THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 22, Number 4/2021, pp. 325–331

OVERPARTITIONS AS SUMS OVER PARTITIONS

Mircea MERCA

University of Craiova, Department of Mathematics
A. I. Cuza 13, Craiova 200585, Romania
Corresponding author: Mircea MERCA, E-mail: mircea.merca@profinfo.edu.ro

Abstract. In this paper, we consider the multiplicity of the odd parts in all the partitions of n and provide a new formula for the number of the overpartitions of n, i.e.,

$$\overline{p}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (1+t_1)(1+t_3)\cdots(1+t_{2\lceil n/2\rceil-1}).$$

Similar results for the number of the overpartitions of n into odd parts are introduce in this context.

Key words: partitions, overpartitions

1. INTRODUCTION

Recall [1] that a composition of a positive integer n is a sequence of natural numbers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose sum is n, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k. \tag{1}$$

When the order of integers λ_i does not matter, the representation (1) is known as an integer partition and can be rewritten as

$$n = t_1 + 2t_2 + \cdots + nt_n,$$

where each positive integer i appears t_i times in the partition. The number of parts of this partition is given by

$$t_1 + t_2 + \cdots + t_n = k.$$

For consistency, we consider a partition of n a non-increasing sequence of natural numbers whose sum is n. For example, the partitions of 4 are given as:

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

The fastest algorithms for enumerating all the partitions of an integer have recently been presented by Merca [7,8]. As usual, we denote by $p(n)$ the number of integer partitions of n and we have the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n>0; \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined [4]. Let $\overline{p}(n)$ denote the number of overpartitions of an integer n. For example, $\overline{p}(4) = 14$ because there are 14 possible overpartitions of 4:

(4), $(\overline{4})$, $(3,1)$, $(3,\overline{1})$, $(\overline{3},1)$, $(\overline{3},\overline{1})$, $(2,2)$, $(\overline{2},2)$, $(2,1,1)$, $(2,\overline{1},1)$, $(\overline{2},1,1)$, $(\overline{2},\overline{1},1)$, $(1,1,1,1)$, $(\overline{1},1,1,1)$.

Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the following generating function for overpartitions,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

In this paper, we consider all the partitions of n in order to introduce a new formula for $\overline{p}(n)$. This formula considers only the multiplicity of the odd parts.

THEOREM 1. *Let n be a non-negative integer. Then*

$$\overline{p}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (1+t_1)(1+t_3)\cdots(1+t_{2\lceil n/2\rceil-1}).$$

Taking into account that

$$\begin{aligned} 4 &= 0\cdot1+0\cdot2+0\cdot3+1\cdot4 = \\ &= 1\cdot1+0\cdot2+1\cdot3+0\cdot4 = \\ &= 0\cdot1+2\cdot2+0\cdot3+0\cdot4 = \\ &= 2\cdot1+1\cdot2+0\cdot3+0\cdot4 = \\ &= 4\cdot1+0\cdot2+0\cdot3+0\cdot4, \end{aligned}$$

the case $n=4$ of Theorem 1 reads as follows

$$\begin{aligned} \overline{p}(4) &= (1+0)(1+0)+(1+1)(1+1)+(1+0)(1+0)+(1+2)(1+0)+(1+4)(1+0) = \\ &= 1+4+1+3+5 = 14. \end{aligned}$$

Let $\overline{p_o}(n)$ be the number of overpartitions of n into odd parts. Then its generating function is

$$\sum_{n=0}^{\infty} \overline{p_o}(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{2}$$

This expression first appeared in the following series-product identity

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty},$$

which was given by Lebesgue [6] in 1840. More recently, the generating function of $\overline{p_o}(n)$ appeared in the works of Bessenrodt [2], Merca [9], Merca, Wang and Yee [10], Santos and Sills [11]. Various arithmetic properties of $\overline{p_o}(n)$ have been investigated by Chen [3], Hirschhorn and Sellers [5].

In analogy with Theorem 1, we have the following result.

THEOREM 2. *Let n be a non-negative integer. Then*

$$\overline{p_o}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_2+t_4+\cdots+t_{2\lfloor n/2\rfloor}} (1+t_1)(1+t_3)\cdots(1+t_{2\lceil n/2\rceil-1}).$$

The case $n=4$ of this theorem reads as

$$\begin{aligned}\overline{p_o}(4) = (-1)^{0+1}(1+0)(1+0)+(-1)^{0+0}(1+1)(1+1)+(-1)^{2+0}(1+0)(1+0)+\\ +(-1)^{1+0}(1+2)(1+0)+(-1)^{0+0}(1+4)(1+0)=\\ =-1+4+1-3+5=6\end{aligned}$$

and the six overpartitions in question are:

$$(3,1),\ (3,\overline{1}),\ (\overline{3},1),\ (\overline{3},\overline{1}),\ (1,1,1,1),\ (\overline{1},1,1,1).$$

In the following result, we consider only the partitions of n in which the odd parts have the multiplicity at most 2 and the even parts have the multiplicity at most 1.

THEOREM 3. *Let n be a non-negative integer. Then*

$$\overline{p_o}(n) = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n\\ t_{2k-1}\leqslant 2,\ t_{2k}\leqslant 1}} (1+t_1 \bmod 2)(1+t_3 \bmod 2)\cdots(1+t_{2\lceil n/2\rceil-1} \bmod 2).$$

For example, the partitions of 4 in which the odd parts have the multiplicity at most 2 and the even parts have the multiplicity at most 1 are:

$$(4),\ (3,1),\ (2,1,1).$$

According to Theorem 3, we can write

$$\overline{p_o}(n) = (1+0)(1+0)+(1+1)(1+1)+(1+0)(1+0) = 1+4+1 = 6.$$

Inspired by Theorem 2, we remark the following connection between the Jacoby theta function

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and the partitions in which the odd parts have the multiplicity at most 2 and the even parts have the multiplicity at most 1.

THEOREM 4. *Let n be a non-negative integer. The coefficient of q^n in the Jacobi theta function $\vartheta_3(q)$ can be expressed as*

$$\sum_{\substack{t_1+2t_2+\cdots+nt_n=n\\ t_{2k-1}\leqslant 2,\ t_{2k}\leqslant 1}} (-1)^{t_2+t_4+\cdots+t_{2\lfloor n/2\rfloor}} (1+t_1 \bmod 2)(1+t_3 \bmod 2)\cdots(1+t_{2\lceil n/2\rceil-1} \bmod 2).$$

For example, the case $n=4$ of Theorem 4 reads as follows

$$(-1)^{0+1}(1+0)(1+0)+(-1)^{0+0}(1+1)(1+1)+(-1)^{1+0}(1+0)(1+0) = -1+4-1 = 2.$$

The rest of the paper continues with the proofs of our theorems.

2. PROOF OF THEOREM 1

Considering Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty},$$

we can write the generating function of $\overline{p}(n)$ as follows

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{1}{(q;q)_\infty (q;q^2)_\infty} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{1+n \bmod 2}}.$$

In order to prove our theorem, we consider the following identity.

LEMMA 1. *Let n be a positive integer. For* $|z|<1$,

$$\prod_{k=1}^{n} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k.$$

Proof. We are to prove this identity by induction on n. For $n=1$, we have

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} (1+k)z^k$$

and the base case of induction is finished. We suppose that the relation

$$\prod_{k=1}^{m} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_m=k} \prod_{i=1}^{m} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k.$$

is true for any integer m, $1 \leqslant m < n$. On the one hand, when n is odd, we can write

$$\begin{aligned}
&\prod_{k=1}^{n} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \frac{1}{(1-q^{n-1}z)^2} \prod_{k=1}^{n-1} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \\
&= \left(\sum_{k=0}^{\infty} (1+k) q^{(n-1)k} z^k \right) \left(\sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_{n-1}=k} \prod_{i=1}^{n-1} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k \right) = \\
&= \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k,
\end{aligned}$$

where we have invoked the well-known Cauchy multiplications of two power series. On the other hand, when n is even, we have

$$\begin{aligned}
&\prod_{k=1}^{n} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \frac{1}{1-q^{n-1}z} \prod_{k=1}^{n-1} \frac{1}{(1-q^{k-1}z)^{1+k \bmod 2}} = \\
&= \left(\sum_{k=0}^{\infty} q^{(n-1)k} z^k \right) \left(\sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_{n-1}=k} \prod_{i=1}^{n-1} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k \right) = \\
&= \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \Big(1+(i \bmod 2)\,t_i\Big) q^{(i-1)t_i} \right) z^k.
\end{aligned}$$

This concludes the proof. □

By this lemma, with z replaced by q, we obtain

$$\prod_{k=1}^{n} \frac{1}{(1-q^k)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \Big(1+(i \bmod 2)t_i\Big) q^{it_i} \right).$$

The limiting case $n \to \infty$ of this relation reads as follows

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+2t_2+\cdots+kt_k=k} \prod_{i=1}^{k} \Big(1+(i \bmod 2)\,t_i\Big) \right) q^k.$$

The proof is finished.

3. PROOF OF THEOREM 2

The proof of this theorem is quite similar to the proof of Theorem 1. Considering the generating function of $\overline{p_o}(n)$, we can write

$$\sum_{n=0}^{\infty} (-1)^n \overline{p_o}(n) q^n = \frac{1}{(-q;q)_\infty(-q;q^2)_\infty} = \prod_{n=1}^{\infty} \frac{1}{(1+q^n)^{1+n \bmod 2}}.$$

By Lemma 1, with z replaced by $-q$, we obtain

$$\prod_{k=1}^{n} \frac{1}{(1+q^k)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} (-1)^k \prod_{i=1}^{n} \Big(1+(i \bmod 2)\,t_i\Big) q^{it_i} \right).$$

The limiting case $n \to \infty$ of this relation reads as follows

$$\prod_{k=1}^{\infty} \frac{1}{(1+q^k)^{1+k \bmod 2}} = \sum_{k=0}^{\infty} \left(\sum_{t_1+2t_2+\cdots+kt_k=k} \prod_{i=1}^{k} (-1)^{t_i} \Big(1+(i \bmod 2)\,t_i\Big) \right) q^k.$$

Thus we deduce that

$$(-1)^n \overline{p_o}(n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+t_2+\cdots+t_n} (1+t_1)(1+t_3)\cdots(1+t_{2\lceil n/2\rceil-1})$$

and the proof is finished.

3. PROOF OF THEOREM 3

The proof of this theorem is quite similar to the proof of Theorem 1. The generating function of $\overline{p_o}(n)$ can be written as

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = (-q;q)_\infty(-q;q^2)_\infty = \prod_{n=1}^{\infty} (1+q^n)^{1+n \bmod 2}.$$

We consider the following identity.

LEMMA 2. *Let n be a positive integer. For $|z|<1$,*

$$\prod_{k=1}^{n} (1+q^{k-1}z)^{1+k \bmod 2} = \sum_{k=0}^{n+\lceil n/2\rceil} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \binom{1+i \bmod 2}{t_i} q^{(i-1)t_i} \right) z^k.$$

Proof. We are to prove this identity by induction on n. For $n=1$, we have

$$(1+z)^2 = \binom{2}{0} + \binom{2}{1} z + \binom{2}{2} z^2$$

and the base case of induction is finished. We suppose that the relation

$$\prod_{k=1}^{m} (1+q^{k-1}z)^{1+k \bmod 2} = \sum_{k=0}^{m+\lceil m/2 \rceil} \left(\sum_{t_1+t_2+\cdots+t_m=k} \prod_{i=1}^{m} \binom{1+i \bmod 2}{t_i} q^{(i-1)t_i} \right) z^k$$

is true for any integer m, $1 \leqslant m < n$. We can write

$$\prod_{k=1}^{n} (1+q^{k-1}z)^{1+k \bmod 2} = (1+q^{n-1}z)^{1+n \bmod 2} \prod_{k=1}^{n-1} (1+q^{k-1}z)^{1+k \bmod 2} =$$
$$= \left(\sum_{k=0}^{1+n \bmod 2} \binom{1+n \bmod 2}{k} q^{(n-1)k} z^k \right) \left(\sum_{k=0}^{n-1+\lceil (n-1)/2 \rceil} \left(\sum_{t_1+t_2+\cdots+t_{n-1}=k} \prod_{i=1}^{n-1} \binom{1+i \bmod 2}{t_i} q^{(i-1)t_i} \right) z^k \right) =$$
$$= \sum_{k=0}^{\infty} \left(\sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \binom{1+i \bmod 2}{t_i} q^{(i-1)t_i} \right) z^k,$$

where we have invoked the well-known Cauchy multiplications of two power series. □

By this lemma, with z replaced by q, we obtain

$$\prod_{k=1}^{n} (1+q^{k})^{1+k \bmod 2} = \sum_{k=0}^{n+\lceil n/2 \rceil} \sum_{t_1+t_2+\cdots+t_n=k} \prod_{i=1}^{n} \binom{1+i \bmod 2}{t_i} q^{t_1+2t_2+\cdots+nt_n}.$$

The limiting case $n \to \infty$ of this relation read as

$$\prod_{k=1}^{\infty} (1+q^{k})^{1+k \bmod 2} = \sum_{k=0}^{\infty} \sum_{t_1+2t_2+\cdots+kt_k=k} \prod_{i=1}^{n} \binom{1+i \bmod 2}{t_i} q^{k}.$$

The proof follows easily considering that $1+i \bmod 2 \in \{1,2\}$.

3. PROOF OF THEOREM 4

Recall that the reciprocal of the generating function of the overpartitions functions $\overline{p}(n)$ appears in a classical theta identity (often attributed to Gauss and sometimes Jacobi) [1, p. 23, eq (2.2.12)]:

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{3}$$

The reciprocal of the generating function of the overpartitions functions $\overline{p}(n)$ can be written as

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = (q;q)_\infty (q;q^2)_\infty = \prod_{n=1}^{\infty} (1-q^n)^{1+n \bmod 2}.$$

By Lemma 2, with z replaced by $-q$, we obtain

$$\prod_{k=1}^{n} (1-q^{k})^{1+k \bmod 2} = \sum_{k=0}^{n+\lceil n/2 \rceil} \left(\sum_{t_1+t_2+\cdots+t_n=k} (-1)^k \prod_{i=1}^{n} \binom{1+i \bmod 2}{t_i} q^{it_i} \right).$$

The limiting case $n \to \infty$ of this relation reads as follows

$$\prod_{k=1}^{\infty}(1-q^k)^{1+k \bmod 2} = \sum_{k=0}^{\infty}\left(\sum_{t_1+2t_2+\cdots+kt_k=k}\prod_{i=1}^{n}(-1)^{t_i}\binom{1+i \bmod 2}{t_i}\right)q^k.$$

Thus we deduce that the coefficient of q^n in (3) is given by

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}(1+t_1 \bmod 2)(1+t_3 \bmod 2)\cdots(1+t_{2\lceil n/2\rceil-1} \bmod 2).$$

The proof follows easily multiplying this expression by $(-1)^n$.

REFERENCES

1. G. E. ANDREWS, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998 (Reprint of the 1976 original).
2. C. BESSENRODT, *On pairs of partitions with steadily decreasing parts*, J. Combin. Theory Ser. A, **99**, pp. 162–174, 2002.
3. S.-C. CHEN, *On the number of overpartitions into odd parts*, Discrete Math., **325**, pp. 32–37, 2014.
4. S. CORTEEL, J. LOVEJOY, *Overpartitions*, Trans. Amer. Math. Soc., **356**, pp. 1623–1635, 2004.
5. M. D. HIRSCHHORN, J. A. SELLERS, *Arithmetic properties of overpartitions into odd parts*, Ann. Comb., **10**, pp. 353–367, 2006.
6. V. A. LEBESGUE, *Sommation de quelques séeries*, J. Math. Pure. Appl., **5**, pp. 42–71, 1840.
7. M. MERCA, *Fast algorithm for generating ascending compositions*, J. Math. Model. Algorithms, **11**. pp. 89–104, 2012.
8. M. MERCA, *Binary diagrams for storing ascending compositions*, Comput. J., **56**, pp. 1320–1327, 2013.
9. M. MERCA, *On the Ramanujan-type congruences modulo 8 for the overpartitions into odd parts*, Quaest. Math., 2021, `https://doi.org/10.2989/16073606.2021.1966543`.
10. M. MERCA, C. WANG, A. J. YEE, *A truncated theta identity of Gauss and overpartitions into odd parts*, Ann. Comb., **23**, pp. 907–915, 2019.
11. J. P. O. SANTOS, D. SILLS, *q-Pell sequences and two identities of V. A. Lebesgue*, Discrete Math., **257**, pp. 125–142, 2002.

Received November 17, 2021