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A PROOF OF THE CENTRAL LIMIT THEOREM FOR C-FREE QUANTUM RANDOM VARIABLES

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Abstract. We give a new proof of the multivariate CLT in the c-free probability theory due to M. Bożejko and R. Speicher [4,3], by extending a combinatorial method exposed by F. Hiai and D. Petz [7](univariate case) or A. Nica and R. Speicher [11] (uni- and multivariate case) for CLT in the frame of D.-V. Voiculescu's free probability theory [15–17].

Key words: non-crossing partition, quantum probability space, non-commutative distribution, φ,ψ-freeness, Wick type formula.

1. INTRODUCTION

Through his investigations on the free group II_1 type factors in the theory of von Neumann algebras, D.-V. Voiculescu created the free probability theory (see, e.g. [15−17] for more information): a quantum probability theory (see, e.g., [6] as an introduction into this field) for "highly" non-commutative random variables, based on free independence (: freeness) as central concept, interpreted as an analogue of the stochastic independence from the classical probability theory. He proved [15] a CLT for freely independent random variables with the famous Wigner's semi-circular law as limit distribution; this key result guided him to reveal a deep connexion with the random matrix theory, transforming then the free probability theory in an expansive and important domain of research with spectacular applications in many fields (see, e.g., [16, 17], but also [7, 11], and the rich bibliography therein). R. Speicher [13] gave a more algebraic proof of Voiculescu's free CLT in W.von Waldenfels' style and discovered the combinatorial structure of freeness based on non-crossing partitions; then, he [14] and A. Nica developed the combinatorial facet of free probability (see, e.g., the monograph [11], and the references therein).

Due to M. Bożejko's previous work on Haagerup type functions on free groups, Bożejko and Speicher [4] introduced a generalization of freeness with respect to two states (: c-freeness), proving a CLT in this frame for identically distributed random variables, with a so-called free Meixner distribution (see, e.g., [1]) as limit. The structure of c-freeness is governed by the non-crossing partitions, but it must distinguish between outer and inner blocks, as they revealed. Thus, they initiated the c-free probability theory, a new and dynamic research topic (see, e.g., [4, 3], the recent [2], and the references therein).

In this Note, we prove the multivariate CLT for φ,ψ-free random variables in Bożejko-Speicher theory, by extending the combinatorial moment method presented in [7] or [11] for the free CLT; but, we focus on the occurrence of the interval blocks in the partition associated now to a product of ψ-centered φ,ψ-free random variables. Alternatively, by cumulants, the proof is shorter. Other limit theorems can be proved. We will detail these elsewhere.

2. PRELIMINARIES

We recall some well-known general information as in, e.g., [4, 11, 14]. (We abbreviate 'such that' by 's.t.', and 'with respect to' by 'w.r.t'). If S is a finite totally ordered set, we denote by $P(S)$ the partitions of S and by $P_{1,2}(S)$ the partitions in $P(S)$ for which every block has at most two elements; calling blocks the non-empty subsets defining a partition (in general). We call pairing a partition in which every block has exactly two elements. For $k,l \in S$, denote by $k \sim_\pi l$ the fact that k and l belong to the same block of $\pi \in P(S)$. Remind that a partition π is called non-crossing if there are no $k_1 < l_1 < k_2 < l_2$ in S s.t. $k_1 \sim_\pi k_2 \nsim_\pi l_1 \sim_\pi l_2$; otherwise, π is crossing. When π is non-crossing, and V is a block of π, say V is inner, if there exist another block W of π, and $k,l \in W$, s.t. $k < v < l$, for all $v \in V$; otherwise, say V is outer. Denote by $\mathrm{I}(\pi)$, and $\mathrm{O}(\pi)$ the inner, and, respectively, outer blocks of π. Denote by $P_2(S)$, and $NC_2(S)$ the pairings, and, respectively, non-crossing pairings of S. When S has m elements, abbreviate the above sets by $P(m)$, $P_{1,2}(m)$, $P_2(m)$, and $NC_2(m)$, respectively ($P_2(m)$ is empty if m is odd). Remind that each non-crossing partition of $\{1,...,m\}$ has at least an interval; i.e., a block of consecutive indices which may be a singleton (:block having a single element). The cardinality of $P_2(2p)$ or $NC_2(2p)$ equals the corresponding moment of a standard Gauss, respectively, semi-circular Wigner distribution; i.e. $(2p)!!$, respectively the Catalan number $c_p := (2p)!/p!(p+1)!$. If S is a disjoint union of non-void subsets S_i, and $\pi \in NC(S)$ s.t. $\pi = \cup \pi_i$, with some $\pi_i \in NC(S_i)$, we write $\pi = \coprod \pi_i$.

We consider a *- algebra as a (complex) associative algebra with an involution * (i.e. a conjugate linear anti-automorphism). A linear functional ϕ of a *- algebra A is positive if $\phi(a^* a) \geq 0$, for all $a \in A$. Let A be unital (complex) (*-) algebra, and φ, ψ be unital linear (positive) functionals of A. We interpret (A,ψ) or (A,φ,ψ) as quantum (*-) probability spaces, and the elements of A as quantum random variables in view of [16, 11]. Let I be an index set and $\mathbb{C} < \xi_i, i \in I >$ be the unital (*-) algebra freely generated by the complex field $\mathbb{C}$ and the non-commuting indeterminates $\xi_i, i \in I$. Let $a = (a_i)_{i \in I}$ be such a random vector with all (self-adjoint) $a_i \in A$. The non-commutative joint distribution of a w.r.t. ϕ is $\phi_a := \phi \circ \tau_a$, where τ_a : $\mathbb{C} < \xi_i, i \in I > \to A$ is the unique unital (*-) homomorphism s.t. $\tau_a(\xi_i) = a_i$. The scalars $\phi(a_{i_1}...a_{i_j})$ are viewed as the joint moments of a w.r.t. ϕ.

If $a_N = (a_N^i)_{i \in I}$ and $a = (a_i)_{i \in I}$ are random vectors in some quantum probability spaces (A_N, φ_N) and (A,φ), we say $(a_N)_N$ converges in distribution to a, denoting $a_N \xrightarrow{distr} a$, if for all $j \geq 1$, and all $i_1,...,i_j \in I$, $\lim_{N\to\infty} \varphi_N(a_N^{i_1}...a_N^{i_j}) = \varphi(a_{i_1}...a_{i_j})$. When $a \in A$ and $\varphi(a) = 0$, say a is centered w.r.t. φ or φ-centered. For $a \in A$ (but, generally, $\psi(a) \neq 0$), we center a w.r.t. ψ, if we decompose $a = \psi(a)\cdot 1 + a^\circ$ via the centering $a^\circ := a - \psi(a)\cdot 1$ of a w.r.t. ψ (see, e.g., [11, Notation 5.14]); 1 being here the unit of A.

When $(1\in)A_i \subset A$, $i \in I$ are unital subalgebras, then every random variable $w = a_1 \cdots a_n \in A$, s.t. all $a_k \in A_{i_k}$, for $i_1,...,i_n \in I$, determines a unique partition π on $\{1,...,n\}$ by $k \sim_\pi l \Leftrightarrow i_k = i_l$; and we call this the partition associated to w. We shall say w is crossing or non-crossing when this partition is crossing or non-crossing. We say $w = a_1 \cdots a_n \in A$, with $a_k \in A_{i_k}$, as before, is a simple random variable in (A,φ,ψ) if w is reduced (i.e., $i_1 \neq i_2 \neq ... \neq i_n$), calling n the length of w, and every a_k is ψ-centered, when $1 \leq k \leq n-1$. If $w = a_2 \cdots a_n \in A$ is a simple random variable, and $a_1 w \in A$ is reduced, we say $a_1 w$ is a quasi-simple random variable in (A,φ,ψ).

The next definition concerning the notion of φ, ψ-free independence (: φ, ψ-freeness) comes from [4,3,8-10](see also [2]).

Definition. Let (A,φ,ψ) be a quantum probability space as above, and $(1\in)A_i \subset A$, $i \in I$ be unital subalgebras. The family $(A_i)_{i \in I}$ is φ, ψ-freely independent (or φ, ψ-free, for short), if $\varphi(a_1 \cdots a_n) = \varphi(a_1)\cdots\varphi(a_n)$, for any $n \geq 2$, all $a_k \in A_{i_k}$, and all $i_1,...,i_n \in I$ s.t. $a_1 \cdots a_n$ is a simple random variable in (A,φ,ψ). If $A \supset S_i$, $i \in I$ are subsets, then $(S_i)_{i \in I}$ is φ, ψ-freely independent, if $(A_i)_{i \in I}$ is φ, ψ-freely independent, A_i being the unital subalgebra of A generated by S_i. □

In particular, the ψ,ψ-freeness is Voiculescu's freeness w.r.t. ψ (according to [16, 17, 6, 7, 11, 13]). The c-freeness w.r.t. (φ,ψ), introduced in [3], involves both freeness w.r.t. ψ, and φ,ψ-freeness.

3. JOINT MOMENTS OF φ,ψ-FREE QUANTUM RANDOM VARIABLES

Let in this section (A,φ,ψ) be a quantum probability space as before, and $(1\in)A_i\subset A$, $i\in I$ be a family of φ,ψ-freely independent unital subalgebras of A. Thus, $(A_i)_{i\in I}$ is weakly independent in (A,φ) in the sense of [5, 8]; the weak-independence having the meaning below.

Definition 3.1. Let (B,ω) be a quantum probability space and $(1\in)B_i\subset B$, $i\in I$ be unital subalgebras. The family $(B_i)_{i\in I}$ is weakly independent if $\omega(x_1...x_n)=\omega(x_1...x_p)\omega(x_{p+1}...x_n)$, for all $n>p\geq 1$, all $i_j\in I$, all $x_j\in B_{i_j}$, s.t. the sets $\{i_1,...,i_p\}$ and $\{i_{p+1},...,i_n\}$ are disjoint. If $B\supset S_i$, $i\in I$ are subsets, then $(S_i)_{i\in I}$ is weakly independent, if $(B_i)_{i\in I}$ is weakly independent; B_i being the unital subalgebra of B generated by S_i. □

For $w=a_1\cdots a_n\in A$ s.t. every $a_j\in A_{i_j}$, we say w has a singleton a_k when $i_j\neq i_k$, for any $j\neq k$.

In the next statement, the a_j, for $j\neq k$ are arbitrary.

LEMMA 3.2. *Let* $w=a_1\cdots a_n\in A$, *s.t. every* $a_j\in A_{i_j}$, *and w has a singleton* a_k *which is centered w.r.t.* φ,ψ. *Then* $\varphi(w)=0$.

Proof. It suffices to suppose w is reduced. If $k\in\{1,n\}$, the assertion follows by the weak-independence and the centeredness of a_k. Thus, it remains to consider $2\leq k\leq n-1$. For $n=3$, we get $\varphi(a_1a_ka_3)=\varphi(a_1)\varphi(a_k)\varphi(a_3)=0$, by φ,ψ-freeness and the centeredness of a_k. Then supposing the statement true for any $a_1\cdots a_r\in A$ of length $r<n$, check it for $w=a_1\cdots a_n\in A$, as follows. Center a_j w.r.t. ψ, for every $n-1\geq j\geq 2$, using $b_j:=\psi(a_j)$, $a_j^\circ:=a_j-b_j\cdot 1$, to get, via the induction hypothesis:

$$\varphi(a_1\cdots a_k\cdots a_n)=b_{n-1}\varphi(a_1\cdots a_k\cdots a_{n-2}a_n)+\varphi(a_1\cdots a_k\cdots a_{n-1}^\circ a_n)=\varphi(a_1\cdots a_k\cdots a_{n-1}^\circ a_n)=...=$$
$$=\varphi(a_1a_2\cdots a_ka_{k+1}^\circ\cdots a_n)=\ldots=\varphi(a_1a_2^\circ\cdots a_k\cdots a_{n-1}^\circ a_n)=\varphi(a_1)\varphi(a_2^\circ)\cdots\varphi(a_k)\cdots\varphi(a_{n-1}^\circ)\varphi(a_n)=0;$$

finally using again the φ,ψ-freeness property and the centeredness of a_k. □

Remark 3.3. If $w=a_1\cdots a_n\in A$, s.t. every $a_j\in A_{i_j}$, is a quasi-simple random variable in (A,φ,ψ), and $\varphi(a_n)=0$, then $\varphi(w)=0$. □

We illustrate the *next statement* by the following partitions π_j in $P_{1,2}(m)$ associated to $a_1xc_{m-1}a_m=w$.

Examples 3.4.

1) If $m=5$, let $\pi_1=\{(1,5),(2,3),(4)\}$ (non-crossing). Its interval gives $a_2a_3=:c_1\in A_{i_2}$. Thus, $w=a_1xc_4a_5=a_1c_1c_4a_5$ as reduced word, with $x:=c_1$. So, center c_1 w.r.t. ψ, denoting $b_1:=\psi(c_1)$, and $c_1^\circ:=c_1-b_1\cdot 1$, to get $w=b_1a_1c_4a_5+a_1c_1^\circ c_4a_5$, a sum of quasi-simple random variables.

2) If $m=7$, let $\pi_2=\{(1,5),(2,7),(3,4),(6)\}$ (crossing), and $\pi_3=\{(1,7),(2,3),(4,5),(6)\}$ (non-crossing). The intervals give: $a_3a_4=:c_1\in A_{i_3}$, for π_2; but, $a_2a_3=:c_2\in A_{i_2}$ and $a_4a_5=:c_1\in A_{i_4}$, for π_3. Express w in reduced form as $w=a_1xc_6a_7=a_1uc_1v_1c_6a_7$, for π_2 (with $x:=uc_1v_1$, where $a_2=:u$, and $a_5=:v_1$), but $w=a_1xc_6a_7=a_1c_2c_1c_6a_7$, for π_3. Then center c_1 w.r.t. ψ, with $b_1:=\psi(c_1)$, and $c_1^\circ:=c_1-b_1\cdot 1$, for π_2, but center c_1 and c_2 w.r.t. ψ, for π_3, with $b_j:=\psi(c_j)$, and $c_j^\circ:=c_j-b_j\cdot 1$, to get $w=b_1a_1uv_1c_6a_7+a_1uc_1^\circ v_1c_6a_7$, and, respectively, $w=b_1a_1c_2c_6a_7+a_1c_2c_1^\circ c_6a_7=b_1a_1c_2c_6a_7+b_2a_1c_1^\circ c_6a_7+a_1c_2^\circ c_1^\circ c_6a_7$, sums of quasi-simple random variables; in view of the example before.

Hence $\varphi(w)=0$, in any of these examples, by the Remark 3.3. □

LEMMA 3.5. *Let* $w=a_1xc_{m-1}a_m\in A$, *s.t.:* $a_1\in A_{i_1}$, $a_m\in A_{i_m}$; x *is void or any product of* $a_j\in A_{i_j}$ *with* $\psi(a_j)=0$; $c_{m-1}\in A_{i_{m-1}}$ *is* ψ *-centered singleton in* w; $\varphi(a_m)=0$; *and the partition associated to* a_1xa_m *is a pairing. Then* $\varphi(w)=0$, *whenever* a_1xa_m *is crossing, or* $i_1=i_m$ *and* x *is non-crossing or void.* □

The proof of the previous Lemma is similar to the next proof and we omit it, because of the page limitation.

LEMMA 3.6. *Let* $w=a_1xc_rya_m\in A$, *s.t.:* $a_1\in A_{i_1}$, $a_m\in A_{i_m}$; x *is void or any product of* $a_j\in A_{i_j}$ *with* $\psi(a_j)=0$; $c_r\in A_{i_r}$ *is* ψ *-centered singleton in w;* ya_m *is a simple random variable;* $\varphi(a_m)=0$; *and the partition associated to* a_1xya_m *is a pairing. Then* $\varphi(w)=0$, *whenever* a_1xya_m *is crossing, or* $i_1=i_m$ *and* xy *is non-crossing.*

Proof. In view of the weak-independence and the Remark 3.3, it suffices to consider the partition associated to w has at least an interval $(l,l+1)$, $l\neq 1$, and $m>5$. Thus, for $m=7$, only the next cases are, for the partition in $P_{1,2}(7)$ associated to $a_1xc_rya_m=w$; namely,

$\pi_1=\{(1,6),(2,3),(4),(5,7)\}$, $\pi_2=\{(1,6),(2,3),(4,7),(5)\}$, $\pi_3=\{(1,6),(2,7),(3,4),(5)\}$, and

$\sigma_1=\{(1,7),(2,6),(3,4),(5)\}$, $\sigma_2=\{(1,7),(2,3),(4,6),(5)\}$; which are crossing, respectively, non-crossing. Denote $a_2a_3=:c_1\in A_{i_2}$, for π_1,π_2, and σ_2, and $a_3a_4=:c_1\in A_{i_3}$, for π_3 and σ_1. Denote by c_4 and c_5 the singleton for π_1, and, respectively, for the other cases. Then, we may express w in reduced form as $w=a_1xc_4ya_7$, for π_1, and $w=a_1xc_5ya_7$, in rest, where: $x:=c_1$, for π_1; $x:=c_1v_1$, with $a_4=:v_1$, for π_2 and σ_2; $x:=uc_1$, with $a_2=:u$, for π_3 and σ_1. Center c_1 w.r.t. ψ; always get w as a sum of quasi-simple random variables. Therefore, $\varphi(w)=0$, by the Remark 3.3.

Letting $m>7$, suppose the assertion true for any word $a_1xc_rya_p$ s.t. the partition associated to a_1xya_p belongs to $P_2(p-1)$ with $p<m$, and check it for m. Consider $w=a_1xc_rya_m$ and the partition associated to a_1xya_m belonging to $P_2(m-1)$. Assume the partition associated to x has exactly k intervals giving singletons $c_k,...,c_1$ and every $c_j\in A_{i_{\tau(j)}}$, so that $x=uc_kv_k\cdots c_1v_1$, with u,v_j as reduced words; otherwise, the argument is similar. Center c_j w.r.t. ψ, denoting $b_j:=\psi(c_j)$, and $c_j^\circ:=c_j-b_j\cdot 1$, to develop

$x=\sum_{j=1}^{k}b_jx^{(j)}+x^\circ$, where $x^\circ:=uc_k^\circ v_kc_{k-1}^\circ v_{k-1}\cdots c_1^\circ v_1$; $x^{(1)}:=uc_kv_k\cdots c_2v_2v_1$;

$x^{(j)}:=uc_kv_k\cdots c_{j+1}v_{j+1}v_jc_{j-1}^\circ v_{j-1}\cdots c_1^\circ v_1$, for $2\le j\le k-1$; and $x^{(k)}:=uv_kc_{k-1}^\circ v_{k-1}\cdots c_1^\circ v_1$.

Thus the partition associated to $a_1x^{(1)}ya_m$ belongs to $P_2(m-3)$, and $x^{(2)},...,x^{(k)}$ may be expressed as algebraic sums (with ± 1 as coefficients) of random variables $\bar{x}$ having the same generic form as x, but the partition associated to each $a_1\bar{x}ya_m$ belongs to $P_2(\bar{p}-1)$, with some $\bar{p}<m$. Therefore $\varphi(a_1x^{(j)}c_rya_m)=0$, for every $j=1,...,k,$ via the inductive hypothesis. Moreover, $\varphi(a_1x^\circ c_rya_m)=0$, since $a_1x^\circ c_rya_m$ is a quasi-simple random variable.

We may conclude by induction. □

LEMMA 3.7. *Let* $w=a_1\cdots a_n\in A$, *s.t. all* $a_j\in A_{i_j}$ *are centered w.r.t.* φ,ψ, *and the partition* π *associated to* w *is a crossing pairing. Then* $\varphi(w)=0$.

Proof. In view of the weak-independence and the Remark 3.3, it remains to consider the partition associated to w has at least an interval, different of $(1,2)$ or $(n-1,n)$. Thus, for $n=6$, only three cases are: the partitions $\pi_1=\{(1,5),(2,3),(4,6)\}$, $\pi_2=\{(1,5),(2,6),(3,4)\}$, $\pi_3=\{(1,3),(2,6),(4,5)\}$.

Denote $a_2a_3 =: c_1 \in A_{i_2}$, for π_1, $a_3a_4 =: c_1 \in A_{i_3}$, for π_2, and $a_4a_5 =: c_1 \in A_{i_4}$, for π_3. We may express w in reduced form as $a_1c_1ya_6$, for π_1, and $a_1uc_1ya_6$, for π_2, but $a_1uc_1a_6$, for π_3; where: $a_4a_5 =: y$, for π_1; $a_2 =: u$, and $a_5 =: y$, for π_2; $a_2a_3 =: u$, for π_3. Center c_1 w.r.t. ψ to get w as a sum of quasi-simple random variables; and thus, $\varphi(w) = 0$, always.

Let $n > 6$, and the statement true for all $p < n$. Then, for $w = a_1 \cdots a_n \in A$, the inferences below help to conclude by induction.

When $a_{n-2}a_{n-1} =: c_{n-1} \in A_{i_{n-1}}$ becomes a singleton in w, we may express $w = a_1xc_{n-1}a_n \in A$, as in Lemma 3.5, s.t. the partition associated to a_1xa_n, is a crossing pairing. Then center c_{n-1} w.r.t. ψ, to get $w = b_{n-1}a_1xa_n + a_1xc^{\circ}_{n-1}a_n$, with $b_{n-1} := \psi(c_{n-1})$, and $c_{n-1} - b_{n-1} \cdot 1 =: c^{\circ}_{n-1}$, and remark the partition associated to a_1xa_n is crossing and belongs to $P_2(n-2)$; hence $\varphi(a_1xa_n) = 0$, by the inductive hypothesis. Moreover, $\varphi(a_1xc^{\circ}_{n-1}a_n) = 0$, by Lemma 3.5.

When $(n-2, n-1) \notin \pi$, we may express $w = a_1xc_rya_n$, as in Lemma 3.6, denoting by $a_ra_{r+1} =: c_r \in A_{i_r}$ the singleton corresponding to the greatest r for which $(r, r+1) \in \pi$. The partition associated to a_1xya_n is crossing and belongs to $P_2(n-2)$. By centering c_r w.r.t. ψ, we get now $w = b_ra_1xya_n + a_1xc^{\circ}_rya_n$, with $b_r := \psi(c_r)$, and $c_r - b_r \cdot 1 =: c^{\circ}_r$, hence $\varphi(w) = 0$; because $\varphi(a_1xya_n) = 0$, via the inductive hypothesis, and $\varphi(a_1xc^{\circ}_rya_n) = 0$, by Lemma 3.6. □

If (A, φ, ψ) is a quantum probability space as before, and $x_1, x_2 \in A$ are random variables s.t. one of them is centered w.r.t. φ, ψ, then $\varphi(x_1x_2) = k_2^{\varphi}(x_1, x_2)$, and $\psi(x_1x_2) = k_2^{\psi}(x_1, x_2)$; whenever, e.g., k_2^{φ} and k_2^{ψ} are the tensor/free/Boolean cumulants (see, e.g., [14]) w.r.t. φ, ψ, respectively, of order two. In the sequel, we may use any of these choices.

In general, the scalars involved below $\overline{k}_{\pi}(x_1, ..., x_n)$, for $\pi \in NC_2(n)$, can be described as follows.

1) If π has a single block, then that is an outer block of π, and $\overline{k}_{\pi}(x_1, x_2) := k_2^{\varphi}(x_1, x_2)$;

2) If $\pi = \sigma \coprod \rho$, with $\sigma \in NC_2(i)$ and $\rho \in NC_2(\{i+1, ..., n\})$, then

$\overline{k}_{\pi}(x_1, ..., x_n) := \overline{k}_{\sigma}(x_1, ..., x_i) \cdot \overline{k}_{\rho}(x_{i+1}, ..., x_n)$;

3) If π contains the block $(1, n)$, and the subpartition $\sigma = \pi \cap \{2, ..., n-1\}$, then

$\overline{k}_{\pi}(x_1, ..., x_n) := k_2^{\varphi}(x_1, x_n)k_{\sigma}(x_2, ..., x_{n-1})$; where, more generally, for a subpartition σ of $\pi \in NC_2(n)$, with $\sigma \in NC_2(S)$, and $S = \{1, ..., s\}$, the scalars $k_{\sigma}(x_i, i \in S)$, can be described in the following way.

1) If σ has a single block, then that is an inner block of π, and $k_{\sigma}(x_1, x_2) := k_2^{\psi}(x_1, x_2)$;

2) If $\sigma = \rho \coprod \tau$, with $\rho \in NC_2(S_1)$, and $\tau \in NC_2(S_2)$, then

$k_{\sigma}(x_1, ..., x_s) := k_{\rho}(x_i, i \in S_1) \cdot k_{\tau}(x_i, i \in S_2)$. □

LEMMA 3.8. *Let $w = a_1 \cdots a_n \in A$, s.t. all $a_j \in A_{i_j}$ are centered w.r.t. φ, ψ, and the partition π associated to w is a non-crossing pairing. Then $\varphi(w) = \overline{k}_{\pi}(a_1, ..., a_n)$.*

Proof. Due to the weak-independence, it remains to consider $(1, n) \in \pi$. For $n = 2$ and $n = 4$, the assertion is trivial, respectively immediate (via Remark 3.3). For $n = 6$, we illustrate only the case $\pi = \{(1,6), (2,3), (4,5)\}$; the case $\pi = \{(1,6), (2,5), (3,4)\}$ is similar. So, $w = a_1xc_5a_6$, where $x := a_2a_3$, and $a_4a_5 =: c_5 \in A_{i_4}$; and $w = b_5a_1xa_6 + a_1xc^{\circ}_5a_6$, with $b_5 := \psi(c_5)$, and $c_5 - b_5 \cdot 1 =: c^{\circ}_5$; then Lemma 3.5 and our assertion for $n = 4$ imply, with $\sigma = \{(1,6), (2,3)\}$,

$$\varphi(w) = b_5\varphi(a_1xa_6) = b_5\overline{k}_{\sigma}(a_1, a_2, a_3, a_6) = b_5k_2^{\varphi}(a_1, a_6)\, k_2^{\psi}(a_2, a_3) = \overline{k}_{\pi}(a_1, ..., a_6).$$

Let $n>6$. Suppose the assertion true for all $p<n$. To conclude by induction, remark alternatively the next facts.

When $a_{n-2}a_{n-1}=:c_{n-1}\in A_{i_{n-1}}$ becomes a singleton in w, we express $w=a_1xc_{n-1}a_n$, as in Lemma 3.5, again, but the partition associated to a_1xa_n is now a non-crossing pairing. Thus, we get $w=b_{n-1}a_1xa_n+a_1xc_{n-1}^{\circ}a_n$, centering c_{n-1} w.r.t. ψ, with $b_{n-1}:=\psi(c_{n-1})$, and $c_{n-1}-b_{n-1}\cdot 1=:c_{n-1}^{\circ}$; and remark $\varphi(a_1xc_{n-1}^{\circ}a_n)=0$, by Lemma 3.5. Let $\rho\in NC_2(\{2,...,n-3\})$ be the partition associated to x. Since $\{(1,n)\}\coprod\rho=:\sigma\in NC_2(\{1,...,n\}\setminus\{n-2,n-1\})$ is associated to a_1xa_n, the induction assumption implies $\varphi(a_1xa_n)=\bar{k}_\sigma(a_1,a_2,...,a_{n-3},a_n)=k_2^{\varphi}(a_1,a_n)k_\rho(a_2,...,a_{n-3})$. But, $\pi=\{(1,n)\}\coprod\tau$, where $\tau:=\rho\coprod\{(n-2,n-1)\}\in NC_2(\{2,...,n-1\})$. Then, $\varphi(w)=k_2^{\varphi}(a_1,a_n)k_\tau(a_2,...,a_{n-1})=\bar{k}_\pi(a_1,...,a_n)$.

When $(n-2,n-1)\notin\pi$, we express $w=a_1xc_rya_n$ as in Lemma 3.6, where $a_ra_{r+1}=:c_r\in A_{i_r}$ is the singleton corresponding to the largest r for which $(r,r+1)\in\pi$. But now the partition associated to a_1xya_n is from $NC_2(n-2)$. Center c_r w.r.t. ψ, to get $w=b_ra_1xya_n+a_1xc_r^{\circ}ya_n$, where $b_r:=\psi(c_r)$, and $c_r-b_r\cdot 1=:c_r^{\circ}$; but $\varphi(a_1xc_r^{\circ}ya_n)=0$, by Lemma 3.6, again. Let $\rho\in NC_2(\{2,...,n-1\}\setminus\{r,r+1\})$ be the partition associated to xy. Since $\{(1,n)\}\coprod\rho=:\sigma\in NC_2(\{1,...,n\}\setminus\{r,r+1\})$ is associated to a_1xya_n, the induction hypothesis implies $\varphi(a_1xya_n)=\bar{k}_\sigma(a_1,a_2,...,a_{r-1},a_{r+2},...,a_{n-1},a_n)=k_2^{\varphi}(a_1,a_n)k_\rho(a_2,...,a_{r-1},a_{r+2},...,a_{n-1})$.

Thus, $\varphi(w)=\bar{k}_\pi(a_1,...,a_n)$; because $\pi=\{(1,n)\}\coprod\tau$, with $\tau:=\rho\coprod\{(r,r+1)\}$ and $\tau\in NC_2(\{2,...,n-1\})$. □

4. C-FREE GAUSSIAN FAMILY AND MULTIVARIATE CLT

We remind a scalar matrix $q=\{q_{ij}\}_{i,j\in I}$ is positive if and only if $\sum_{k,l=1}^{n}q_{i_k,i_l}\bar{\lambda}_k\lambda_l\geq 0$, for all n, all $i_1,...,i_n\in I$, and all $\lambda_1,...,\lambda_n\in\mathbb{C}$.

The following definition comes from [4, 11].

Definition 4.1. Let $q=\{q_{ij}\}_{i,j\in I}$ and $r=\{r_{ij}\}_{i,j\in I}$ be (positive) scalar matrices. Let (A,φ) be a quantum (*-) probability space. A family of (selfadjoint) random variables $g=(g_i)_{i\in I}$ in this is called a centered c-free Gaussian family of covariances q and r, if its distribution is of the following form, for all $j\in\mathbb{N}$ and all $i_1,...,i_j\in I$:

$$\varphi(g_{i_1}...g_{i_j})=\sum_{\pi\in NC_2(j)}\bar{k}_\pi(g_{i_1},...,g_{i_j})\text{; where }\bar{k}_\pi(g_{i_1},...,g_{i_j}):=\prod_{(k,l)\in O(\pi)}q_{i_ki_l}\prod_{(k,l)\in I(\pi)}r_{i_ki_l}\,.\ \square$$

THEOREM 4.2. *Let* (A,φ,ψ) *be a quantum (*-)probability space, and* $\{X_r^i, i\in I\}\subset A, r\in\mathbb{N}$ *be a sequence of* φ,ψ *-freely independent sets of (selfadjoint) random variables in this, s.t.* $X_r=(X_r^i)_{i\in I}$ *has the same joint distribution for all* $r\in\mathbb{N}$*, and all variables are centered, both w.r.t.* φ,ψ*. Consider, for every* $N\geq 1$*, the sums* $S_N^i:=\frac{1}{\sqrt{N}}\sum_{r=1}^{N}X_r^i\in A$*, and* $S_N:=(S_N^i)_{i\in I}$ *as random vector in* (A,φ)*. Denote the covariances of the variables w.r.t.* φ,ψ *by* $q=\{q_{ij}\}_{i,j\in I}$ *and* $r=\{r_{ij}\}_{i,j\in I}$*; i.e.,* $q_{ij}:=\varphi(X_r^iX_r^j)$*, and* $r_{ij}:=\psi(X_r^iX_r^j)$*. Then* $S_N\xrightarrow{distr}g$*; where* $g=(g_i)_{i\in I}$ *is a centered c-free Gaussian family of (positive) covariances* q *and* r*.*

Proof. Since all X_r have the same joint distribution w.r.t. φ, ψ, and the φ, ψ-freeness gives a rule for computing joint moments w.r.t. φ, from the values of the moments of the individual variables w.r.t. φ, ψ, for all fixed $j \in \mathbb{N}$ and all $i_1,...,i_j \in I$, the moment $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j})$ depends only on the partition $\pi \in P(j)$ corresponding to $(r_1,...,r_j) \in \mathbb{N}^j$, and uniquely defining an equivalence relation $\sim_\pi$ on $\{1,..., j\}$ by $k \sim_\pi l \Leftrightarrow r_k = r_l$. We may denote $\varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) =: \varphi(\pi; i_1,...,i_j)$.

Thus,

$$\varphi(S_N^{i_1}...S_N^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{r_1,...,r_j=1}^{N} \varphi(X_{r_1}^{i_1}...X_{r_j}^{i_j}) = (\tfrac{1}{\sqrt{N}})^j \sum_{\pi\in P(j)} A_N^{|\pi|}\varphi(\pi; i_1,...,i_j),$$

as in [4, 8, 13, 11] (see also [7]); where $|\pi|$ denotes the number of blocks in π; and the number of representatives of the equivalence class (w.r.t. $\sim_\pi$) corresponding to the involved partition $A_N^{|\pi|} := N(N-1)\cdots(N-|\pi|+1)$ grows asymptotically like $N^{|\pi|}$ for large N. Lemma 3.2 implies that every partition with singletons has null contribution in the sum above. But the partitions without singletons have $|\pi| \le \frac{j}{2}$ blocks, and the limit of the factor $(\frac{1}{\sqrt{N}})^j A_N^{|\pi|}$ is 0, if $|\pi| < \frac{j}{2}$. So $\lim_{N\to\infty} \varphi(S_N^{i_1}...S_N^{i_j}) = \sum_{\pi\in P_2(j)} \varphi(\pi; i_1,...,i_j)$, because π is a pairing, if $\pi \in P(j)$ has no singletons and its number of blocks is equal to $\dfrac{j}{2}$. Thus, the odd moments vanish, since $\pi \in P_2(j)$ is void, when j is odd. We may conclude, by Lemmata 3.7 – 3.8, because the crossing pairings have null contribution in the previous sum, and, respectively, the non-crossing pairings give the desired contribution. □

Remarks 4.3. 1) As in the classical or free cases [7, 11] (see also [6, 16], for simple proofs), the assumption of being identically distributed for the involved random vectors may be replaced by the pair (i)&(ii) below, with essentially the same proof as above, but we do detail this elsewhere:

i) $\sup_{r\in\mathbb{N}} \left|\varphi(X_r^{i_1}...X_r^{i_j})\right| < \infty$, $\sup_{r\in\mathbb{N}} \left|\psi(X_r^{i_1}...X_r^{i_j})\right| < \infty$ (for all j, and all $i_1,...,i_j \in I$);

ii) there exist $q_{ij} = \lim_{N\to\infty} \dfrac{1}{N}\sum_{r=1}^{N} \varphi(X_r^i X_r^j)$ and $r_{ij} = \lim_{N\to\infty} \dfrac{1}{N}\sum_{r=1}^{N} \psi(X_r^i X_r^j)$.

2) The combinatorial description of the joint moments of a Gaussian family (: multivariate normal distribution) involving all pairings instead of non-crossing pairings (as, in particular, a semicircular family [11, 13] in the free probability theory) is usually named the Wick formula in the quantum field theory (see, e.g., [12]). By analogy, the above formula (see also [4]) describing the joint moments of such a c-free Gaussian family may be interpreted as a c-free Wick formula. □

We send to [9, 10] for some operator-valued versions of these facts or other generalizations.

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