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BINDING NUMBER FOR PATH-FACTOR UNIFORM GRAPHS

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Abstract. A path-factor of a graph G is a spanning subgraph of G whose components are paths. A $P_{\geq d}$-factor of a graph G is a path-factor of G whose components are paths with at least d vertices, where $d \geq 2$ is an integer. A graph G is called a $P_{\geq d}$-factor uniform graph if for any two different edges e_1 and e_2 of G, G admits a $P_{\geq d}$-factor containing e_1 and avoiding e_2. The binding number of G is defined by $\text{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}$. In this paper, we prove that (i) a 3-connected graph G is a $P_{\geq 2}$-factor uniform graph if $\text{bind}(G) > 1$; (ii) a 3-connected graph G is a $P_{\geq 3}$-factor uniform graph if $\text{bind}(G) > \frac{10}{7}$.

Key words: graph; binding number; path-factor; $P_{\geq 2}$-factor uniform graph; $P_{\geq 3}$-factor uniform graph.

1. INTRODUCTION

We deal with finite undirected graphs which have neither loops nor multiple edges. For a graph G, let $V(G)$, $E(G)$, $I(G)$, $i(G)$ and $c(G)$ be the vertex set, the edge set, the set of isolated vertices, the number of isolated vertices and the number of connected components of G, respectively. For $x \in V(G)$, the set of neighbours of x is denoted by $N_G(x)$. The degree of $x \in V(G)$ in G is denoted by $d_G(x)$. Note that $d_G(x) = |N_G(x)|$. For $X \subseteq V(G)$, we write $N_G(X)$ for $\bigcup_{x \in X} N_G(x)$, and $G - X$ for the subgraph derived from G by deleting all vertices in X. We call $e = uv$ an independent edge of G if $N_G(\{u, v\}) = \{u, v\}$. For $E' \subseteq E(G)$, we write $G - E'$ for the subgraph obtained from G by deleting all edges in E'. For $X \subseteq V(G)$, we say that X is independent if no two elements in X are adjacent. The binding number of G is defined by Woodall [1] as

$$\text{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

For two graphs G_1 and G_2, we denote by $G_1 \vee G_2$ the join of G_1 and G_2, and by $G_1 \cup G_2$ the union of G_1 and G_2. We denote the path and the complete graph of order n by P_n and K_n, respectively.

A path-factor of a graph G is a spanning subgraph of G whose components are paths. A $P_{\geq d}$-factor of a graph G is a path-factor of G whose components are paths with at least d vertices, where $d \geq 2$ is an integer.

Kano, Lu and Yu [2] presented a sufficient condition for a graph admitting a $P_{\geq 3}$-factor. Zhou et al [3–6] investigated the existence of $P_{\geq 3}$-factors in graphs. Kawarabayashi, Matsuda, Oda and Ota [7] verified that a 2-connected cubic graph with at least six vertices admits a $P_{\geq 6}$-factor. Kano, Lee and Suzuki [8] proved that a connected cubic bipartite graph of order at least 8 contains a $P_{\geq 8}$-factor. Ando, Egawa, Kaneko, Kawarabayashi and Matsuda [9] showed a sufficient condition for a claw-free graph to have a $P_{\geq d+1}$-factor. Las Vergnas [10] derived a characterization of a graph with a $P_{\geq 2}$-factor.

THEOREM 1 ([10]). *A graph G contains a* $P_{\geq 2}$*-factor if and only if* $i(G-X) \leq 2|X|$ *for all* $X \subseteq V(G)$.

To characterize a graph with a $P_{\geq 3}$-factor, Kaneko [11] posed the concept of a sun. A graph H is factor-critical if any induced subgraph with $|V(H)|-1$ vertices admits a perfect matching. Let H be a factor-critical graph with vertex set $V(H) = \{u_1, u_2, \cdots, u_n\}$. By adding n new vertices $v_1, v_2, \cdots, v_n$ together with n new edges $u_1v_1, u_2v_2, \cdots, u_nv_n$ to H, we derive a new graph R, which is called a sun. By Kaneko, K_1 and K_2 are also suns. A big sun is a sun with at least six vertices. If a component of G is isomorphic to a sun, it is called a sun component of G. We write $\mathrm{Sun}(G)$ for the set of sun components of G, and let $\mathrm{sun}(G) = |\mathrm{Sun}(G)|$ be the number of sun components of G.

Kaneko [11] provided a criterion for a graph with a $P_{\geq 3}$-factor. Kano, Katona and Király [12] presented a simple proof.

THEOREM 2 ([11, 12]). *A graph G contains a* $P_{\geq 3}$*-factor if and only if* $\mathrm{sun}(G-X) \leq 2|X|$ *for all* $X \subseteq V(G)$.

Later, Zhang and Zhou [13] defined a graph G being $P_{\geq d}$-factor covered if for any $e \in E(G)$, G has a $P_{\geq d}$-factor covering e. Furthermore, they posed two characterizations for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs.

THEOREM 3 ([13]). *A connected graph G is a* $P_{\geq 2}$*-factor covered graph if and only if* $i(G-X) \leq 2|X| - \varepsilon_1(X)$ *for any* $X \subseteq V(G)$*, where* $\varepsilon_1(X)$ *is defined by*

$$\varepsilon_1(X) = \begin{cases} 2, & \textit{if X is not an independent set}; \\ 1, & \textit{if X is a nonempty independent set and } G-X \textit{ has} \\ & \textit{a nontrivial component}; \\ 0, & \textit{otherwise}. \end{cases}$$

THEOREM 4 ([13]). *A connected graph G is a* $P_{\geq 3}$*-factor covered graph if and only if* $\mathrm{sun}(G-X) \leq 2|X| - \varepsilon_2(X)$ *for any* $X \subseteq V(G)$*, where* $\varepsilon_2(X)$ *is defined by*

$$\varepsilon_2(X) = \begin{cases} 2, & \textit{if X is not an independent set}; \\ 1, & \textit{if X is a nonempty independent set and } G-X \textit{ has} \\ & \textit{a non-sun component}; \\ 0, & \textit{otherwise}. \end{cases}$$

Recently, Zhou and Sun [14] posed the concept of a $P_{\geq d}$-factor uniform graph. A graph G is called a $P_{\geq d}$-factor uniform graph if for any two different edges e_1 and e_2 of G, G admits a $P_{\geq d}$-factor containing e_1 and avoiding e_2. In other words, a graph G is called a $P_{\geq d}$-factor uniform graph if for any $e \in E(G)$, $G-e$ is a $P_{\geq d}$-factor covered graph. Furthermore, they showed two binding number conditions for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor uniform graphs.

THEOREM 5 ([14]). *Let G be a 2-edge-connected graph. If* $\mathrm{bind}(G) > \frac{4}{3}$*, then G is a* $P_{\geq 2}$*-factor uniform graph.*

THEOREM 6 ([14]). *Let G be a 2-edge-connected graph. If* $\mathrm{bind}(G) > \frac{9}{4}$*, then G is a* $P_{\geq 3}$*-factor uniform graph.*

Gao and Wang [15] improved the binding number condition of Theorem 6.

THEOREM 7 ([15]). *Let G be a 2-edge-connected graph. If* $\mathrm{bind}(G) > \frac{5}{3}$*, then G is a* $P_{\geq 3}$*-factor uniform graph.*

Kano and Tokushige [16], Plummer and Saito [17], Wang and Zhang [18], Zhou [19], Zhou, Xu and Xu [20] established some relationships between binding numbers and graph factors. Some other results on graph factors can be found in [21–26].

The purpose of this paper is to weaken the binding number conditions in Theorems 5–7 by assuming that G is 3-connected.

THEOREM 8. *A 3-connected graph G is a $P_{\geq 2}$-factor uniform graph if* $\text{bind}(G) > 1$.

THEOREM 9. *A 3-connected graph G is a $P_{\geq 3}$-factor uniform graph if* $\text{bind}(G) > \frac{10}{7}$.

2. PROOF OF THEOREM 8

Proof of Theorem 8. We prove Theorem 8 by contradiction. Assume that $G' = G - e$ is not a $P_{\geq 2}$-factor covered graph for some $e = uv \in E(G)$. Then by Theorem 3,

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{1}$$

for some $X \subseteq V(G')$.

Claim 1. $|X| \geq 3$.

Proof. If $0 \leq |X| \leq 1$, then it follows from (1) and $\varepsilon_1(X) \leq |X|$ that

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq |X| + 1 \geq 1. \tag{2}$$

On the other hand, since G is 3-connected, $G' - X$ is connected. Thus, we have $i(G' - X) = 0$, which contradicts (2).

If $|X| = 2$, then by (1) and $\varepsilon_1(X) \leq 2$,

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 = 3. \tag{3}$$

Note that G is 3-connected. Then $G - X$ is connected, and so $i(G - X) = 0$. Thus, we have

$$i(G' - X) = i(G - X - e) \leq i(G - X) + 2 = 2,$$

which contradicts (3). Hence, we get $|X| \geq 3$. This completes the proof of Claim 1. □

We shall distinguish between the following three cases.

Case 1. $u, v \in I(G' - X)$. In this case, $e = uv$ is an independent edge of $G - X$. Then we deduce $|N_G(I(G' - X))| \leq |X| + 2$. In terms of (1), $\varepsilon_1(X) \leq 2$ and Claim 1, we derive

$$\begin{aligned} \text{bind}(G) &\leq \frac{|N_G(I(G' - X))|}{|I(G' - X)|} = \frac{|N_G(I(G' - X))|}{i(G' - X)} \leq \frac{|X| + 2}{2|X| - \varepsilon_1(X) + 1} \\ &\leq \frac{|X| + 2}{2|X| - 1} = \frac{1}{2} + \frac{5}{4|X| - 2} \leq \frac{1}{2} + \frac{5}{4 \times 3 - 2} = 1, \end{aligned}$$

which contradicts $\text{bind}(G) > 1$.

Case 2. $u, v \notin I(G' - X)$. In this case, $i(G' - X) = i(G - X - e) = i(G - X)$. Combining this with (1), $\varepsilon_1(X) \leq 2$ and Claim 1,

$$i(G - X) = i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 5,$$

which implies $I(G - X) \neq \emptyset$ and $N_G(I(G - X)) \neq V(G)$. Thus, we infer

$$1 < \text{bind}(G) \leq \frac{|N_G(I(G - X))|}{|I(G - X)|} \leq \frac{|X|}{i(G - X)} = \frac{|X|}{i(G' - X)}. \tag{4}$$

According to (4) and $\varepsilon_1(X) \leq |X|$,

$$i(G' - X) < |X| \leq 2|X| - \varepsilon_1(X),$$

which contradicts (1).

Case 3. $u \in I(G'-X)$ and $v \notin I(G'-X)$, or $u \notin I(G'-X)$ and $v \in I(G'-X)$. Without loss of generality, let $u \in I(G'-X)$ and $v \notin I(G'-X)$. Combining this with (1), $\varepsilon_1(X) \le 2$ and Claim 1, we derive $i(G-X-v) = i(G-X-v-e) = i(G'-X-v) \ge i(G'-X) \ge 2|X| - \varepsilon_1(X) + 1 \ge 2|X| - 1 \ge 5$, which implies that $I(G-X-v) \ne \emptyset$ and $N_G(I(G-X-v)) \ne V(G)$. In light of (1), $\varepsilon_1(X) \le 2$, $\text{bind}(G) > 1$ and the definition of $\text{bind}(G)$,

$$\begin{aligned} 1 &< \text{bind}(G) \le \frac{|N_G(I(G-X-v))|}{|I(G-X-v)|} \le \frac{|X|+1}{i(G-X-v)} \\ &\le \frac{|X|+1}{i(G'-X)} \le \frac{|X|+1}{2|X|-\varepsilon_1(X)+1} \le \frac{|X|+1}{2|X|-1}, \end{aligned}$$

namely,

$$|X| < 2,$$

which contradicts Claim 1. Theorem 8 is verified. □

Remark 1. We now claim that $\text{bind}(G) > 1$ in Theorem 8 is sharp. We construct a 3-connected graph $G = K_3 \vee ((3K_1) \cup K_2)$. Then $\text{bind}(G) = \frac{|N_G(V((3K_1)\cup K_2))|}{|(3K_1)\cup K_2|} = 1$. Select $e \in E(K_2)$. Let $G' = G - e$ and $X = V(K_3)$. Then $\varepsilon_1(X) = 2$. Hence, we have $i(G'-X) = 5 > 4 = 2|X| - \varepsilon_1(X)$. It follows from Theorem 3 that G' is not a $P_{\ge 2}$-factor covered graph, which implies that G is not a $P_{\ge 2}$-factor uniform graph.

Remark 2. Next, We show that 3-connected in Theorem 8 is sharp. We construct a graph $G = H \vee (K_1 \cup K_2)$ with $\text{bind}(G) = \frac{|N_G(V(K_1\cup K_2))|}{|V(K_1\cup K_2)|} = \frac{4}{3} > 1$, where $H = K_2$. Obviously, G is 2-connected. Select $e \in E(K_1 \cup K_2)$. Let $G' = G - e$ and $X = V(H)$. Then $\varepsilon_1(X) = 2$. Therefore, we derive $i(G'-X) = 3 > 2 = 2|X| - \varepsilon_1(X)$. By Theorem 3, G' is not a $P_{\ge 2}$-factor covered graph, and so G is not a $P_{\ge 2}$-factor uniform graph.

3. PROOF OF THEOREM 9

Proof of Theorem 9. We verify Theorem 9 by contradiction. Assume that $G' = G - e$ is not a $P_{\ge 3}$-factor covered graph for some $e = uv \in E(G)$. Then it follows from Theorem 4 that

$$\text{sun}(G'-X) \ge 2|X| - \varepsilon_2(X) + 1 \tag{5}$$

for some vertex subset X of G'.

Suppose that there exist a isolated vertices, b K_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \ge 6$, in $G'-X$. Then

$$\text{sun}(G'-X) = a + b + c. \tag{6}$$

We write $G_1 = (aK_1) \cup (bK_2) \cup H_1 \cup \cdots \cup H_c$.

Claim 2. $|X| \ge 3$.

Proof. If $|X| = 0$, then $\varepsilon_2(X) = 0$. According to (5),

$$\text{sun}(G') \ge 1. \tag{7}$$

Since G is 3-connected, G' is 2-connected. Hence, we obtain $\text{sun}(G') = 0$, which contradicts (7).

If $|X| = 1$, then $\varepsilon_2(X) \le 1$. In terms of (5),

$$\text{sun}(G'-X) \ge 2|X| - \varepsilon_2(X) + 1 \ge 2|X| = 2. \tag{8}$$

Since G is 3-connected, $G'-X$ is connected. Therefore, $\mathrm{sun}(G'-X)\le\omega(G'-X)=1$, which contradicts (8).

If $|X|=2$, then $\varepsilon_2(X)\le 2$. It follows from (5) that

$$\mathrm{sun}(G'-X)\ge 2|X|-\varepsilon_2(X)+1\ge 2|X|-1=3,$$

and so

$$\omega(G-X)\ge\omega(G-X-e)-1=\omega(G'-X)-1\ge\mathrm{sun}(G'-X)-1\ge 2. \tag{9}$$

On the other hand, since $|X|=2$ and G is 3-connected, we derive $\omega(G-X)=1$, which contradicts (9). This completes the proof of Claim 2. □

It follows from (5), (6), Claim 2 and $\varepsilon_2(X)\le 2$ that

$$a+b+c=\mathrm{sun}(G'-X)\ge 2|X|-\varepsilon_2(X)+1\ge 2\times 3-2+1=5. \tag{10}$$

Claim 3. $a\ge 1$.

Proof. Assume that $a=0$. Then by (10), we get $b+c\ge 5$, which implies that there exists one vertex x_1 with degree 1 in G_1. Let x_2 be the vertex adjacent to x_1 in G_1. Then

$$|N_G(V(G_1)\setminus\{x_2\})|\le|X|+2b+\sum_{i=1}^{c}|V(H_i)|-1.$$

Combining this with $\mathrm{bind}(G)>\frac{10}{7}$ and the definition of $\mathrm{bind}(G)$,

$$\frac{10}{7}<\mathrm{bind}(G)\le\frac{|N_G(V(G_1)\setminus\{x_2\})|}{|V(G_1)\setminus\{x_2\}|}\le\frac{|X|+2b+\sum_{i=1}^{c}|V(H_i)|-1}{2b+\sum_{i=1}^{c}|V(H_i)|-1},$$

which implies

$$7|X|>6b+3\sum_{i=1}^{c}|V(H_i)|-3. \tag{11}$$

In view of (10), (11), $a=0$, $|V(H_i)|\ge 6$ and $\varepsilon_2(X)\le 2$, we infer

$$\begin{aligned}7|X| &> 6b+3\sum_{i=1}^{c}|V(H_i)|-3\ge 6b+18c-3\ge 6(b+c)-3\\ &\ge 6(2|X|-\varepsilon_2(X)+1)-3\ge 6(2|X|-1)-3=12|X|-9,\end{aligned}$$

namely,

$$|X|<\frac{9}{5}<2,$$

which contradicts Claim 2. We completes the proof of Claim 3. □

In what follows, we consider three cases.

Case 1. $u,v\notin V(G_1)$. In this case, we admit $V(G_1)\ne\emptyset$ by (10) and $|N_G(V(G_1))|\le|X|+2b+\sum_{i=1}^{c}|V(H_i)|$. From $\mathrm{bind}(G)>\frac{10}{7}$ and the definition of $\mathrm{bind}(G)$,

$$\frac{10}{7}<\mathrm{bind}(G)\le\frac{|N_G(V(G_1))|}{|V(G_1)|}\le\frac{|X|+2b+\sum_{i=1}^{c}|V(H_i)|}{a+2b+\sum_{i=1}^{c}|V(H_i)|},$$

which implies

$$10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7|X|<0. \tag{12}$$

In terms of (10), (12), $|V(H_i)|\geq 6$, Claims 2–3 and $\varepsilon_2(X)\leq 2$,

$$\begin{aligned} 0 &> 10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7|X|\geq 10a+6b+18c-7|X|\geq 6(a+b+c)+4-7|X| \\ &\geq 6(2|X|-\varepsilon_2(X)+1)+4-7|X|\geq 6(2|X|-1)+4-7|X|=5|X|-2>0, \end{aligned}$$

which is a contradiction.

Case 2. $u,v\in V(G_1)$.

Subcase 2.1. $u\in V(aK_1)$ and $v\notin V(aK_1)$, or $u\notin V(aK_1)$ and $v\in V(aK_1)$. Without loss of generality, let $u\in V(aK_1)$ and $v\notin V(aK_1)$. Then we derive $V(G_1)\setminus\{v\}\neq\emptyset$ by (10) and $|N_G(V(G_1)\setminus\{v\})|\leq |X|+2b+\sum\limits_{i=1}^{c}|V(H_i)|$. It follows from $\operatorname{bind}(G)>\frac{10}{7}$ and the definition of $\operatorname{bind}(G)$ that

$$\frac{10}{7}<\operatorname{bind}(G)\leq\frac{|N_G(V(G_1)\setminus\{v\})|}{|V(G_1)\setminus\{v\}|}\leq\frac{|X|+2b+\sum\limits_{i=1}^{c}|V(H_i)|}{a+2b+\sum\limits_{i=1}^{c}|V(H_i)|-1},$$

which implies

$$10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7|X|-10<0. \tag{13}$$

In light of (10), (13), $|V(H_i)|\geq 6$, Claims 2–3 and $\varepsilon_2(X)\leq 2$, we deduce

$$\begin{aligned} 0 &> 10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7|X|-10\geq 10a+6b+18c-7|X|-10\geq 6(a+b+c)-7|X|-6 \\ &\geq 6(2|X|-\varepsilon_2(X)+1)-7|X|-6\geq 6(2|X|-1)-7|X|-6=5|X|-12>0, \end{aligned}$$

which is a contradiction.

Subcase 2.2. $u,v\in V(aK_1)$. In this case, $a\geq 2$. We have $V(G_1)\setminus\{v\}\neq\emptyset$ by (10) and $|N_G(V(G_1)\setminus\{v\})|\leq |X|+2b+\sum\limits_{i=1}^{c}|V(H_i)|+1$. According to $\operatorname{bind}(G)>\frac{10}{7}$ and the definition of $\operatorname{bind}(G)$, we yield

$$\frac{10}{7}<\operatorname{bind}(G)\leq\frac{|N_G(V(G_1)\setminus\{v\})|}{|V(G_1)\setminus\{v\}|}\leq\frac{|X|+2b+\sum\limits_{i=1}^{c}|V(H_i)|+1}{a+2b+\sum\limits_{i=1}^{c}|V(H_i)|-1},$$

that is,

$$10a+6b+3\sum_{i=1}^{c}|V(H_i)|-17<7|X|. \tag{14}$$

Using (10), (14), $a\geq 2$, $|V(H_i)|\geq 6$ and $\varepsilon_2(X)\leq 2$,

$$\begin{aligned} 7|X| &> 10a+6b+3\sum_{i=1}^{c}|V(H_i)|-17\geq 10a+6b+18c-17\geq 6(a+b+c)-9 \\ &\geq 6(2|X|-\varepsilon_2(X)+1)-9\geq 6(2|X|-1)-9, \end{aligned}$$

Combining this with Claim 2, we derive $3\leq |X|<3$, a contradiction.

Subcase 2.3. $u,v\notin V(aK_1)$. We admit $V(G_1)\neq\emptyset$ by (10) and $|N_G(V(G_1))|\leq|X|+2b+\sum_{i=1}^{c}|V(H_i)|$. By $\mathrm{bind}(G)>\frac{10}{7}$ and the definition of $\mathrm{bind}(G)$,

$$\frac{10}{7}<\mathrm{bind}(G)\leq\frac{|N_G(V(G_1))|}{|V(G_1)|}\leq\frac{|X|+2b+\sum_{i=1}^{c}|V(H_i)|}{a+2b+\sum_{i=1}^{c}|V(H_i)|},$$

namely,

$$10a+6b+3\sum_{i=1}^{c}|V(H_i)|<7|X|. \tag{15}$$

In light of (10), (15), $|V(H_i)|\geq 6$, Claim 3 and $\varepsilon_2(X)\leq 2$,

$$\begin{aligned}7|X| &> 10a+6b+3\sum_{i=1}^{c}|V(H_i)|\geq 10a+6b+18c>6(a+b+c)\\ &\geq 6(2|X|-\varepsilon_2(X)+1)\geq 6(2|X|-1),\end{aligned}$$

which implies $|X|<2$, which contradicts Claim 2.

Case 3. $u\in V(G_1)$ and $v\notin V(G_1)$, or $u\notin V(G_1)$ and $v\in V(G_1)$. Without loss of generality, let $u\in V(G_1)$ and $v\notin V(G_1)$. We know $V(G_1)\neq\emptyset$ by (10) and $|N_G(V(G_1))|\leq|X|+2b+\sum_{i=1}^{c}|V(H_i)|+1$. From $\mathrm{bind}(G)>\frac{10}{7}$ and the definition of $\mathrm{bind}(G)$, we infer

$$\frac{10}{7}<\mathrm{bind}(G)\leq\frac{|N_G(V(G_1))|}{|V(G_1)|}\leq\frac{|X|+2b+\sum_{i=1}^{c}|V(H_i)|+1}{a+2b+\sum_{i=1}^{c}|V(H_i)|},$$

which implies

$$10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7<7|X|. \tag{16}$$

It follows from (10), (16), $|V(H_i)|\geq 6$, Claim 2 and $\varepsilon_2(X)\leq 2$ that

$$\begin{aligned}7|X| &> 10a+6b+3\sum_{i=1}^{c}|V(H_i)|-7\geq 10a+6b+18c-7\geq 6(a+b+c)-7\\ &\geq 6(2|X|-\varepsilon_2(X)+1)-7\geq 6(2|X|-1)-7=12|X|-13\geq 7|X|+2,\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 9. □

Remark 3. Next, we claim that $\mathrm{bind}(G)>\frac{10}{7}$ in Theorem 9 cannot be replaced by $\mathrm{bind}(G)\geq\frac{10}{7}$. We construct a 3-connected graph $G=K_3\vee(4K_2)$. Then we have $\mathrm{bind}(G)=\frac{|N_G(V(4K_2)\setminus\{v\})|}{|V(4K_2)\setminus\{v\}|}=\frac{10}{7}$, where $v\in V(4K_2)$. Select $e\in E(4K_2)$. Let $G'=G-e$ and $X=V(K_3)$. Then $\varepsilon_2(X)=2$. Therefore, we admit $\mathrm{sun}(G'-X)=5>4=2|X|-\varepsilon_2(X)$. In view of Theorem 4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

Remark 4. In what follows, We show that 3-connected in Theorem 9 is sharp. We construct a graph $G=H\vee(2K_2)$ with $\mathrm{bind}(G)=\frac{|N_G(V(2K_2)\setminus\{v\})|}{|V(2K_2)\setminus\{v\}|}=\frac{5}{3}>\frac{10}{7}$, where $H=K_2$ and $v\in V(2K_2)$. Obviously, G is 2-connected. Select $e\in E(2K_2)$. Let $G'=G-e$ and $X=V(H)$. Then $\varepsilon_2(X)=2$. Thus, we obtain $\mathrm{sun}(G'-X)=3>2=2|X|-\varepsilon_2(X)$. In light of Theorem 4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $P_{\geq 3}$-factor uniform graph.

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