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NULL-CONTROLLABILITY PROPERTIES OF THE HEAT EQUATION WITH MEMORY

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Abstract: This article studies the controllability properties of the heat equation having a memory term with higher order derivatives. It is know that in the usual setting the diffusion equation with a memory term has poor controllability properties. We show that these difficulties can be overcome if a moving control is considered.

Key words: heat equation, memory term, null-controllability, biorthogonals.

1. INTRODUCTION AND MAIN RESULTS

Let $M,T>0$, $\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}$ be the one-dimensional torus and let $Q:=(0,T)\times\mathbb{T}$. The aim of this article is to analyze the controllability properties of the following heat equation involving a memory term:

$$\begin{cases} u_t(t,x)-u_{xx}(t,x)+M\displaystyle\int_0^t u_{xx}(s,x)\,\mathrm{ds}=\mathbf{1}_{w(t)}h(t,x) & (t,x)\in Q \\ u(0,x)=u^0(x) & x\in\mathbb{T}, \end{cases} \tag{1}$$

where the support $w(t)$ of the control h at time t moves in space with a constant velocity c, that is,

$$w(t)=w_0-ct, \tag{2}$$

with $w_0\subset\mathbb{T}$ a reference set, open and non empty. The control $h\in L^2(\mathscr{P})$ is then an applied force localized, at each instant $t\geq 0$, in the set $w(t)$, where $\mathscr{P}:=\{(t,x)|\,t\in(0,T),x\in w(t)\}$. Our equation being defined on the torus $\mathbb{T}$, there is no need of specifying the boundary conditions which will automatically be of periodic type.

In many problems arising in mathematical physics such as flow of fluid through fissured rocks, diffusion process of gas in a transparent tube, heat conduction in materials, and viscoelasticity, one may encounter memory effects that are relevant from a physical point of view and can be modeled by nonlocal terms. Equation (1) models the heat transfer in isotropic media in which the heat flux depends both on the present value of the temperature gradient and its history (see, for instance, [3,4,12]). The well-posedness of (1) can be investigated by using a spectral approach and it will be addressed elsewhere.

Controllability results for the heat equation with memory have been obtained at the beginning by asking that the temperature be identically equal to zero at time T,

$$u(T,\cdot)=0, \tag{3}$$

(or to a pre-established state; see, for instance, [1,5,6]). As remarked by [7] (see, also, [9] in the context of viscoelasticity), (3) represents a kind of "relative null controllability", which however need not be controllability to rest. Indeed, even if the trajectory hits the zero, the solution may leave it in the future due to the memory effect. Therefore, the following definition of the null-controllability property for problem (1) will be used:

Definition 1. *Equation* (1) *is said to be memory-type null controllable in time* $T > 0$ *if, for each initial data* $u^0 \in L^2(\mathbb{T})$*, there exists a control function* $h \in L^2(\mathscr{P})$ *such that the corresponding solution of* (1) *verifies*

$$u(T,x) = \int_0^T u_{xx}(s,x)\,\mathrm{ds} = 0 \qquad (x \in \mathbb{T}). \tag{4}$$

Notice that in (4) to the classical notion of null controllability (3) an extra condition, demanding that the memory term to be equal to zero, has been added. As mentioned before, this guarantees that, once driven to zero, the solution stays there for any $t > T$.

The controllability results for (1) when the support of the control is fixed ($c = 0$) are known to be very poor and even the spectral controllability may fail (see Remark 1). To overcome this difficulty, in [2] a new strategy was proposed, consisting in considering a control with moving support as in (2) with $c > 0$. With this approach, [2] proved the memory-type null controllability in time T when the memory term is of the form $M\int_0^t u(s,x)\,\mathrm{ds}$. The main ingredients used to show this result are Carleman type-inequalities for the typical heat equation and the compactness of the memory term. Our aim is to study the controllability properties of equation (1), in which the memory term depends on the second derivative of the temperature. This model is more natural and closer to the original modified Fourier law proposed in the seventies. As in [2], we shall consider a moving control with the support given by (2) and $c > 0$. The main result of this article is the following.

Theorem 1. *Suppose that* $c > 0$ *and the control support* w *is given by* (2)*. Then equation* (1) *is memory-type null controllable in any time* $T > \frac{2\pi}{c}$*.*

It is usual in the context of linear controllability problems, like Theorem 1, to reduce their study to an observability inequality for the adjoint system. However, the analysis is facilitated if we first fix the support of the control. To do this, we consider the change of variable $x + ct = y$ and we define the new unknown functions

$$\xi(t,y) := u(t,x), \qquad \zeta(t,y) := z(t,x) = \int_0^t u_{xx}(s,x)\,\mathrm{ds}.$$

We obtain that (1) is equivalent to

$$\begin{cases} \xi_t(t,y) + c\xi_y(t,y) - \xi_{yy}(t,y) + M\zeta(t,y) = \mathbf{1}_{w_0}\widetilde{h}(t,y), & (t,y) \in Q \\ \zeta_t(t,y) + c\zeta_y(t,y) = \xi_{yy}(t,y), & (t,y) \in Q \\ \xi(0,y) = \xi^0(y), \quad \zeta(0,y) = 0, & y \in \mathbb{T}, \end{cases} \tag{5}$$

where $\widetilde{h}(t,y) := h(t, y - ct)$. Now the memory-type null controllability Definition 1 reads as follows.

Definition 2. *System* (5) *is said to be memory-type null controllable in time* $T > 0$ *if, for each initial data* $\xi^0 \in L^2(\mathbb{T})$*, there exists a control* $\widetilde{h} \in L^2((0,T) \times w_0)$ *such that the corresponding solution of* (5) *verifies*

$$\xi(T,y) = \zeta(T,y) = 0 \qquad (y \in \mathbb{T}). \tag{6}$$

We remark that the control $\widetilde{h}$ in Definition 2 has fixed support in time, w_0. Moreover, let us emphasize that (1) is memory-type null controllable in time $T > 0$ if and only if (5) is so.

As mentioned before, the proof of Theorem 1 uses the well-known duality of the concepts of controllability and observability. More precisely we have the following result whose proof is omitted.

Theorem 2. *System* (5) *is memory-type null controllable in a time* $T > 0$ *if there exists a constant* $K > 0$ *such that the following observability inequality holds*

$$\|\varphi(0)\|^2_{L^2(\mathbb{T})} \le K \int_0^T \int_{w_0} |\varphi(t,y)|^2\,\mathrm{dy}\,\mathrm{dt}, \tag{7}$$

for any $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \in \left(L^2(\mathbb{T})\right)^2$*, where* $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ *is the unique solution of the following adjoint system*

$$\begin{cases} -\varphi_t(t,y) - c\varphi_y(t,y) - \varphi_{yy}(t,y) + \psi_{yy}(t,y) = 0, & (t,y) \in Q \\ -\psi_t(t,y) - c\psi_y(t,y) - M\varphi(t,y) = 0, & (t,y) \in Q \\ \varphi(T,y) = \varphi^T(y), \quad \psi(T,y) = \psi^T(y), & y \in \mathbb{T}. \end{cases} \tag{8}$$

The remaining part of the paper is devoted to the proof of the observability inequality (7) under the assumption that $T > \frac{2\pi}{c}$. In this endeavor, the spectral analysis of the differential operator corresponding to the adjoint system (8) will play a fundamental role. Firstly, we notice that system (8) can be equivalently written as

$$-\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t + \mathscr{A}^\star \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(T) = \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}, \tag{9}$$

where

$$\mathscr{A}^\star = \begin{pmatrix} -\partial^2_{yy} - c\partial_y & \partial^2_{yy} \\ -M\mathrm{I}_d & -c\partial_y \end{pmatrix}.$$

In the following theorem we introduce the spectral properties of the operator $\mathscr{A}^\star$. Its proof is mostly a straightforward, if somehow tedious, computation and we omit it.

Theorem 3. *The eigenvalues of the operator $\mathscr{A}^\star$ are given by the family $(\lambda_n^\pm)_{n\in\mathbb{Z}^*} \cup \{\lambda_0\}$, where $\lambda_0 = 0$ and*

$$\lambda_n^\pm = \frac{1}{2}\left(n^2 \pm |n|\sqrt{n^2+4M}\right) - icn := \mu^\pm_{|n|} - icn \qquad (n \in \mathbb{Z}^*). \tag{10}$$

Each eigenvalue $\lambda_n^\pm$ has an associated eigenvector

$$\Phi_n^{\star,\pm} = \gamma_n^\pm \begin{pmatrix} 1 \\ -\frac{M}{\mu^\pm_{|n|}} \end{pmatrix} e^{iny}, \quad \textit{where } \gamma_n^\pm = \left(\left(\frac{M}{\mu^\pm_{|n|}}\right)^2 + 1\right)^{-\frac{1}{2}} \qquad (n \in \mathbb{Z}^*). \tag{11}$$

The eigenvalue $\lambda_0 = 0$ is algebraically double, with one eigenvector, $\Phi_0^{\star,+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and one generalized eigenvector, $\Phi_0^{\star,-} = \frac{M}{\sqrt{M^2+1}} \begin{pmatrix} -\frac{1}{M} \\ 1 \end{pmatrix}$, verifying $\mathscr{A}^\star \Phi_0^{\star,-} = \frac{M}{\sqrt{M^2+1}} \Phi_0^{\star,+}$. Moreover, $\left(\Phi_n^{\star,\pm}\right)_{n\in\mathbb{Z}}$ is a Riesz basis in the space $\left(L^2(\mathbb{T})\right)^2$.

Remark 1. *In the case $c = 0$ and $M \neq 0$, the spectrum of the operator $\mathscr{A}^\star$ is purely real and the subfamily of eigenvalues $(\lambda_n^-)_{n\geq 1}$ has an accumulation point at $-M$. It is known that the presence of an accumulation point in the spectrum makes a system not even spectrally controllable. This is why the controllability properties of the heat equation with memory* (1) *and fixed support control are very poor. The situation changes when $c > 0$, i. e. when a moving control is considered. In this case, due to its imaginary part, the subfamily $(\lambda_n^-)_{n\geq 1}$, becomes well-separated and we are able to show that the resulting system is controllable, if the time T is sufficiently large. From the spectral point of view, the problem becomes similar to the one studied in the context of viscoelasticity [10].*

According to Theorem 3, given the initial data $\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} = \sum_{n\in\mathbb{Z}} a_n^\pm \Phi_n^{\star,\pm}$, the corresponding solution of the adjoint system (9) is given by

$$\begin{pmatrix} \varphi(t,\cdot) \\ \psi(t,\cdot) \end{pmatrix} = \sum_{n\in\mathbb{Z}^*} a_n^\pm e^{\lambda_n^\pm (t-T)} \Phi_n^{\star,\pm} + a_0^+ \Phi_0^{\star,+} + a_0^- \left(\Phi_0^{\star,-} + \frac{M}{\sqrt{M^2+1}}(t-T)\Phi_0^{\star,+}\right), \tag{12}$$

and the observability inequality (7) is equivalent to

$$\sum_{n\in\mathbb{Z}^*} \left| a_n^+ \gamma_n^+ e^{-T\mu^+_{|n|}} + a_n^- \gamma_n^- e^{-T\mu^-_{|n|}} \right|^2 + \frac{\left|a_0^-\right|^2}{M^2+1} \leq K \int_0^T \int_{w_0} \left| \sum_{n\in\mathbb{Z}^*} a_n^\pm \gamma_n^\pm e^{\lambda_n^\pm (t-T)+iny} - \frac{a_0^-}{\sqrt{M^2+1}} \right|^2 \mathrm{d}y\,\mathrm{d}t. \tag{13}$$

In the sequel C represents a positive constant which may change from one row to another and we denote $\Lambda := \left(e^{\lambda_n^\pm t}\right)_{n\in\mathbb{Z}^*} \cup \left\{e^{\lambda_0 t}\right\}$. We recall that a sequence $(\Theta_m^\pm)_{m\in\mathbb{Z}^*} \cup \{\Theta_0\} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is biorthogonal to the family Λ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if the following relations hold

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m^\pm(t) e^{\overline{\lambda_m^\pm} t}\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_0(t) = 1, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m^\pm(t) e^{\overline{\lambda_m^\mp} t}\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m^\pm(t)\,\mathrm{d}t = 0 \quad (m\in\mathbb{Z}^*),$$
$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m^\pm(t) e^{\overline{\lambda_n^\pm} t}\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m^\pm(t) e^{\overline{\lambda_n^\mp} t}\,\mathrm{d}t = \int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_0(t) e^{\overline{\lambda_m^\pm} t}\,\mathrm{d}t = 0 \quad (m,n\in\mathbb{Z}^*,\, n\neq m). \tag{14}$$

The concept of biorthogonal sequence will be useful to prove the observability inequality (13).

2. CONSTRUCTION OF A BIORTHOGONAL SEQUENCE

In this section we construct and evaluate a biorthogonal sequence to the family of exponential functions Λ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$. Firstly, let us define the infinite products

$$P_0(z) := \underbrace{\prod_{n\in\mathbb{Z}^*} \frac{\overline{\lambda_n^+} + zi}{\overline{\lambda_n^+}}}_{\widetilde{P}_0^+(z)} \underbrace{\prod_{n\in\mathbb{Z}^*} \frac{\overline{\lambda_n^-} + zi}{\overline{\lambda_n^-}}}_{\widetilde{P}_0^-(z)}, \qquad P_m^\pm(z) := -\frac{zi}{\overline{\lambda_m^\pm}} \underbrace{\prod_{\substack{n\in\mathbb{Z}^*\\ n\neq m}} \frac{\overline{\lambda_n^\pm} + zi}{\overline{\lambda_n^\pm} - \overline{\lambda_m^\pm}}}_{\widetilde{P}_m^\pm(z)} \underbrace{\prod_{n\in\mathbb{Z}^*} \frac{\overline{\lambda_n^\mp} + zi}{\overline{\lambda_n^\mp} - \overline{\lambda_m^\pm}}}_{\widetilde{\widetilde{P}}_m^\pm(z)} \quad (m\in\mathbb{Z}^*), \tag{15}$$

where $(\lambda_n^\pm)_{n\in\mathbb{Z}^*}$ are the eigenvalues given by (10). The convergence of products (15) is a consequence of the estimates given by the following result.

Theorem 4. *There exist positive constants C_1, C_2 and C_3, independent of m, such that the products $P_m^\pm$ and P_0 given by* (15) *verify the following estimates*

$$\left|P_m^\pm(z)\right| \le C_1 \left|z - i\overline{\lambda_m^\mp}\right|^{C_2} \exp\left(C_3\sqrt{\left|z - i\overline{\lambda_m^\pm}\right|} + \frac{\pi}{c}\left|\Im(z - i\overline{\lambda_m^-})\right|\right) \qquad (z\in\mathbb{C}, \quad m\in\mathbb{Z}^*), \tag{16}$$
$$|P_0(z)| \le C_1 \exp\left(C_3\sqrt{|z|} + \frac{\pi}{c}|\Im(z)|\right) \qquad (z\in\mathbb{C}). \tag{17}$$

Proof. Let us consider that $m\in\mathbb{Z}^*$ and $Z := \frac{z - i\overline{\lambda_m^-}}{c}$. With the notation introduced in (15) we have

$$\begin{aligned}
\left|\widetilde{P}_m^-(z)\right| &= \left|\prod_{\substack{n\in\mathbb{Z}^*\\ n\neq m}} \left(1 - \frac{cZ}{i\overline{\lambda_n^-} - i\overline{\lambda_m^-}}\right)\right| = \left|\prod_{\substack{n\in\mathbb{Z}^*\\ n\le m-1}} \prod_{\substack{n\in\mathbb{Z}^*\\ n\ge m+1}} \left(1 - \frac{Z}{m-n} + Z\frac{i\left(\mu_{|n|}^- - \mu_{|m|}^-\right)}{(m-n)(i\overline{\lambda_n^-} - i\overline{\lambda_m^-})}\right)\right| \\
&= \left|\prod_{\substack{k=1\\ k\neq |m|}}^\infty \left(1 - \frac{Z}{k} + Z\frac{i\left(\mu_{|m-k|}^- - \mu_{|m|}^-\right)}{k(i\overline{\lambda_{m-k}^-} - i\overline{\lambda_m^-})}\right)\left(1 + \frac{Z}{k} - Z\frac{i\left(\mu_{|m+k|}^- - \mu_{|m|}^-\right)}{k(i\overline{\lambda_{m+k}^-} - i\overline{\lambda_m^-})}\right)\right| \\
&\le \prod_{\substack{k=1\\ k\neq |m|}}^\infty \left(\left|1 - \left(\frac{Z}{k}\right)^2\right| + \frac{4M}{ck^2}|Z| + \frac{2cM + 4M^2}{c^2k^3}|Z|^2\right).
\end{aligned} \tag{18}$$

From (18) it follows immediately that $\widetilde{P}_m^-$ verifies the following estimate

$$\left|\frac{zi}{\overline{\lambda_m^-}}\widetilde{P}_m^-(z)\right| \le C \qquad (|Z|<1). \tag{19}$$

Now, suppose that $|Z| \geq 1$ and let $\lfloor |Z| \rfloor := k_z$, $\left\lfloor \frac{|Z|}{2} \right\rfloor := k_z^1$, $\left\lfloor \frac{3|Z|}{2} \right\rfloor := k_z^2$. From (18) we deduce that

$$
\begin{aligned}
\left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \widetilde{P}_m^-(z) \right| &\leq \left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \right| \prod_{\substack{k=1 \\ k \neq |m|}}^{\infty} \left(\left| 1 - \left(\frac{Z}{k} \right)^2 \right| + Ck^{-2} |Z| + Ck^{-3} |Z|^2 \right) \\
&\leq C \left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \right| \prod_{\substack{k=1 \\ k \neq k_z, |m|}}^{\infty} \left| 1 - \left(\frac{Z}{k} \right)^2 \right| \left(1 + \frac{k^2}{|k^2 - Z^2|} \left(Ck^{-2} |Z| + Ck^{-3} |Z|^2 \right) \right) \\
&\leq C \left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \right| \left| \frac{\sin(\pi Z)}{\pi Z} \right| \frac{(k_z)^2}{\left| (k_z)^2 - Z^2 \right|} \frac{m^2}{|m^2 - Z^2|} \underbrace{\prod_{\substack{k=1 \\ k \neq k_z}}^{\infty} \left(1 + \frac{k^2}{|k^2 - Z^2|} \left(Ck^{-2} |Z| + Ck^{-3} |Z|^2 \right) \right)}_{P_m^1(z)}.
\end{aligned} \tag{20}
$$

We remark that

$$
\left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \right| \left| \frac{\sin(\pi Z)}{\pi Z} \right| \frac{(k_z)^2}{\left| (k_z)^2 - Z^2 \right|} \frac{m^2}{|m^2 - Z^2|} \leq C \exp(\pi |\Im(Z)|), \tag{21}
$$

and it remains to estimate the product P_m^1 from (20). We remark that

$$
\ln \left(1 + \frac{k^2}{|k^2 - Z^2|} \left(Ck^{-2} |Z| + Ck^{-3} |Z|^2 \right) \right) \leq C \times \begin{cases} \frac{1}{k} & k \leq k_z^1 - 1 \\ \frac{1}{||Z| - k|} & k_z^1 \leq k \leq k_z^2 \\ \frac{|Z|}{k^2} & k_z^2 + 1 \leq k. \end{cases}
$$

From the above estimates we immediately deduce that

$$
\left| P_m^1(z) \right| \leq C |Z|^C \qquad (|Z| \geq 1). \tag{22}
$$

Now, taking into account (20), (21) and (22) we obtain that

$$
\left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \widetilde{P}_m^-(z) \right| \leq C |Z|^C \exp(\pi |\Im(Z)|) \qquad (|Z| \geq 1). \tag{23}
$$

From this and (19) we deduce that there exists a positive constant C, independent of m, such that

$$
\left| \frac{zi}{\overline{\overline{\lambda_m^-}}} \widetilde{P}_m^-(z) \right| \leq \begin{cases} C \left| z - i\overline{\lambda_m^-} \right|^C \exp\left(\frac{\pi}{c} \left| \Im(z - i\overline{\lambda_m^-}) \right| \right) & \text{if } \left| z - i\overline{\lambda_m^-} \right| \geq c \\ C & \text{if } \left| z - i\overline{\lambda_m^-} \right| < c \end{cases} \qquad (z \in \mathbb{C}, \quad m \in \mathbb{Z}^*). \tag{24}
$$

To estimate the product $\widetilde{\widetilde{P}}_m^-(z)$ we proceed as follows

$$
\begin{aligned}
\left| \widetilde{\widetilde{P}}_m^-(z) \right| &\leq \left| z - i\overline{\lambda_m^+} \right| \prod_{\substack{n \in \mathbb{Z}^* \\ n \neq m}} \left| 1 + \frac{\overline{\lambda_m^-} + zi}{\lambda_n^+ - \overline{\lambda_m^-}} \right| \leq \left| z - i\overline{\lambda_m^+} \right| \exp\left(\sum_{\substack{n \in \mathbb{Z}^* \\ n \neq m}} \ln\left(1 + \left| \frac{\overline{\lambda_m^-} + zi}{\lambda_n^+ - \lambda_m^-} \right| \right) \right) \\
&\leq \left| z - i\overline{\lambda_m^+} \right| \exp\left(\sum_{n \in \mathbb{Z}^*} \int_0^{\left| \overline{\lambda_m^-} + zi \right|} \frac{\mathrm{d}t}{t + \left| \lambda_n^+ - \lambda_m^- \right|} \right) = \left| z - i\overline{\lambda_m^+} \right| \exp\left(\int_0^{\left| \overline{\lambda_m^-} + zi \right|} \sum_{n \in \mathbb{Z}^*} \int_{\left| \lambda_n^+ - \lambda_m^- \right|}^{\infty} \frac{\mathrm{d}s}{(t+s)^2} \,\mathrm{d}t \right) \\
&\leq \left| z - i\overline{\lambda_m^+} \right| \exp\left(2 \int_0^{\left| \overline{\lambda_m^-} + zi \right|} \int_0^{\infty} \frac{K(s)}{(t+s)^2} \,\mathrm{d}s \,\mathrm{d}t \right) \leq \left| z - i\overline{\lambda_m^+} \right| \exp\left(C \int_0^{\left| \overline{\lambda_m^-} + zi \right|} \frac{1}{\sqrt{t}} \,\mathrm{d}t \right),
\end{aligned}
$$

where $K(s) := \sum_{\substack{n^2 + \widetilde{r} < s \\ n \in \mathbb{N}}} 1 \leq \left(1 + \sqrt{\frac{2M+1}{M}} \right) \sqrt{s}$ and $\widetilde{r} = \frac{2M}{1 + \sqrt{1 + 4M}}$. From the above estimate we obtain that

$$
\left| \widetilde{\widetilde{P}}_m^-(z) \right| \leq \left| z - i\overline{\lambda_m^+} \right| \exp\left(C \sqrt{\left| \overline{\lambda_m^-} + zi \right|} \right) \tag{25}
$$

The estimate of the product $\widetilde{P}_m^+(z)$ is similar to the one of the product $\widetilde{\widetilde{P}}_m^-(z)$ and we obtain that

$$\left|\widetilde{P}_m^+(z)\right| \leq \exp\left(C\sqrt{\left|\overline{\lambda_m^+}+zi\right|}\right). \tag{26}$$

To estimate the product $\widetilde{\widetilde{P}}_m^+(z)$ we remark that

$$\left|\frac{zi}{\overline{\lambda_m^+}}\widetilde{\widetilde{P}}_m^+(z)\right| = \left|\frac{zi}{\overline{\lambda_m^+}}\frac{\overline{\lambda_m^-}+zi}{\overline{\lambda_m^-}-\overline{\lambda_m^+}}\widetilde{P}_m^-(z)\prod_{\substack{n\in\mathbb{Z}^*\\ n\neq m}}\frac{\overline{\lambda_n^-}-\overline{\lambda_m^-}}{\overline{\lambda_n^-}-\overline{\lambda_m^+}}\right| \leq \left|\frac{zi}{\overline{\lambda_m^+}}\frac{\overline{\lambda_m^-}+zi}{\overline{\lambda_m^-}-\overline{\lambda_m^+}}\widetilde{P}_m^-(z)\right| \leq \left|z-i\overline{\lambda_m^-}\right|\left|\frac{zi}{\overline{\lambda_m^-}}\widetilde{P}_m^-(z)\right|.$$

From the last estimate, (24), (25) and (26) we deduce that the product $P_m^{\pm}$ verify (16).

Finally, in the case case $m=0$, we remark that $\widetilde{P}_0^+(z)$ can be estimated with the same arguments as $\widetilde{P}_m^+(z)$ while $\widetilde{P}_0^-(z)$ is similar to the term $-\frac{zi}{\overline{\lambda_m^-}}\widetilde{P}_m^-(z)$. Hence, estimate (17) follows immediately. The proof of the theorem is complete. □

Remark 2. *According to* (16) *and* (17), *$P_m^{\pm}$ and P_0 are entire exponential functions of type $\frac{\pi}{c}$.*

We can pass to construct the desired biorthogonal sequence $(\Theta_m^{\pm})_{m\in\mathbb{Z}^*}\cup\{\Theta_0\}$ to the family Λ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$.

Theorem 5. *Let $T>\frac{2\pi}{c}$ and $\varepsilon>0$ such that $T>\frac{2\pi}{c}+2\varepsilon$. There exist a biorthogonal sequence $(\Theta_m^{\pm})_{m\in\mathbb{Z}^*}\cup\{\Theta_0\}$ to the family Λ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ and a positive constant $\widetilde{K}$ such that the following inequality holds*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{m\in\mathbb{Z}^*}\beta_m^{\pm}\Theta_m^{\pm}(t)+\beta_0\Theta_0(t)\right|^2 \mathrm{dt} \leq \widetilde{K}\left[\sum_{m\in\mathbb{Z}^*}\left|\beta_m^{\pm}\right|^2\exp\left(2\varepsilon\left|\mu_{|m|}^{\pm}\right|\right)+|\beta_0|^2\right], \tag{27}$$

for any finite sequence $(\beta_m^{\pm})_{m\in\mathbb{Z}^*}\cup\{\beta_0\}\subset\mathbb{C}$.

Proof. Let $T'>0$ such that $T>T'>\frac{2\pi}{c}+2\varepsilon$. Let $P_m^{\pm}$ and P_0 be the products given by (15). Taking into account Theorem 4 and the fact that $\left|x-i\overline{\lambda_m^-}\right|\leq\left|x-i\overline{\lambda_m^+}\right|$ we deduce immediately that there exist positive constants C_4 and C_5 such that the products $P_m^{\pm}$ and P_0 verify the following estimates on the real axis

$$|P_0(x)|\leq C_4\exp\left(C_5\sqrt{|x|}\right),\quad \left|P_m^{\pm}(x)\right|\leq C_4\exp\left(C_5\sqrt{\left|x-i\overline{\lambda_m^{\pm}}\right|}\right)\qquad (x\in\mathbb{R},\ m\in\mathbb{Z}^*). \tag{28}$$

Now, given $\beta\in(0,1)$, we consider the entire function $\widehat{h}$ from [8, Lemma 2.1] with the properties that

$$\widehat{h}(0)=1,\qquad \left|\widehat{h}(z)\right|\leq C\exp\left(-\varepsilon|z|^{\beta}+\varepsilon|y|\right)\qquad (z=x+iy\in\mathbb{C}), \tag{29}$$

and we define the functions

$$\Psi_0(z)=P_0(z)\widehat{h}(z),\qquad \Psi_m^{\pm}(z)=P_m^{\pm}(z)\widehat{h}(z-i\overline{\lambda}_m^{\pm})\qquad (m\in\mathbb{Z}^*). \tag{30}$$

Since $P_m^{\pm}$ and P_0 are functions of exponential type $\frac{\pi}{c}$ and $\widehat{h}$ is a function of exponential type ε we deduce that $\Psi_m^{\pm}$ and Ψ_0 are entire functions of exponential type $\frac{T'}{2}$. Furthermore,

$$\Psi_0\left(i\overline{\lambda}_n^{\pm}\right)=0,\quad \Psi_0(0)=1,\qquad \Psi_m^{\pm}\left(i\overline{\lambda}_n^{\pm}\right)=\delta_{mn}^{\pm},\quad \Psi_m^{\pm}(0)=0\qquad (m,n\in\mathbb{Z}^*). \tag{31}$$

From estimate (28) of the product $P_m^{\pm}$ on the real axis and property (29) of the multiplier $\widehat{h}$, it follows that

$$\int_{\mathbb{R}}\left|\Psi_m^{\pm}(x)\right|^2\mathrm{dx}\leq C\exp\left(2\varepsilon\left|\mu_{|m|}^{\pm}\right|\right)\int_{\mathbb{R}}\exp\left(2C_5\left|x-i\overline{\lambda}_m^{\pm}\right|^{\frac{1}{2}}-2\varepsilon\left|x-i\overline{\lambda}_m^{\pm}\right|^{\beta}\right)\mathrm{dx}.$$

By choosing $\beta > \frac{1}{2}$, we obtain that

$$\int_{\mathbb{R}} |\Psi_0(x)|^2 \,\mathrm{dx} \leq C, \quad \int_{\mathbb{R}} \left|\Psi_m^{\pm}(x)\right|^2 \mathrm{dx} \leq C\exp\left(2\varepsilon \left|\mu_{|m|}^{\pm}\right|\right) \qquad (m \in \mathbb{Z}^*). \tag{32}$$

By using Paley-Wiener Theorem, we deduce that there exists $(\theta_m^{\pm})_{m\in\mathbb{Z}^*} \cup \{\theta_0\} \subset L^2\left(-\frac{T'}{2}, \frac{T'}{2}\right)$ such that

$$\Psi_0(z) = \int_{-\frac{T'}{2}}^{\frac{T'}{2}} \theta_0(t)e^{-itz}\,\mathrm{dt}, \quad \Psi_m^{\pm}(z) = \int_{-\frac{T'}{2}}^{\frac{T'}{2}} \theta_m^{\pm}(t)e^{-itz}\,\mathrm{dt} \quad (m \in \mathbb{Z}^*).$$

From (31) and (32) it follows that $(\theta_m^{\pm})_{m\in\mathbb{Z}^*} \cup \{\theta_0\}$ is a biorthogonal sequence to Λ in $L^2\left(-\frac{T'}{2}, \frac{T'}{2}\right)$ verifying

$$\|\theta_0\|_{L^2\left(-\frac{T'}{2},\frac{T'}{2}\right)} \leq C, \quad \left\|\theta_m^{\pm}\right\|_{L^2\left(-\frac{T'}{2},\frac{T'}{2}\right)} \leq C\exp\left(\varepsilon \left|\mu_{|m|}^{\pm}\right|\right) \qquad (m \in \mathbb{Z}^*). \tag{33}$$

Since $T > T'$, a new biorthogonal $(\Theta_m^{\pm})_{m\in\mathbb{Z}^*} \cup \{\Theta_0\}$ to the family Λ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ verifying (27) can be constructed as in [11, Theorem 3.2]. The proof of the theorem is complete. □

3. PROOF OF THE OBSERVABILITY INEQUALITY (13) AND OF THEOREM 1

Let $T > \frac{2\pi}{c}$. The biorthogonal sequence $(\Theta_m^{\pm})_{m\in\mathbb{Z}^*} \cup \{\Theta_0\}$ to Λ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ given by Theorem 5 allows us to conclude the proof of the observability inequality (13). Let $(a_n^{\pm})_{n\in\mathbb{Z}^*} \cup \{a_0\}$ be a finite sequence, $\beta_n^{\pm} = a_n^{\pm}\exp\left(-\overline{\lambda_n^{\pm}}\frac{T}{2}\right)$ and $\beta_0 = a_0$. From the biorthogonality properties (14), we deduce that

$$\left(\sum_{m\in\mathbb{Z}^*} \beta_m^{\pm}\Theta_m^{\pm} + \beta_0\Theta_0, \sum_{n\in\mathbb{Z}^*} a_n^{\pm}\exp\left(\lambda_n^{\pm}\frac{T}{2} + \lambda_n^{\pm}t\right) + a_0\right)_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} = \sum_{n\in\mathbb{Z}^*} \left|a_n^{\pm}\right|^2 + |a_0|^2. \tag{34}$$

Moreover, by using inequality (27) and the fact that $-T\mu_{|m|}^{\pm} + 2\varepsilon|\mu_{|m|}^{\pm}| \leq 2MT$, it follows that

$$\begin{aligned}
&\left|\left(\sum_{m\in\mathbb{Z}^*} \beta_m^{\pm}\Theta_m^{\pm} + \beta_0\Theta_0, \sum_{n\in\mathbb{Z}^*} a_n^{\pm}\exp\left(\lambda_n^{\pm}\frac{T}{2} + \lambda_n^{\pm}t\right) + a_0\right)_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}\right| \\
&\leq \sqrt{\widetilde{K}}\left(\sum_{m\in\mathbb{Z}^*} \left|\beta_m^{\pm}\right|^2 \exp\left(2\varepsilon\left|\mu_{|m|}^{\pm}\right|\right) + |\beta_0|^2\right)^{1/2} \left\|\sum_{n\in\mathbb{Z}^*} a_n^{\pm}\exp\left(\lambda_n^{\pm}\frac{T}{2} + \lambda_n^{\pm}t\right) + a_0\right\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \\
&\leq \sqrt{\widetilde{K}}e^{MT}\left(\sum_{m\in\mathbb{Z}^*} \left|a_m^{\pm}\right|^2 + |a_0|^2\right)^{1/2} \left\|\sum_{n\in\mathbb{Z}^*} a_n^{\pm}\exp\left(\lambda_n^{\pm}\frac{T}{2} + \lambda_n^{\pm}t\right) + a_0\right\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)},
\end{aligned} \tag{35}$$

where $\widetilde{K}$ is the constant in (27). From relations (34)-(35) we obtain that the following inequality holds

$$\left(\sum_{n\in\mathbb{Z}^*} \left|a_n^{\pm}\right|^2 + |a_0|^2\right)^{1/2} \leq \sqrt{\widetilde{K}}e^{MT} \left\|\sum_{n\in\mathbb{Z}^*} a_n^{\pm}\exp\left(\lambda_n^{\pm}\frac{T}{2} + \lambda_n^{\pm}t\right) + a_0\right\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}, \tag{36}$$

for any finite sequence of scalars $(a_n^{\pm})_{n\in\mathbb{Z}^*} \cup \{a_0\}$. From (36) it follows that

$$\int_0^T \int_{w_0} \left|\sum_{n\in\mathbb{Z}^*} a_n^{\pm}\gamma_n^{\pm}e^{\lambda_n^{\pm}(t-T)+iny} - \frac{a_0^-}{\sqrt{M^2+1}}\right|^2 \mathrm{dy}\,\mathrm{dt} = \int_{w_0} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\sum_{n\in\mathbb{Z}^*} a_n^{\pm}\gamma_n^{\pm}e^{\lambda_n^{\pm}\left(t-\frac{T}{2}\right)+iny} - \frac{a_0^-}{\sqrt{M^2+1}}\right|^2 \mathrm{dy}\,\mathrm{dt}$$

$$\geq \frac{e^{-2MT}}{\widetilde{K}} \int_{w_0} \left(\sum_{n\in\mathbb{Z}^*} \left| a_n^{\pm} \gamma_n^{\pm} e^{-T\lambda_n^{\pm}+iny} \right|^2 + \frac{1}{M^2+1} |a_0^-|^2 \right) \mathrm{dy} \geq \frac{2}{K} \left(\sum_{n\in\mathbb{Z}^*} e^{-2T\mu_{|n|}^{\pm}} \left| a_n^{\pm} \right|^2 + |a_0^-|^2 \right),$$

where $K = \frac{\widetilde{K}e^{2MT}}{|w_0|} \max\left\{2M^2+2, \left(1+\sqrt{1+4M}\right)^2\right\}$. On the other hand, we have

$$\sum_{n\in\mathbb{Z}^*} \left| a_n^+ \gamma_n^+ e^{-T\mu_{|n|}^+} + a_n^- \gamma_n^- e^{-T\mu_{|n|}^-} \right|^2 + \frac{\left|a_0^-\right|^2}{M^2+1} \leq 2 \left(\sum_{n\in\mathbb{Z}^*} e^{-2T\mu_{|n|}^{\pm}} \left| a_n^{\pm} \right|^2 + |a_0^-|^2 \right).$$

From the above inequalities we deduce that the observability inequality (13) holds. Consequently, by taking into account Theorem 2, the main result given by Theorem 1 is also proved.

Remark 3. *Since the asymptotic gap of the spectral subfamily $(\lambda_n^-)_{n\in\mathbb{Z}^*}$ is equal to c, the control time T should be greater than $\frac{2\pi}{c}$. Consequently, Theorem 1 gives the optimal control time.*

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