THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A*
*Volume xx, Number yy/2021, pp. xx–xx*

# A Fan-type result for the existence of restricted fractional $(g, f)$-factors

Sizhong Zhou

School of Science, Jiangsu University of Science and Technology,
Mengxi Road 2, Zhenjiang, Jiangsu 212003, China
Corresponding author: Sizhong Zhou, E-mail: zsz_cumt@163.com

**Abstract:** Let $h$ be a function defined on $E(G)$ with $h(e) \in [0,1]$ for all $e \in E(G)$. If $g(u) \leq \sum_{e \ni u} h(e) \leq f(u)$ for every $u \in V(G)$, then a graph $F_h$ with vertex set $V(G)$ and edge set $E_h$ is called a fractional $(g, f)$-factor of $G$ with indicator function $h$, where $E_h = \{e \in E(G) : h(e) > 0\}$. Let $M$ and $N$ be two sets of independent edges of $G$ such that $|M| = m$, $|N| = n$ and $M \cap N = \emptyset$. We say that $G$ admits a fractional $(g, f)$-factor with the property $E(m,n)$ if $G$ has a fractional $(g, f)$-factor $F_h$ satisfying $h(e) = 1$ for any $e \in M$ and $h(e) = 0$ for any $e \in N$. In this paper, we give a lower bound on Fan-type condition which guarantees graphs to admit fractional $(g, f)$-factors with the property $E(1,n)$, which is a generalization of Yu and Liu's previous result.

*Key words:* graph; Fan-type condition; fractional $(g, f)$-factor; restricted fractional $(g, f)$-factors.

## 1. INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops nor multiple edges. Let $G=(V(G),E(G))$ be a graph, where $V(G)$ denotes the set of vertices of $G$ and $E(G)$ denotes the set of edges of $G$. For any $u\in V(G)$, we use $N_G(u)$ to denote the set of vertices adjacent to $u$ in $G$, and $d_G(u)=|N_G(u)|$ is the degree of $u$ in $G$. For any $X\subseteq V(G)$, $N_G(X)=\cup_{u\in X}N_G(u)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$, and set $G-X=G[V(G)\setminus X]$. For a subset $E'$ of $E(G)$, we use $G-E'$ to denote the graph obtained from $G$ by deleting edges of $E'$. A subset $X$ of $V(G)$ is independent if $N_G(X)\cap X=\emptyset$. For two disjoint subsets $X$ and $Y$ of $V(G)$, we use $e_G(X,Y)$ to denote the number of edges joining $X$ to $Y$. We define the distance $d_G(u,v)$ between two vertices $u$ and $v$ as the minimum of the lengths of the $(u,v)$ paths of $G$. We use $\delta(G)$ to denote the minimum degree of $G$ and use $\Delta(G)$ to denote the maximum degree of $G$.

Let $g$ and $f$ be two integer-valued functions defined on $V(G)$ such that $0\le g(u)\le f(u)$ for every $u\in V(G)$. A $(g,f)$-factor of $G$ is a spanning subgraph $F$ of $G$ such that $g(u)\le d_F(u)\le f(u)$ for all $u\in V(G)$. If $g(u)=a$ and $f(u)=b$ for every $u\in V(G)$, then a $(g,f)$-factor is an $[a,b]$-factor. A $[k,k]$-factor is simply called a $k$-factor.

Let $h$ be a function defined on $E(G)$ with $h(e)\in[0,1]$ for all $e\in E(G)$. If $g(u)\le\sum_{e\ni u}h(e)\le f(u)$ for every $u\in V(G)$, then a graph $F_h$ with vertex set $V(G)$ and edge set $E_h$ is called a fractional $(g,f)$-factor of $G$ with indicator function $h$, where $E_h=\{e\in E(G):h(e)>0\}$. A fractional $(g,f)$-factor is a fractional $[a,b]$-factor if $g(u)=a$ and $f(u)=b$ for all $u\in V(G)$. A fractional $[k,k]$-factor is simply called a fractional $k$-factor.

Let $M$ and $N$ be two sets of independent edges of $G$ such that $|M|=m$, $|N|=n$ and $M\cap N=\emptyset$. We say that $G$ admits a fractional $(g,f)$-factor with the property $E(m,n)$ if $G$ has a fractional $(g,f)$-factor $F_h$ satisfying $h(e)=1$ for any $e\in M$ and $h(e)=0$ for any $e\in N$.

We first introduce a well-known result on a Hamiltonian cycle (or 2-factor) of graph depending on Fan-type condition.

**THEOREM 1** ( [3]). Let $G$ be a 2-connected graph of order $p\ge 3$. If

$$\max\{d_G(u),d_G(v)\}\ge\frac{p}{2}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ admits a Hamiltonian cycle (or 2-factor).

Niessen [11] generalized Theorem 1 to $k$-factors, which is shown in the following.

**THEOREM 2** ( [11]). Let $k$ be an integer with $k\ge 1$ and $G$ a connected graph of order $p$ with $p\ge 8k^2+12k+6$, $kp$ is even. If $\delta(G)\ge k$ and

$$\max\{d_G(u),d_G(v)\}\ge\frac{p}{2}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ admits a $k$-factor.

Yu and Liu [15] put forward a Fan-type condition for the existence of fractional $k$-factors in graphs.

**THEOREM 3** ( [15]). Let $G$ a connected graph of order $p$ with $p\ge 8k^2+12k+6$, where $k$ is a positive integer. If $\delta(G)\ge k$ and

$$\max\{d_G(u),d_G(v)\}\ge\frac{p}{2}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ admits a fractional $k$-factor.

For other results on graph factors see [1, 2, 4–6, 8–10, 12–14, 16–29]. In this paper, we investigate the existence of restricted fractional $(g,f)$-factors in graphs, and obtain a Fan-type condition for graphs having restricted fractional $(g,f)$-factors, which is shown in Section 2.

## 2. MAIN RESULTS

Motivated by Theorems 1–3, we verify the following theorem.

**THEOREM 4.** Let $a,b,\lambda$ and $n$ be nonnegative integers with $2 \le a \le b-\lambda$, let $G$ be a graph of order $p$ with $p \ge \frac{(a+b)((a+b)(b-\lambda+1)+2n-1)}{a+\lambda-1} + \frac{(a+b)(b-\lambda+1)-2}{b-\lambda} + \frac{a+b}{(a+\lambda)(b-\lambda)}$, and let $g,f : V(G) \to Z$ be two functions such that $a \le g(x) \le f(x)-\lambda \le b-\lambda$ for all $x \in V(G)$. If $\delta(G) \ge \frac{(b-\lambda)(b+2)}{a+\lambda-1}+1$ and

$$\max\{d_G(u),d_G(v)\} \ge \frac{(b-\lambda)p+2}{a+b}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,n)$.

*Remark.* The condition $\max\{d_G(u),d_G(v)\} \ge \frac{(b-\lambda)p+2}{a+b}$ in Theorem 4 is sharp, i.e., we cannot replace $\frac{(b-\lambda)p+2}{a+b}$ by $\frac{(b-\lambda)p+2}{a+b}-1$.

Let $a,b,\lambda$ and $n$ be nonnegative integers with $2 \le a = b-\lambda$ and $\beta$ be a sufficiently large integer with $\beta > 0$ and $n < (b-\lambda)\beta$. Set $G = K_{(b-\lambda)\beta} \vee (a+\lambda)\beta K_1$. Then we have $p = (b-\lambda)\beta + (a+\lambda)\beta = (a+b)\beta$ and

$$\frac{(b-\lambda)p+2}{a+b} - 1 < \max\{d_G(u),d_G(v)\} = (b-\lambda)\beta = \frac{(b-\lambda)p}{a+b} < \frac{(b-\lambda)p+2}{a+b}$$

for any two vertices $u,v \in V((a+\lambda)\beta K_1)$ with $d_G(u,v)=2$. Let $g,f : V(G) \to Z$ be two functions with $g(u) = b-\lambda$ and $f(u) = a+\lambda$ for any $u \in V(G)$. Let $X = V(K_{(b-\lambda)\beta})$, $Y = V((a+\lambda)\beta K_1)$, $N = \{e_1,e_2,\cdots,e_n\}$ being a set of independent edges in $G$ and $H = G-N$. Then it follows that $|X| = (b-\lambda)\beta$, $|Y| = (a+\lambda)\beta$, $d_{H-X}(Y)=0$ and $\varepsilon(X,Y)=2$. Hence, we obtain

$$\begin{aligned}
\gamma_H(X,Y) &= f(X)+d_{H-X}(Y)-g(Y)\\
&= (a+\lambda)|X|-(b-\lambda)|Y|\\
&= (a+\lambda)(b-\lambda)\beta-(b-\lambda)(a+\lambda)\beta\\
&= 0 < 2 = \varepsilon(X,Y).
\end{aligned}$$

In light of Theorem 5, $H$ has no fractional $(g,f)$-factor with the property $E(1,0)$, that is, $G$ has no fractional $(g,f)$-factor with the property $E(1,n)$.

Let $n=0$ in Theorem 4. Then we get the following corollary.

**COROLLARY 1.** Let $a,b,\lambda$ be nonnegative integers with $2 \le a \le b-\lambda$, let $G$ be a graph of order $p$ with $p \ge \frac{(a+b)((a+b)(b-\lambda+1)-1)}{a+\lambda-1} + \frac{(a+b)(b-\lambda+1)-2}{b-\lambda} + \frac{a+b}{(a+\lambda)(b-\lambda)}$, and let $g,f : V(G) \to Z$ be two functions such that $a \le g(x) \le f(x)-\lambda \le b-\lambda$ for all $x \in V(G)$. If $\delta(G) \ge \frac{(b-\lambda)(b+2)}{a+\lambda-1}+1$ and

$$\max\{d_G(u),d_G(v)\} \ge \frac{(b-\lambda)p+2}{a+b}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,0)$.

Let $\lambda = 0$ in Theorem 4. Then we obtain the following result.

**COROLLARY 2.** Let $a,b$ and $n$ be nonnegative integers with $2 \le a \le b$, let $G$ be a graph of order $p$ with $p \ge \frac{(a+b)((a+b)(b+1)+2n-1)}{a-1} + \frac{(a+b)(b+1)-2}{b} + \frac{a+b}{ab}$, and let $g,f : V(G) \to Z$ be two functions such that $a \le g(x) \le f(x) \le b$ for all $x \in V(G)$. If $\delta(G) \ge \frac{b(b+2)}{a-1}+1$ and

$$\max\{d_G(u),d_G(v)\} \ge \frac{bp+2}{a+b}$$

for any two vertices $u$ and $v$ of $G$ with $d_G(u,v)=2$, then $G$ has a fractional $(g,f)$-factor with the property $E(1,n)$.

## 3. PROOF OF THEOREM 4

For any $X\subseteq V(G)$, let $\varphi(X)=\sum_{u\in X}\varphi(u)$, where $\varphi$ is a function defined on $V(G)$. Especially, $\varphi(\emptyset)=0$. Li, Yan and Zhang [7] put forward a characterization for graphs to have fractional $(g,f)$-factors with the property $E(1,0)$, which is used in the proof of Theorem 4.

**THEOREM 5** ( [7]). Let $G$ be a graph, and let $g,f:V(G)\to Z$ be two functions with $0\le g(x)\le f(x)$ for all $x\in V(G)$. Then $G$ has a fractional $(g,f)$-factor with the property $E(1,0)$ if and only if

$$\gamma_G(X,Y)=f(X)+d_{G-X}(Y)-g(Y)\ge\varepsilon(X,Y)$$

for any $X\subseteq V(G)$, where $Y=\{y:y\in V(G)\setminus X,d_{G-X}(y)\le g(y)\}$ and $\varepsilon(X,Y)$ is defined as follows:

$$\varepsilon(X,Y)=\begin{cases}2, & if\ X\ is\ not\ independent,\\ 1, & if\ X\ is\ independent\ and\ there\ is\ an\ edge\ joining\ X\ and\ V(G)\setminus(X\cup Y),\ or\\ & there\ is\ an\ edge\ e=uv\ joining\ X\ and\ Y\ such\ that\ d_{G-X}(v)=g(v)\ for\ v\in Y,\\ 0, & otherwise.\end{cases}$$

*Proof of Theorem 4.* Assume that $G$ has no fractional $(g,f)$-factor with the property $E(1,n)$. Then there exist an edge $e$ and a set of independent edges $\{e_1,e_2,\cdots,e_n\}$ of $G$ such that $G$ has no fractional $(g,f)$-factor $F_h$ with $h(e)=1$ and $h(e_i)=0$ for $1\le i\le n$. Set $N=\{e_1,e_2,\cdots,e_n\}$ and $H=G-N$. Then $H$ has no fractional $(g,f)$-factor with the property $E(1,0)$. In view of Theorem 5, there exists a subset $X$ of $V(H)$ such that

$$\gamma_H(X,Y)=f(X)+d_{H-X}(Y)-g(Y)\le\varepsilon(X,Y)-1,\tag{1}$$

where $Y=\{y:y\in V(H)\setminus X,d_{H-X}(y)\le g(y)\}$.
*Claim 1.* $Y\neq\emptyset$.
*Proof.* If $Y=\emptyset$, then by (1) we obtain

$$\varepsilon(X,Y)-1\ge\gamma_H(X,Y)=f(X)\ge(a+\lambda)|X|\ge2|X|\ge|X|\ge\varepsilon(X,Y),$$

which is a contradiction. □
*Claim 2.* $d_{H-X}(Y)\ge d_{G-X}(Y)-\min\{2n,|Y|\}$.
*Proof.* Let $D=V(G)\setminus(X\cup Y)$ and $E_G(Y)=\{e:e=uv\in E(G),u,v\in Y\}$. Since $N=\{e_1,e_2,\cdots,e_n\}$ is a set of independent edges of $G$, we easily obtain

$$2|N\cap E_G(Y)|+|N\cap E_G(Y,D)|\le\min\{2n,|Y|\}.\tag{2}$$

It follows from (2) and $H=G-N$ that

$$\begin{aligned}d_{H-X}(Y)&=d_{G-N-X}(Y)\\&=d_{G-X}(Y)-(2|N\cap E_G(Y)|+|N\cap E_G(Y,D)|)\\&\ge d_{G-X}(Y)-\min\{2n,|Y|\}.\end{aligned}$$

The proof of Claim 2 is finished. □
*Claim 3.* $|Y|\ge b+3$.

*Proof.* Since $Y \neq \emptyset$ (by Claim 1), we may define

$$d = \min\{d_{G-X}(u) : u \in Y\},$$

and choose $u_1 \in Y$ with $d_{G-X}(u_1) = d$. Clearly, $0 \le d \le b-\lambda+1$ by $H = G-N$ and the definition of $Y$. Moreover, we have

$$|X| + d = |X| + d_{G-X}(u_1) \ge d_G(u_1) \ge \delta(G),$$

that is,

$$|X| \ge \delta(G) - d \ge \frac{(b-\lambda)(b+2)}{a+\lambda-1} + 1 - d. \tag{3}$$

Let $|Y| \le b+2$. We shall consider two cases by the value of $d$.

*Case 1.* $d = 0$.

In light of (1), (3), $2 \le a \le b-\lambda$ and $\varepsilon(X,Y) \le 2$, we obtain

$$\begin{aligned}
\varepsilon(X,Y) - 1 &\ge \gamma_H(X,Y) = f(X) + d_{H-X}(Y) - g(Y) \\
&\ge f(X) - g(Y) \ge (a+\lambda)|X| - (b-\lambda)|Y| \\
&\ge (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1} + 1 - d\Big) - (b-\lambda)(b+2) \\
&> a+\lambda \ge a \ge 2 \ge \varepsilon(X,Y),
\end{aligned}$$

which is a contradiction.

*Case 2.* $1 \le d \le b-\lambda+1$.

It follows from (3), Claim 2, $2 \le a \le b-\lambda$ and $1 \le d \le b-\lambda+1$ that

$$\begin{aligned}
\gamma_H(X,Y) &= f(X) + d_{H-X}(Y) - g(Y) \\
&\ge f(X) + d_{G-X}(Y) - \min\{2n, |Y|\} - g(Y) \\
&\ge f(X) + d_{G-X}(Y) - |Y| - g(Y) \\
&\ge (a+\lambda)|X| + d|Y| - |Y| - (b-\lambda)|Y| \\
&= (a+\lambda)|X| - (b-\lambda-d+1)|Y| \\
&\ge (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1} + 1 - d\Big) - (b-\lambda-d+1)(b+2) \\
&= (d-1)(b+2-a-\lambda) + \frac{(b-\lambda)(b+2)}{a+\lambda-1} \ge \frac{(b-\lambda)(b+2)}{a+\lambda-1} \\
&\ge \frac{a(a+\lambda+2)}{a+\lambda-1} > a \ge 2 \ge \varepsilon(X,Y),
\end{aligned}$$

which contradicts (1). Hence, we have $|Y| \ge b+3$. Claim 3 is proved. □

*Claim 4.* $d_{G-X}(u) \le b-\lambda+1 \le b+1$ for any $u \in Y$.

*Proof.* In view of the definitions of $Y$ and $N$, $H = G-N$, we have

$$d_{G-X}(u) = d_{H+N-X}(u) \le d_{H-X}(u) + 1 \le g(u) + 1 \le b-\lambda+1 \le b+1$$

for any $u \in Y$. □

*Claim 5.* $1 \le |X| \le \frac{(b-\lambda)p+1}{a+b}$.

*Proof.* If $X = \emptyset$, then it follows from (1), $2 \le a \le b-\lambda$ and Claims 2–3 that

$$\begin{aligned}
\varepsilon(X,Y)-1 &\ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-\min\{2n,|Y|\}-g(Y)\\
&\ge f(X)+d_{G-X}(Y)-|Y|-g(Y)\\
&= d_G(Y)-|Y|-g(Y) \ge \delta(G)|Y|-|Y|-(b-\lambda)|Y|\\
&= (\delta(G)-(b-\lambda+1))|Y| \ge \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y|\\
&\ge \Big(\frac{(b-\lambda)(a+\lambda+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y|\\
&= \frac{3(b-\lambda)}{a+\lambda-1}|Y| \ge \frac{3(b-\lambda)}{a+\lambda-1}(b+3)\\
&\ge \frac{3(b-\lambda)}{a+\lambda-1}(a+\lambda+3) > 3(b-\lambda) > 2 \ge \varepsilon(X,Y),
\end{aligned}$$

which is a contradiction. Therefore, $|X| \ge 1$.

On the other hand, by (1), $\varepsilon(X,Y) \le 2$ and $|X|+|Y| \le p$, we have

$$\begin{aligned}
1 &\ge \varepsilon(X,Y)-1 \ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)-g(Y) \ge (a+\lambda)|X|-(b-\lambda)|Y|\\
&\ge (a+\lambda)|X|-(b-\lambda)(p-|X|)\\
&= (a+b)|X|-(b-\lambda)p,
\end{aligned}$$

which implies

$$|X| \le \frac{(b-\lambda)p+1}{a+b}.$$

Hence, we obtain that $1 \le |X| \le \frac{(b-\lambda)p+1}{a+b}$. □

*Claim 6.* $(b-\lambda)|Y| \ge (a+\lambda)|X|-1$.

*Proof.* In terms of (1) and $\varepsilon(X,Y) \le 2$, we get

$$\begin{aligned}
1 &\ge \varepsilon(X,Y)-1 \ge \gamma_H(X,Y) = f(X)+d_{H-X}(Y)-g(Y)\\
&\ge f(X)-g(Y) \ge (a+\lambda)|X|-(b-\lambda)|Y|,
\end{aligned}$$

that is,

$$(b-\lambda)|Y| \ge (a+\lambda)|X|-1.$$

Claim 6 is verified. □

*Claim 7.* $|X| < \frac{(b-\lambda)p+2}{a+b}-(b-\lambda+1)$.

*Proof.* Assume that $|X| \ge \frac{(b-\lambda)p+2}{a+b}-(b-\lambda+1)$, that is, $(b-\lambda)p-(a+b)|X| \le (a+b)(b-\lambda+1)-2$. According to (1), $\varepsilon(X,Y) \le 2$, Claim 2 and $|X|+|Y| \le p$, we have

$$\begin{aligned}
d_{G-X}(Y) &\le d_{H-X}(Y)+\min\{2n,|Y|\}\\
&\le g(Y)-f(X)+\varepsilon(X,Y)-1+2n\\
&\le (b-\lambda)|Y|-(a+\lambda)|X|+1+2n\\
&\le (b-\lambda)(p-|X|)-(a+\lambda)|X|+2n+1\\
&= (b-\lambda)p-(a+b)|X|+2n+1\\
&\le (a+b)(b-\lambda+1)+2n-1.
\end{aligned}$$

Combining this with Claim 6 and $p \geq \frac{(a+b)((a+b)(b-\lambda+1)+2n-1)}{a+\lambda-1} + \frac{(a+b)(b-\lambda+1)-2}{b-\lambda} + \frac{a+b}{(a+\lambda)(b-\lambda)}$, we obtain

$$\begin{aligned}
\frac{d_{G-X}(Y)}{(b-\lambda)|Y|} &\leq \frac{(a+b)(b-\lambda+1)+2n-1}{(a+\lambda)|X|-1} \\
&\leq \frac{(a+b)(b-\lambda+1)+2n-1}{(a+\lambda)\cdot\frac{(b-\lambda)p+2}{a+b}-(a+\lambda)(b-\lambda+1)-1} \\
&\leq \frac{1}{b-\lambda}\Big(1-\frac{1}{a+\lambda}\Big),
\end{aligned}$$

which implies

$$d_{G-X}(Y) \leq \Big(1-\frac{1}{a+\lambda}\Big)|Y| = |Y| - \frac{1}{a+\lambda}|Y|. \tag{4}$$

It follows from (4), $2 \leq a \leq b-\lambda$ and Claim 3 that

$$d_{G-X}(Y) \leq |Y| - \frac{1}{a+\lambda}|Y| \leq |Y| - \frac{b+3}{a+\lambda} < |Y|-1. \tag{5}$$

Set $Y_0 = \{y \in Y : d_{G-X}(y) = 0\}$. It is easy to see that $|Y_0| \geq 2$ holds by (5). For any $y \in Y_0$, $d_G(y) \leq |X| \leq \frac{(b-\lambda)p+1}{a+b}$ by Claim 5. Note that $Y_0$ is an independent set of $G$. Combining this with the assumption of Theorem 4, the neighborhoods of the vertices in $Y_0$ are disjoint. Therefore, we obtain

$$|X| \geq |\cup_{y\in Y_0} N_G(y)| \geq \delta(G)|Y_0| \geq \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y_0|. \tag{6}$$

On the other hand, it follows from (4) that

$$\Big(1-\frac{1}{a+\lambda}\Big)|Y| \geq d_{G-X}(Y) \geq |Y| - |Y_0|,$$

which implies

$$|Y_0| \geq \frac{1}{a+\lambda}|Y|. \tag{7}$$

In light of (6), (7), $2 \leq a \leq b-\lambda$ and Claim 1, we have

$$\begin{aligned}
(a+\lambda)|X| &\geq (a+\lambda)\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y_0| \\
&\geq \Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1\Big)|Y| \\
&> (b-\lambda)|Y| + |Y| \geq (b-\lambda)|Y|+1,
\end{aligned}$$

which contradicts Claim 6. Hence, Claim 7 holds. □

*Claim 8.* $e_G(X,Y) \leq (b-\lambda+2)|X|$.

*Proof.* Since $|Y| \geq b+3$ by Claim 3 and $d_{G-X}(u) \leq b-\lambda+1 \leq b+1$ for every $u \in Y$ by Claim 4, there exist at least two independent vertices $u,v \in Y$. Moreover, it follows from Claims 4 and 7 that

$$\begin{aligned}
\max\{d_G(u), d_G(v)\} &\leq \max\{d_{G-X}(u)+|X|, d_{G-X}(v)+|X|\} \\
&\leq (b-\lambda+1)+|X| < (b-\lambda+1) + \frac{(b-\lambda)p+2}{a+b} - (b-\lambda+1) \\
&= \frac{(b-\lambda)p+2}{a+b}
\end{aligned}$$

for any two vertices $u,v \in Y$. In terms of the above inequalities and the hypothesis of Theorem 4, $G[N_G(x)\cap Y]$ is complete for every $x \in X$. Note that $X \neq \emptyset$ by Claim 5. Combining this with Claim 4, we have $e_G(x,Y) \leq$

$\Delta(G[Y])+1 \leq b-\lambda+2$. Therefore, $e_G(X,Y) \leq (b-\lambda+2)|X|$ holds. $\square$

Note that $\varepsilon(X,Y) \leq |X|$. It follows from (1), Claims 2, 5, 6, 8 and $\delta(G) \geq \frac{(b-\lambda)(b+2)}{a+\lambda-1}+1$ that

$$\begin{aligned}
\varepsilon(X,Y)-1 &\geq \gamma_H(X,Y)=f(X)+d_{H-X}(Y)-g(Y)\\
&\geq f(X)+d_{G-X}(Y)-\min\{2n,|Y|\}-g(Y)\\
&\geq f(X)+d_{G-X}(Y)-|Y|-g(Y)\\
&\geq (a+\lambda)|X|+d_{G-X}(Y)-|Y|-(b-\lambda)|Y|\\
&= (a+\lambda)|X|+d_G(Y)-e_G(X,Y)-(b-\lambda+1)|Y|\\
&\geq (a+\lambda)|X|+\delta(G)|Y|-(b-\lambda+2)|X|-(b-\lambda+1)|Y|\\
&= (a-b+2\lambda-2)|X|+(\delta(G)-(b-\lambda+1))|Y|\\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{(b-\lambda)(b+2)}{a+\lambda-1}+1-(b-\lambda+1)\Big)|Y|\\
&= (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)(b-\lambda)|Y|\\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)((a+\lambda)|X|-1)\\
&\geq (a-b+2\lambda-2)|X|+\Big(\frac{b+2}{a+\lambda-1}-1\Big)(a+\lambda-1)|X|\\
&= (\lambda+1)|X| \geq |X| \geq \varepsilon(X,Y),
\end{aligned}$$

which is a contradiction. Theorem 4 is verified. $\square$

## ACKNOWLEDGEMENTS

The author is grateful to the anonymous referees for giving many helpful comments and suggestions in improving this paper. This work is supported by Six Big Talent Peak of Jiangsu Province (Grant No. JY–022) and 333 Project of Jiangsu Province.

## REFERENCES

1. M. AXENOVICH, J. ROLLIN, *Brooks type results for conflict-free colorings and $\{a,b\}$-factors in graphs*, Discrete Mathematics, **338**, pp. 2295–2301, 2015.
2. R. CYMER, M. KANO, *Generalizations of Marriage Theorem for degree factors*, Graphs and Combinatorics, **32**, pp. 2315–2322, 2016.
3. G. FAN, *New sufficient conditions for cycles in graphs*, Journal of Combinatorial Theory Series B, **37**, pp. 221–227, 1984.
4. W. GAO, J. GUIRAO, Y. CHEN, *A toughness condition for fractional $(k,m)$-deleted graphs revisited*, Acta Mathematica Sinica, English Series, **35**, pp. 1227–1237, 2019.
5. W. GAO, J. GUIRAO, H. WU, *Two tight independent set conditions for fractional $(g,f,m)$-deleted graphs systems*, Qualitative Theory of Dynamical Systems, **17**(1), pp. 231–243, 2018.
6. W. GAO, L. LIANG, Y. CHEN, *An isolated toughness condition for graphs to be fractional $(k,m)$-deleted graphs*, Utilitas Mathematica, **105**, pp. 303–316, 2017.
7. Z. LI, G. YAN, X. ZHANG, *On fractional $(g,f)$-covered graphs*, OR Transactions (China), **6**(4), pp. 65–68, 2002.
8. G. LIU, Q. YU, L. ZHANG, *Maximum fractional factors in graphs*, Applied Mathematics Letters, **20**, pp. 1237–1243, 2007.

9. X. LV, *A degree condition for fractional $(g,f,n)$-critical covered graphs*, AIMS Mathematics, **5**(2), pp. 872–878, 2020.

10. H. MATSUDA, *Fan-type results for the existence of $[a,b]$-factors*, Discrete Mathematics, **306**, pp. 688–693, 2006.

11. T. NIESSEN, *A Fan-type result for regular factor*, Ars Combinatoria, **46**, pp. 277–285, 1997.

12. M. PLUMMER, *Graph factors and factorizations: 1985–2003: A survey*, Discrete Mathematics, **307**, pp. 791–821, 2007.

13. Z. SUN, S. ZHOU, *A generalization of orthogonal factorizations in digraphs*, Information Processing Letters, **132**, pp. 49–54, 2018.

14. S. WANG, W. ZHANG, *Research on fractional critical covered graphs*, Problems of Information Transmission, **56**(3), pp. 270–277, 2020.

15. J. YU, G. LIU, *Fractional k-factors of graphs*, Chinese Journal of Engineering Mathematics, **22**(2), pp. 377–380, 2005.

16. Y. YUAN, R. HAO, *A neighborhood union condition for fractional ID-$[a,b]$-factor-critical graphs*, Acta Mathematicae Applicatae Sinica, English Series, **34**, pp. 775–781, 2018.

17. S. ZHOU, *Binding numbers and restricted fractional $(g,f)$-factors in graphs*, Discrete Applied Mathematics, DOI: 10.1016/j.dam.2020.10.017.

18. S. ZHOU, *Remarks on orthogonal factorizations of digraphs*, International Journal of Computer Mathematics, **91**(10), pp. 2109–2117, 2014.

19. S. ZHOU, *Remarks on path factors in graphs*, RAIRO-Operations Research, **54**(6), pp. 1827–1834, 2020.

20. S. ZHOU, *Some new sufficient conditions for graphs to have fractional k-factors*, International Journal of Computer Mathematics, **88**(3), pp. 484–490, 2011.

21. S. ZHOU, *Some results about component factors in graphs*, RAIRO-Operations Research, **53**(3), pp. 723–730, 2019.

22. S. ZHOU, *Some results on path-factor critical avoidable graphs*, Discussiones Mathematicae Graph Theory, DOI: 10.7151/dmgt.2364.

23. S. ZHOU, Z. SUN, *Binding number conditions for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs*, Discrete Mathematics, **343**(3), 111715, 2020.

24. S. ZHOU, Z. SUN, *Some existence theorems on path factors with given properties in graphs*, Acta Mathematica Sinica, English Series, **36**(8), pp. 917–928, 2020.

25. S. ZHOU, Z. SUN, Q. PAN, *A sufficient condition for the existence of restricted fractional $(g,f)$-factors in graphs*, Problems of Information Transmission, **56**(4), pp. 334–346, 2020.

26. S. ZHOU, Z. SUN, H. YE, *A toughness condition for fractional $(k,m)$-deleted graphs*, Information Processing Letters, **113**(8), pp. 255–259, 2013.

27. S. ZHOU, Y. XU, Z. SUN, *Degree conditions for fractional $(a,b,k)$-critical covered graphs*, Information Processing Letters, **152**, 105838, 2019.

28. S. ZHOU, F. YANG, L. XU, *Two sufficient conditions for the existence of path factors in graphs*, Scientia Iranica, **26**(6), pp. 3510–3514, 2019.

29. S. ZHOU, T. ZHANG, Z. XU, *Subgraphs with orthogonal factorizations in graphs*, Discrete Applied Mathematics, **286**, pp. 29–34, 2020.


*Received on May 22, 2019*