D_+(\Lambda^k)$, $k = 1,2,3$, and $\sum_{n\in\mathbb{N}}(1+\lambda_n)^k q_n^{+}(F) < \infty$, $k = 1,2,\dots$, $\forall\, F \in \cap_{k\geq 1}\mathscr{D}_+(\Lambda^k)$. In addition, we assume:

1° $\exists\, a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ non-decreasing and convex such that, for each $n = 0,1,2,\dots$,

$$q_n^-(F) \leq \lambda_n a(\sum_{i\in\mathbb{N}}(1+\lambda_i)F_{i,i}) F_{n,n}, \quad \forall F \in D_+(\Lambda)$$

and the mapping $D_+(\Lambda) \ni F \mapsto \lambda_n a(\sum_{i\in\mathbb{N}}(1+\lambda_i)F_{i,i})F_{n,n} - q_n^-(F) \in \mathbb{R}_+$ is isotone;

2°

$$\Delta(F) = \sum_{n\in\mathbb{N}} \lambda_n (q_n^-(F) - q_n^+(F)) \geq 0, \quad \forall F \in D_+(\Lambda^2), \tag{26}$$

and the mapping $D_+(\Lambda^2) \ni F \mapsto \Delta(F) \in \mathbb{R}_+$ is isotone;

3°

$$\sum_{n\in\mathbb{N}} (1+\lambda_n)^2 [q_n^+(F) - q_n^-(F)] \leq \rho(\sum_{n\in\mathbb{N}}(1+\lambda_n)F_{n,n})(\sum_{n\in\mathbb{N}}(1+\lambda_n)^2 F_{n,n}), \quad \forall F \in D_+(\Lambda^3) \tag{27}$$

for some positive non-decreasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Motivated by (23), we also suppose

4°

$$\sum_{n\in\mathbb{N}} q_n^+(F) = \sum_{n\in\mathbb{N}} q_n^-(F), \quad \forall F \in D_+(\Lambda). \tag{28}$$

Notice that 4° formally implies (23).

By the above hypotheses, for each $F \in \mathscr{D}_+(\Lambda)$, we can define the operators $Q^{\pm}(F) \in \mathscr{T}_+$ as

$$Q^{\pm}(F) := \sum_{n\in\mathbb{N}} q_n^{\pm}(F)(e_n, \cdot)e_n, \tag{29}$$

[4] E.g, H is the Hamiltonian of the one-dimensional, non-relativistic quantum oscillator in L^2.

and put $Q = Q^+ - Q^-$ in (22).

It can be checked that A, Λ and $Q^{\pm}(F)$ (as defined above) satisfy the assumptions behind Theorem 1(b). Therefore the theorem can be directly applied to (22). Due to 4°, the solution of (22) provided by Theorem 1(b) satisfies (23).

On the other hand, it should be remarked that, expressed in terms of $F_{n,m}$, Eq. (22) yields an ODE system, which can be decoupled into a trivial part, with solutions $F_{n,m}(t) = F_{0n,m}\exp[i(\lambda_m - \lambda_n)t]$, when $n \neq m$, and a nontrivial one, otherwise,

$$\frac{\mathrm{d}F_{n,n}}{\mathrm{d}t} = q_n^+(F) - q_n^-(F) \quad F_{n,n}(0) = F_{0n,n} \geq 0, \quad n = 0,1,2,...,$$

where the mappings $q_n^{\pm}$ satisfy conditions $1^\circ - 4^\circ$.

We finally emphasize that the above model reduces to the caricature of von Neumann - Boltzmann equation considered in [8] if, for $n = 0,1,2$, $q_n^{\pm}$ are of the form $q_n^{\pm}(F) = \varepsilon_n^{\pm} F_{0,0} Tr(\Lambda F)$, with $\varepsilon_n^{\pm}$ suitable (not all vanishing) positive constants, while $q_n^{\pm} \equiv 0$, for $n \geq 3$.

APPENDIX

In the following, we briefly recap known definitions and facts, needed in previous sections. Although some of the below statements are valid in larger contexts, here we keep the assumption that X is an abstract state space, in the sense of Section 1, with norm $||\cdot||$, order $\leq$ and positive cone X_+.

First recall that, in our setting (X - abstract state space), the following properties hold: the norm $||\cdot||$ is monotone, i.e., if $0 \leq h \leq g$, then $\|h\| \leq \|g\|$; X_+ satisfies the strong Levi property, i.e., every norm bounded increasing sequence in X_+ is convergent (see [13, Definition 2.44]); if $g : \mathfrak{S} \mapsto X_+$ is Bochner integrable, then

$$\left\| \int_{\mathfrak{S}} g(s)\mathrm{d}s \right\| = \int_{\mathfrak{S}} \|g(s)\| \,\mathrm{d}s, \tag{30}$$

(the integral in the r.h.s. of (30) being in the sense of Lebesgue). Moreover, if a set $\mathfrak{S} \subset \mathbb{R}$ is (Lebesgue) measurable and $g : \mathfrak{S} \mapsto X_+$ is Bochner integrable, then

$$\int_{\mathfrak{S}} g(s)\mathrm{d}s \in X_+,$$

where ds is the Lebesgue measure on the real line.

A set $\emptyset \neq \mathscr{M} \subset X$ is called positively saturated (p-saturated) [4] if for all $h \in \mathscr{M}$ and $g \in X_+$,

$$g \leq h \Rightarrow g \in \mathscr{M}.$$

Consider a mapping $\Gamma : \mathscr{D}(\Gamma) \subset X \mapsto X$, with $\mathscr{D}(\Gamma) \cap X_+ \neq \emptyset$. The mapping Γ is called *positive* if $\mathscr{D}(\Gamma) \cap X_+ \subset X_+$. One says that Γ is isotone (monotone) if $g, h \in \mathscr{D}(\Gamma)$ and $g \leq h$, imply $\Gamma(g) \leq \Gamma(h)$. Γ is called closed with respect to the order (o-closed) [4] if for every increasing sequence $\{g_n\} \subset \mathscr{D}(\Gamma)$ we have that g_n converges to g (in symbols, $g_n \nearrow g$) and $\Gamma(g_n) \to h$ imply $g \in \mathscr{D}(\Gamma)$ and $\Gamma(g) = h$. Similar definitions may be introduces for mappings between two different abstract state spaces, or more generally, ordered spaces, in particular between X and $\mathbb{R}$ endowed with the usual order.

If $\Gamma : \mathscr{D}(\Gamma) \subset X \mapsto X$ is a closed linear operator, $\mathfrak{S} \subset \mathbb{R}$ is measurable, $g : \mathfrak{S} \mapsto \mathscr{D}(\Gamma)$ is Bochner integrable, and Γg is also Bochner integrable, then

$$\Gamma \int_{\mathfrak{S}} g(s)\mathrm{d}s = \int_{\mathfrak{S}} \Gamma g(s)\mathrm{d}s. \tag{31}$$

If $\{S^t\}_{t\geq 0}$ is a one-parameter C_0-semigroup of linear operators (on short, C_0-semigroup) on X, then its infinitesimal generator $G : \mathscr{D}(G) \subset X \mapsto X$ is a closed linear operator, with the domain $\mathscr{D}(G)$ dense in X. The same is true for the positive integral powers G^k (defined by $G^1 := G$, $\mathscr{D}(G^k) := \{g : g \in \mathscr{D}(G^{k-1}),\ G^{k-1}g \in \mathscr{D}(G)\}$, $G^k g := G(G^{k-1}g)$, $k = 2, 3...$).

Let $G^0 := I$, $\mathscr{D}(G^0) := X$, where I is the identity operator on X. Then

$$\int_0^t S^s g \mathrm{d}s \in \mathscr{D}(G^{k+1}), \quad \forall g \in \mathscr{D}(G^k), \quad \forall t > 0, \quad k = 0, 1, 2, ... \tag{32}$$

Then $\mathscr{D}(G^\infty) := \cap_{n\geq 1}\mathscr{D}(G^n)$ is dense in X. Indeed, following [9, Theorem 10.3.4]), let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$, indefinitely differentiable on $(0, \infty)$, with compact support, and satisfying, $\int_0^\infty \varphi(t)dt = 1$. For $g \in X$, let

$$\mathscr{D}(G^\infty) \ni g_n := n \int_0^\infty \varphi(nt) S^t g \mathrm{d}t, \quad n = 1, 2, ... \tag{33}$$

Then $g_n \to g$ as $n \to \infty$.

If the C_0-semigroup $\{S^t\}_{t\geq 0}$ is positive, i.e., $S^t X_+ \subset X_+$, $\forall\ t > 0$, then $\mathscr{D}(G^\infty) \cap X_+$ is dense in X_+, as can be seen by choosing $\varphi \geq 0$ and $g \in X_+$ in (33).

REFERENCES

1. E. B. DAVIES, *Quantum theory of open systems*, Academic Press, 1976.
2. L. ARLOTTI, B. LODS, M. MOKHTAR-KHARROUBI, *On perturbed substochastic semigroups in abstract state spaces*, Zeitschrift für Analysis und ihre Anwendungen, **30**, pp. 457–495, 2011.
3. L. ARLOTTI, B. LODS, M. MOKHTAR-KHARROUBI, *Non-autonomous Honesty theory in abstract state spaces with applications to linear kinetic equations*, Communications on Pure and Applied Analysis, **13**, *2*, pp. 729–771, 2014.
4. C. P. GRÜNFELD, *A nonlinear evolution equation in an ordered space, arising from kinetic theory*, Communications in Contemporary Mathematics, **09**, pp. 217–251, 2007.
5. L. ARKERYD, *On the Boltzmann equation I & II*, Archive for Rational Mechanics and Analysis, **45**, pp. 1–34, 1972.
6. A. J. POVZNER, *The Boltzmann equation in the kinetic theory of gases*, Matematicheskii Sbornik (N.S.), **58**, *100*, pp. 65–86, 1962.
7. C. P. GRÜNFELD *A nonlinear evolution equation in an ordered Banach space reconsidered*, arXiv preprint, arXiv:1905.00193 [math.DS], 2019.
8. C. P. GRÜNFELD, *On a nonlinear evolution equation in an ordered space*, Analele Universității București (Matematică), **55**, *2*, pp. 183–194, 2006.
9. E. HILLE, R. S. PHILLIPS, *Functional analysis and semi-groups*, American Mathematical Society, 1974.
10. A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. **44**, Springer-Verlag, 1983.
11. J. MESSER, B. BAUMAGRTNER, *Nonlinear von Neumann equations for quantum dissipative systems*, Zeitschrift fr Physik B Condensed Matter, **32**, pp. 103–105,1978.
12. M. HENSEL, H J KORSCH, *Dissipative quantum dynamics: solution of the generalized von Neumann equation for the damped harmonic oscillator*, Journal of Physics A: Mathematical and General, **25**, pp. 2043–2064, 1992.
13. D. CHARALAMBOS, D ALIPRANTIS, R TOURKY, *Cones and duality*, Graduate studies in mathematics, Vol. **84**, American Mathematical Society, 2007.

Received November 4, 2021