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DYNAMICAL ANALYSIS OF THE CONFORMABLE FRACTIONAL ORDER HOST-PARASITE MODEL

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Abstract. In this study, the differential equation system with a mathematical model of parasites is examined in cases where infection does not depend on transmission and defense, but on the level of infectivity and defense of the parasite and host. When discretization is applied to the differential equation, a two-dimensional discrete system is obtained in the range of $t \in [n, n+1]$ then the stability of the Neimark-Sacker bifurcation of the positive equilibrium point of this discrete system is investigated. Finally, MAPLE and MATLAB package program are used to show the accuracy of the results obtained.

***Key words**:* host-parasite model, Neimark-Sacker bifurcation, discrete dynamical system.

1. INTRODUCTION

Great attention has been given to difference equations used in discrete dynamical systems. There are many studies in mathematical biology which is one of the domains of applications of difference equations obtained from continuous models the differential equation system. In Mathematical Biology, the host-parasite model has been attracting a lot of attention in recent years [1-6]. There are two basic theories for modeling the dynamics of host-parasite systems. Gene for gene and matching allele model describe how host-parasite interaction takes place and how host genetics and gene follows the frequency domain parasites. Parasites with certain alleles may or may not infect host-carried genes [11]. This model, which is evolutionary, suggests that only some parasites can infect certain hosts, and the basis of contagion is controlled by only a few genes. Evolutionary ecology models therefore provide more transfer of certain parasite species against any host type than other species. In response to this, the success of infection of different parasite species is significantly linked to the defensive characteristics of the common host type [12]. Discrete-time equations are suitable for describing nonlinear dynamics and chaotic behaviors and are used to obtain dynamic results [13]. Fractional order differential equations have been gaining attention in the recent years. Mathematical models created with fractional order ordinary differential equations give better results than integer order Ordinary Differential Equations [14]. Fractional integral and derivative have many definitions, such as Riemann-Liouville fractional integration and Caputo fractional derivative. According to these definitions, the Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ is defined as [32–33]

$$ {}_{0}^{RL}D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (x-t)^{-\alpha} f(t)\,\mathrm{d}t $$

and the Caputo fractional derivative is derived in order to obtain possible solutions

$$ {}_{0}^{C}D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\,\mathrm{d}t $$

In 2014, Khalil et al. introduced a new fractional derivative called conformable fractional derivative [16]. In 2015, it was named as the left and right conformable fractional derivatives [17]. According to this definition, the left conformable fractional derivative of order $\alpha \in (0,1]$ with $f:[\alpha,\infty) \to IR$ is given as

$$\left(T_{\alpha}^{a}\right)(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon} \tag{1}$$

and right conformable fractional derivative is defined as

$$\left({}_{\alpha}^{b}T\right)(f)(t)=-\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{2}$$

To obtain the discretized version of the fractional ordered Lotka-Volterra host-parasite model [18]

$$\begin{cases} D^{\alpha}x(t)=(a-b)x(t)-(q+\beta)x(t)y(t)+\gamma y(t)-qx^{2}(t) \\ D^{\alpha}y(t)=\beta x(t)y(t)-(c+b+\gamma)y(t) \end{cases} \tag{3}$$

the greatest integer function is added to equation (3), which has the Caputo fractional order derivative

$$\begin{cases} D^{\alpha}x(t)=(a-b)\,x\left(\left[\frac{t}{h}\right]h\right)-(q+\beta)\,x\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)+\gamma\, y\left(\left[\frac{t}{h}\right]h\right)-qx^{2}\left(\left[\frac{t}{h}\right]h\right) \\ D^{\alpha}y(t)=\beta\, x\left(\left[\frac{t}{h}\right]h\right)y\left(\left[\frac{t}{h}\right]h\right)-(c+b+\gamma)\,y\left(\left[\frac{t}{h}\right]h\right). \end{cases} \tag{4}$$

Here a system of difference equations is obtained from the solution for $t>0$ in the sub-interval $t\in\left[nh,(n+1)h\right)$.

$$\begin{cases} x_{n+1}=x_{n}+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left((a-b)x_{n}-(q+\beta)x_{n}y_{n}+\gamma y_{n}-qx_{n}^{2}\right) \\ y_{n+1}=y_{n}+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left(\beta x_{n}y_{n}-(c+b+\gamma)y_{n}\right). \end{cases} \tag{5}$$

In addition, it is biologically important to ensure the continuity of some terms by adding piecewise constant arguments to differential equations. The aim of this study is to examine the dynamic behavior of conformable fractional order host parasites with the greatest integer function. The model (3) with conformable fractional order is therefore considered as follows:

$$\begin{cases} T_{\alpha}x(t)=(a-b)x(t)-(q+\beta)x(t)y\left(\left[\frac{t}{h}\right]h\right)+\gamma y\left(\left[\frac{t}{h}\right]h\right)-qx^{2}(t) \\ T_{\alpha}y(t)=\beta\, x\left(\left[\frac{t}{h}\right]h\right)y(t)-(c+b+\gamma)y(t). \end{cases} \tag{6}$$

Here $t\in\left[0,\infty\right)$ denotes the integer part and $h>0$ is a discretization parameter. This model combines the properties of both continuous and discrete-time equations for parasites [19–21]. In Model (6), uninfected hosts reproduce at a rate of a, which is reduced by a factor q modeling traffic due to crowding, and have a natural mortality rate of b. The transmission coefficient of infection is β, and c is an additional death rate due to infection of the infected host. Also, γ with improvement potential is included, but for analytical results presented for mathematical reasons, $\gamma=0$ is used [24]. The paper is organized as follows: the process of discretization of the model is performed and the system of difference equations is obtained in section 2. The stability of the equilibrium points of the model is given in section 3. The Neimark-Sacker bifurcation around the positive equilibrium point of the model was shown in section 4 and also the numerical simulations of the stability of the Neimark-sacker bifurcation and theoretical results were obtained using the central manifold bifurcation theory.

2. THE DISCRETIZATION PROCESS

In this section,(6) we will discretize equation [26]. Let $t\in\left[nh,(n+1)h\right)$, $n=0,1,2,...$ and using the left conformable fractional derivative, the first equation in system (6) is obtained as:

$$(t-nh)^{1-\alpha}\frac{\mathrm{d}x(t)}{\mathrm{d}t}+\left(b-a+(q+\beta)y(nh)\right)x(t)=-qx^2(t)+\gamma y(nh)\ \ (\gamma=0) \tag{7}$$

Bernoulli differential equation is obtained

$$-\frac{x'(t)}{x^2(t)}+\frac{\left(a-b-(q+\beta)y(nh)\right)}{(t-nh)^{1-\alpha}x(t)}=\frac{q}{(t-nh)^{1-\alpha}}. \tag{8}$$

Multiplying both sides of equation (8) by $e^{\left(a-b-(q+\beta)y(nh)\right)\frac{(t-nh)^{1-\alpha}}{\alpha}}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x(t)}e^{\left(a-b-(q+\beta)y(nh)\right)\frac{(t-nh)^{1-\alpha}}{\alpha}}\right)=\frac{q}{(t-nh)^{1-\alpha}}e^{\left(a-b-(q+\beta)y(nh)\right)\frac{(t-nh)^{1-\alpha}}{\alpha}},\quad t\in\left[nh,(n+1)h\right). \tag{9}$$

The integral of both sides of the equation(9) in the interval $\left[nh,t\right)$ is taken with respect to t,

$$\frac{1}{x(t)}e^{\left(a-b-(q+\beta)y(nh)\right)\frac{(t-nh)^{1-\alpha}}{\alpha}}-\frac{1}{x(nh)}=\frac{q}{\left(a-b-(q+\beta)y(nh)\right)}\left[e^{\left(a-b-(q+\beta)y(nh)\right)\frac{(t-nh)^{1-\alpha}}{\alpha}}-1\right]. \tag{10}$$

While $t\to(n+1)h$ in (10), replacing $x(nh)$ with $x(n)$:

$$x_{n+1}=\frac{x(n)\left(a-b-(q+\beta)y(n)\right)}{\left(a-b-(q+\beta)y(n)-qx(n)\right)e^{-\left(a-b-(q+\beta)y(nh)\right)\frac{h^\alpha}{\alpha}}+qx(n)} \tag{11}$$

Similarly, we obtain the second equation of system (6):

$$(t-nh)^{1-\alpha}\frac{\mathrm{d}y(t)}{\mathrm{d}t}=\beta x(nh)y(t)-(c+b)y(t) \tag{12}$$

from here

$$\frac{\mathrm{d}y(t)}{y(t)}=\frac{\beta x(nh)-(c+b)}{(t-nh)^{1-\alpha}}\mathrm{d}t, \tag{13}$$

when the integral of both sides of (13) in the interval $\left[nh,t\right)$ is taken with respect to t,

$$\ln y(t)-\ln y(nh)=\left(\beta x(nh)-(c+b)\right)\frac{(t-nh)^{1-\alpha}}{\alpha},\quad t\in\left[nh,(n+1)h\right), \tag{14}$$

if we write instead of the equation (14) for $t\to(n+1)h$,

$$y_{n+1}=y(n)e^{\left(\beta x(n)-(c+b)\right)\frac{h^\alpha}{\alpha}}. \tag{15}$$

As a result, a two-dimensional discrete system was obtained by discretizing the (6) model

$$\begin{cases} x_{n+1}=\dfrac{x(n)\left(a-b-(q+\beta)y(n)\right)}{\left(a-b-(q+\beta)y(n)-qx(n)\right)e^{-\left(a-b-(q+\beta)y(nh)\right)\frac{h^\alpha}{\alpha}}+qx(n)} \\ y_{n+1}=y(n)e^{\left(\beta x(n)-(c+b)\right)\frac{h^\alpha}{\alpha}} \end{cases} \tag{16}$$

3. STABILITY OF EQUILIBRIUM POINTS

The system (16) has three equilibrium points $E_0=(0,0)$, $E_1=\left(\frac{a-b}{q},0\right)$ and $E_2=\left(\frac{c+b}{\beta},\frac{(a-b)\beta-(c+b)q}{(q+\beta)\beta}\right)$. Of these, E_0 is the trivial state, E_1 is the axial state, and E_2 is the coexistence equilibrium point for $\beta>\frac{q(c+b)}{(a-b)}$.

THEOREM 1. *The equilibrium point* $E_0=(0,0)$ *is a saddle point*.

Proof. Jacobian matrix of system (16) around (0,0) is

$$J(E_0)=\begin{pmatrix} e^{(a-b)\frac{h^\alpha}{\alpha}} & 0 \\ 0 & e^{-(c+b)\frac{h^\alpha}{\alpha}} \end{pmatrix}$$

and its eigenvalues are $\lambda_1=e^{(a-b)\frac{h^\alpha}{\alpha}}$ and $\lambda_2=e^{-(c+b)\frac{h^\alpha}{\alpha}}$. $|\lambda_1|>1$ and $|\lambda_2|<1$ are obtained. So the equilibrium point E_0 is a saddle point.

THEOREM 2. *The following results are given for the equilibrium point* $E_1=\left(\frac{a-b}{q},0\right)$*:*

i. *if* $\beta>\frac{q(c+b)}{(a-b)}$, E_1 *is a saddle point.*

ii. *if* $\beta<\frac{q(c+b)}{(a-b)}$, E_1 *is a sink.*

iii. *if* $\beta=\frac{q(c+b)}{(a-b)}$*, then* E_1 *is not hyperbolic.*

Proof. Jacobian matrix of system (16) around $\left(\frac{a-b}{q},0\right)$ is

$$J(E_1)=\begin{pmatrix} e^{-(a-b)\frac{h^\alpha}{\alpha}} & \dfrac{(q+\beta)\left(1+e^{-(a-b)\frac{h^\alpha}{\alpha}}\right)}{q} \\ 0 & e^{\left(\frac{\beta(a-b)}{q}-(c+b)\right)\frac{h^\alpha}{\alpha}} \end{pmatrix}$$

Eigenvalues are $\lambda_1=e^{-(a-b)\frac{h^\alpha}{\alpha}}$, $|\lambda_1|<1$ and $\lambda_2=e^{\left(\frac{\beta(a-b)}{q}-(c+b)\right)\frac{h^\alpha}{\alpha}}$. Here the following results are obtained:

i. if $\beta>\frac{q(c+b)}{(a-b)}$, $|\lambda_2|>1$, the equilibrium point is a saddle point;

ii. if $\beta<\frac{q(c+b)}{(a-b)}$, $|\lambda_2|<1$, the equilibrium point is a sink;

iii. if $\beta=\frac{q(c+b)}{(a-b)}$, $|\lambda_2|=1$, the equilibrium point is not hyperbolic.

THEOREM 3. *The positive equilibrium point of the system (16), which is* $E_2=\left(\frac{c+b}{\beta},\frac{(a-b)\beta-(c+b)q}{(q+\beta)\beta}\right)$, *is local asymptotic stable if and only if* $\beta<\left(\frac{q}{(a-b)}\right)\left(\left(\alpha/h^{\alpha}\right)+c+b\right)$.

Proof. Jacobian matrix of equilibrium point E_2 is

$$J\left(E_2\right)=\begin{pmatrix} e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^{\alpha}}{\alpha}} & \dfrac{(q+\beta)\left(-1+e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^{\alpha}}{\alpha}}\right)}{q} \\ \left(\dfrac{\beta(a-b)-(c+b)q}{(q+\beta)}\right)\dfrac{h^{\alpha}}{\alpha} & 1 \end{pmatrix}$$

and the characteristic equation is,

$$\lambda^2+p_1\lambda+p_0=0. \tag{17}$$

Here,

$$p_0=\frac{e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}\left[q\alpha-\left(-1+e^{\frac{q(c+b)h^{\alpha}}{\beta\alpha}}\right)\left((c+b)q-\beta(a-b)\right)h^{\alpha}\right]}{q\alpha}$$

and

$$p_1=-1-e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}.$$

Local asymptotic stability of system (16) using the conditions of Jury criterion [29] is given as:

$$1+p_1+p_0>0$$
$$1-p_1+p_0>0$$
$$1-p_0>0$$

$\beta>\frac{q(c+b)}{(a-b)}$, taking into account condition (i)

$$\frac{\left(1-e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}\right)\left(\beta(a-b)-(c+b)q\right)h^{\alpha}}{q\alpha}>0$$

and for the condition (ii)

$$\frac{2q\alpha\left(1+e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}\right)+\left(1-e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}\right)\left(\beta(a-b)-(c+b)q\right)h^{\alpha}}{q\alpha}>0$$

is obtained. Condition (iii) gives

$$\frac{\left(1-e^{\frac{-q(c+b)h^{\alpha}}{\beta\alpha}}\right)\left(q\alpha+\left(\beta(a-b)-(c+b)q\right)h^{\alpha}\right)}{q\alpha}>0.$$

From here, the proof is complete. Let $a=3$, $b=0.5$, $c=1.5$, $\beta=0.5$, $q=0.5$, and α and h parameters can vary. Figure 1 and Figure 2 below, respectively, show the stability dynamic behaviors at equilibrium point E_2 for the fractional order α and the h parameter in equation system (16) with increasing discretization.

4. NEIMARK-SACKER BIFURCATION ANALYSIS

Neimark-Sacker bifurcation analysis is a very important issue in discrete-time systems. Here, the positive equilibrium point $\left(x^*, y^*\right)$ of the system (16) yields the Neimark-Sacker bifurcation and as it is chosen as the bifurcation parameter [30].

THEOREM 4. *If* $\beta<\left(\dfrac{q}{(a-b)}\right)\left(\left(\alpha/h^{\alpha}\right)+c+b\right)$ *then Neimark-Sacker bifurcation occurs at the equilibrium point* $\left(\dfrac{c+b}{\beta}, \dfrac{(a-b)\beta-(c+b)q}{(q+\beta)\beta}\right)$ *of system (16). Also, if* $q<0$ *the bifurcation of system (16) is a supercritical Neimark-Sacker bifurcation and* $q>0$ *is subcritical Neimark-Sacker bifurcation.*

Proof. The characteristic equation with the linearized system (16) at the equilibrium point $\left(x^*(\beta), y^*(\beta)\right)$ is given as follows:

$$\lambda^2+p(\beta)\lambda+q(\beta)=0 . \tag{18}$$

The eigenvalues of equation (18) are given as:

$$\lambda_{1,2}(\beta)=\frac{-p(\beta)\pm\sqrt{p(\beta)^2-4q(\beta)}}{2} .$$

Here

$$p(\beta)=-\frac{1}{\left(\left(a-b-(q+\beta)y-qx\right)e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}+qx\right)^2}\times\Bigg[\left(a-b-(q+\beta)y\right)^2e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}+q^2x^2e^{(\beta x-(c+b))\frac{h^{\alpha}}{\alpha}}+$$

$$+\left(a-b-(q+\beta)y-qx\right)^2e^{-2\left((a-b-(q+\beta)y)+(\beta x-(c+b))\right)\frac{h^{\alpha}}{\alpha}}+2\left(a-b-(q+\beta)y-qx\right)qxe^{\left(-(a-b-(q+\beta)y)+(\beta x-(c+b))\right)\frac{h^{\alpha}}{\alpha}}\Bigg]$$

and

$$q(\beta)=\frac{e^{(\beta x-(c+b))\frac{h^{\alpha}}{\alpha}}}{\left(\alpha\left(\left(a-b-(q+\beta)y-qx\right)e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}+qx\right)\right)^2}\times\alpha^2\left(a-b-(q+\beta)y\right)^2e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}-$$

$$-\alpha qx^2yh^{\alpha}\beta\left(q+\beta\right)e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}+\alpha qx^2y\beta h^{\alpha}\left(q+\beta\right)-$$

$$-xy\beta h^{2\alpha}\left(q+\beta\right)e^{-(a-b-(q+\beta)y)\frac{h^{\alpha}}{\alpha}}\left(a-b-(q+\beta)y-qx\right)\left(a-b-(q+\beta)y\right)$$

is found, where,

$$\beta^* = \left(\frac{q}{(a-b)}\right)\left(\left(\frac{\alpha}{h^\alpha}\right)+c+b\right) \tag{19}$$

and

$$\lambda_{1,2}(\beta) = \frac{1+e^{-\frac{(c+b)(a-b)h^\alpha}{\left(c+b+\frac{\alpha}{h^\alpha}\right)\alpha}}}{2} \pm i\frac{\sqrt{\left(e^{-\frac{(c+b)(a-b)h^\alpha}{\left(c+b+\frac{\alpha}{h^\alpha}\right)\alpha}}-1\right)\left(e^{-\frac{(c+b)(a-b)h^\alpha}{\left(c+b+\frac{\alpha}{h^\alpha}\right)\alpha}}+3\right)}}{2}$$

are obtained. From here, it is seen that $\left|\lambda_{1,2}(\beta)\right|=1$ and from the Transversality condition [31]

$$\left.\frac{\mathrm{d}\left|\lambda_{1,2}(\beta)\right|}{\mathrm{d}\beta}\right|_{\beta=\beta^*} = \frac{(a-b)^2 h^\alpha e^{-\frac{(a-b)h^\alpha}{\alpha}}}{q(c+b)\alpha}\left[1-(c+b)+e^{-\frac{(a-b)h^\alpha}{\alpha}}\left(2+c+b\right)\right] \neq 0 .$$

Also, from the Nonresonance condition $p(\beta^*) \neq 0,1$, $\lambda_{1,2}^n(\beta^*) \neq 1, n=1,2,3,4$ is found. From here the Neimark-Sacker bifurcation occurs under the conditions obtained from the system (16).

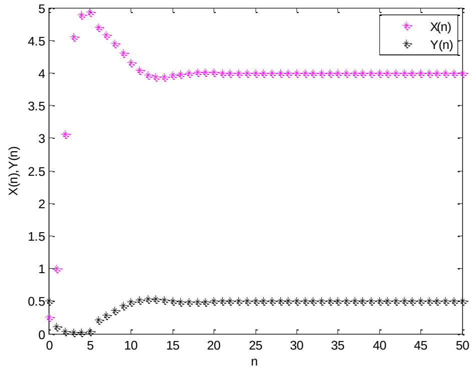

Fig. 1 – Behavior of the system (16) for initial conditions $\left(x(0), y(0)\right) = (0.25, 0.5)$ with parameters $a = 3$, $b = 0.5$, $c = 1.5$, $\beta = 0.5$, $q = 0.5$, $\alpha = 0.95$, $h = 0.75$.

Let $\tilde{x} = x - x^*$ and $\tilde{y} = y - y^*$ transform the fixed point $\left(x^*, y^*\right)$ of the system (16)

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \to \begin{pmatrix} e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^\alpha}{\alpha}} & \dfrac{(q+\beta)\left(-1+e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^\alpha}{\alpha}}\right)}{q} \\ \dfrac{a-b}{a+c+\dfrac{\alpha}{h^\alpha}} & 1 \end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} + \begin{pmatrix} f_1(\tilde{x},\tilde{y})\\ f_2(\tilde{x},\tilde{y})\end{pmatrix} \tag{20}$$

Here $A=\left(c+b+\dfrac{\alpha}{h^\alpha}\right)$, $B=\left(a+c+\dfrac{\alpha}{h^\alpha}\right)$, $uv=(a-b)(c+b)$

$$F_1(\tilde{x},\tilde{y})=\left(\frac{-qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)A}{uv}\right)\tilde{x}^2+\left(\frac{qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}B}{\alpha u^2v}\left[h^\alpha uv-2\alpha\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)A\right]\right)\tilde{x}\tilde{y}+$$

$$+\left(\frac{-qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{\alpha u^2v}\left[BA\alpha\left[1+qBe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+qB^2uvh^\alpha\right]\right]\right)\tilde{y}^2+\left(\frac{q^2\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)^2A^2}{u^2v^2}\right)\tilde{x}^3+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{2\alpha u^3v^2}\left(\begin{array}{c}6q^2\alpha A^2B^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(4q^2h^\alpha AB^2-10q^2\alpha A^2B^2\right)+\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(4q^2\alpha A^2B^2-2q^2h^\alpha AB^2\right)\end{array}\right)\right)\tilde{x}^2\tilde{y}+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{2\alpha^2u^4v^2}\left(\begin{array}{c}6q^2\alpha^2A^2B^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(8q^2\alpha h^\alpha AB^2uv-8q^2\alpha^2A^2B^2\right)+\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(2q^2\alpha^2A^2B^2+q^2h^{2\alpha}-2q^2h^\alpha B^2u^2v^2-4q^2\alpha h^\alpha AB^2uv\right)\end{array}\right)\right)\tilde{x}\tilde{y}^2+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{6\alpha^2u^5v^2}\left(\begin{array}{c}6q^2A^2B^3\alpha^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(12q^2\alpha h^\alpha uvAB^3-6q^2\alpha^2A^2B^3\right)+\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(3q^2B^3h^{2\alpha}uv^2-6q^2h^\alpha AB^3\alpha uv\right)\end{array}\right)\right)\tilde{y}^3+O\left(\|X\|^4\right)$$

$$F_2(\tilde{x},\tilde{y})=\left(\frac{qAh^\alpha}{2\alpha B}\right)\tilde{x}^2+\left(\frac{qA}{\alpha u}\right)\tilde{x}\tilde{y}+\left(\frac{q^2A^2h^{2\alpha}}{6\alpha^2uA}\right)\tilde{x}^3+\left(\frac{q^2A^2h^{2\alpha}}{2\alpha^2u^2}\right)\tilde{x}^2\tilde{y}+O\left(\|X\|^4\right) \tag{21}$$

and for $X=(\tilde{x},\tilde{y})^T$

$$B_1(x,y)=\sum_{i,j=1}^{2}\frac{\partial F_1(\xi,\delta)}{\partial\xi_i\partial\xi_j}\bigg|_{\xi=0}x_iy_j=\left(\frac{-2qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)A}{uv}\right)x_1y_1$$

$$+\left(\frac{qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}B}{\alpha u^2v}\left[h^\alpha uv-2\alpha\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)A\right]\right)(x_1y_2+x_2y_1)+$$

$$+\left(\frac{qe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{\alpha u^2v}\left[BA\alpha\left[1+qBe^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+qB^2uvh^\alpha\right]\right]\right)x_2y_2$$

$$B_2(x,y)=\sum_{i,j=1}^{2}\frac{\partial F_2(\xi,\delta)}{\partial\xi_i\partial\xi_j}\bigg|_{\xi=0}x_iy_j=\left(\frac{qAh^\alpha}{\alpha B}\right)x_1y_1+\left(\frac{qA}{\alpha u}\right)(x_1y_2+x_2y_1)$$

$$C_1(x,y,t)=\sum_{i,j,k=1}^{2}\frac{\partial F_1(\xi,\delta)}{\partial\xi_i\partial\xi_j}\bigg|_{\xi=0}x_iy_jt_k=\left(\frac{6q^2\left(-e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}+1\right)^2A^2}{u^2v^2}\right)x_1y_1t_1+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{\alpha^2u^4v^2}\left(\begin{array}{l}6q^2\alpha^2A^2B^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(8q^2\alpha h^\alpha AB^2uv-8q^2\alpha^2A^2B^2\right)\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(2q^2\alpha^2A^2B^2+q^2h^{2\alpha}-2q^2h^\alpha B^2u^2v^2-4q^2\alpha h^\alpha AB^2uv\right)\end{array}\right)\right)\left(x_1y_1t_2+x_1y_2t_1+x_2y_1t_1\right)+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{\alpha^2u^4v^2}\left(\begin{array}{l}6q^2\alpha^2A^2B^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(8q^2\alpha h^\alpha AB^2uv-8q^2\alpha^2A^2B^2\right)\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(2q^2\alpha^2A^2B^2+q^2h^{2\alpha}-2q^2h^\alpha B^2u^2v^2-4q^2\alpha h^\alpha AB^2uv\right)\end{array}\right)\right)\left(x_1y_2t_2+x_2y_1t_2+x_2y_2t_1\right)+$$

$$+\left(\frac{e^{-3\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}}{\alpha^2u^5v^2}\left(\begin{array}{c}6q^2A^2B^3\alpha^2+e^{-\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(12q^2\alpha h^\alpha uvAB^3-6q^2\alpha^2A^2B^3\right)\\+e^{-2\left(\frac{uv}{A}\right)\frac{h^\alpha}{\alpha}}\left(3q^2B^3h^{2\alpha}uv^2-6q^2h^\alpha AB^3\alpha uv\right)\end{array}\right)\right)x_2y_2t_2$$

$$C_2(x,y,t)=\sum_{i,j,k=1}^{2}\frac{\partial F_2(\xi,\delta)}{\partial\xi_i\partial\xi_j}\bigg|_{\xi=0}x_iy_jt_k=\left(\frac{q^2A^2h^{2\alpha}}{\alpha^2uA}\right)x_1y_1t_1+\left(\frac{q^2A^2h^{2\alpha}}{\alpha^2u^2}\right)\left(x_1y_1t_2+x_1y_2t_1+x_2y_1t_1\right).$$

Hence, where $B(x,y)=\begin{pmatrix}B_1(x,y)\\B_2(x,y)\end{pmatrix}$ and $C(x,y)=\begin{pmatrix}C_1(x,y)\\C_2(x,y)\end{pmatrix}$, $x,y\in\mathbb{R}^2$, are symmetrical to linear vector functions. The simple eigenvalue of $J(E_2)$ is $\lambda_1=1$, and the corresponding eigenspace E^c is one-dimensional. $J(E_2)q=-q$ is produced by an eigenvector $q\in\mathbb{R}^2$. Let $p\in\mathbb{R}^2$ be the combined eigenvector, i.e. $J(E_2)p=-p$. It is obtained by direct calculation as

$$q\sim\left(-2,\left(\frac{\beta(a-b)-(c+b)q}{(q+\beta)}\right)\frac{h^\alpha}{\alpha}\right)^T$$

$$p\sim\left(-2,\frac{(q+\beta)\left(-1+e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^\alpha}{\alpha}}\right)}{q}\right)^T$$

Normalizing p according to q gives

$$p=\gamma_1\left(-2,\frac{(q+\beta)\left(-1+e^{-q\left(\frac{c+b}{\beta}\right)\frac{h^\alpha}{\alpha}}\right)}{q}\right)^T$$

and from here

$$\gamma_1=\frac{1}{4-\frac{(e-1)\left((a-b)\beta-(c+b)q\right)h^\alpha}{q\alpha}}$$

it is seen $p,q=1$, where .,. is $\mathbb{R}^2: p,q=p_1q_1+p_2q_2$, which stands for standard scalar product.

The critical normal form coefficient, which determines the direction of flip bifurcation, is obtained by the following formula [9]

$$D=\frac{1}{6}p,C(q,q,q)-\frac{1}{2}p,B\left(q\left(A-I\right)^{-1}B(q,q)\right).$$

Numerical Simulations. In this section, the above theoretical results are verified. Suppose that the parameters $a=3,\ b=0.5,\ c=1.5,\ q=0.5,\ \gamma=0,\ \alpha=0.95,\ h=0.75$ are constant. The Neimark-Sacker bifurcation point is $\beta^*=1.6497$. In this case, $J\left(E_2\right)$ at the equilibrium point is

$$J\left(E_2\right)=\begin{pmatrix}0.2915 & -1.6291\\ 0.4349 & 1.0000\end{pmatrix}$$

and the eigenvalues are

$$\left|\lambda_{1,2}(\beta^*)\right|=\left|0.6457\pm 0.7635i\right|=1.$$

Also,

$$\left.\frac{\mathrm{d}\left|\lambda_{1,2}(\beta)\right|}{\mathrm{d}\beta}\right|_{\beta=\beta^*}=-0.3108\neq 0,\quad \lambda_{1,2}^{n}(\beta^*)\neq 1,\quad n=1,2,3,4 \text{ is obtained.}$$

Now $F_1\left(\tilde{x},\tilde{y}\right)$ and $F_2\left(\tilde{x},\tilde{y}\right)$ in equation (20) are

$$F_1\left(\tilde{x},\tilde{y}\right)=-0.0671\tilde{x}^2-0.0401\tilde{x}\tilde{y}+1.1717\tilde{y}^2+0.1760\tilde{x}^3+0.0634\tilde{x}^2\tilde{y}-0.0153\tilde{x}\tilde{y}^2+0.0428\tilde{y}^3+O\left(\|X\|^4\right)$$

$$F_2\left(\tilde{x},\tilde{y}\right)=1.0011\tilde{x}^2+0.6839\tilde{x}\tilde{y}+0.0347\tilde{x}^3+0.1354\tilde{x}^2\tilde{y}+O\left(\|X\|^4\right).$$

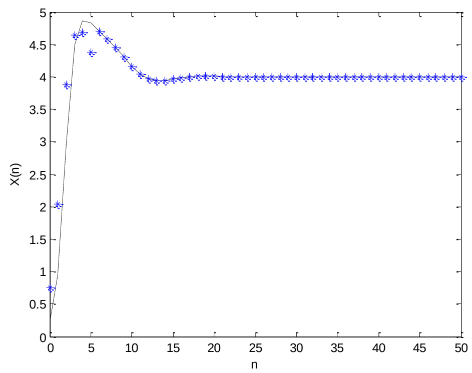

Fig. 2 – Two trajectories for x coordinates are drawn according to the number of iterations, for straight $(-),\left(x_0,y_0\right)=\left(0.25,0.5\right)$; for dashed line $(-,-)\left(x_0+0.5,y_0\right)$.

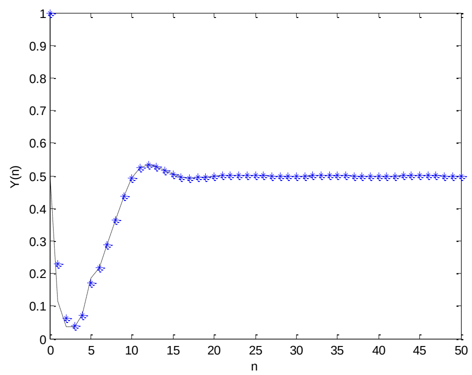

Fig. 3 – Two trajectories for the y coordinates are plotted according to the number of iterations, for straight $(-),\left(x_0,y_0\right)=\left(0.25,0.5\right)$; for dashed line $(-,-)\left(x_0+0.5,y_0\right)$.

5. CONCLUSION

In this study, the host-parasite model with conformable fractional order and greatest integer function was discussed and the stability of equilibrium points was investigated by discretizing this model. The system (16) of difference equations was obtained from the solution of the model in the sub-interval $\left[nh,(n+1)h\right)$. It was found that $\beta<\left(\frac{q}{(a-b)}\right)\left(\left(\alpha/h^{\alpha}\right)+c+b\right)$ ensures that the E_2 equilibrium point is locally asymptotically stable. For $\alpha=0.95,\ h=0.75$, the local asymptotic stability range from this condition was determined as $\beta<0.6497$. The transcript, flip, and Neimark-Sacker bifurcation point at the positive equilibrium point of the discrete systems is β. From here our discrete system (16) forms the Neimark-Sacker bifurcation. Stability analysis was performed by applying the Jury criterion to the system (16). From Figure 1, when β is selected to remain within this interval $(\beta=0.5)$, it can be seen that the positive equilibrium point of the system is locally asymptotic stable. By Neimark-Sacker bifurcation analysis, it was shown that the critical bifurcation value is $\beta^{*}=\left(\frac{q}{(a-b)}\right)\left(\left(\alpha/h^{\alpha}\right)+c+b\right)$. In Figures 2–3, the parameter values $a=3,\ b=0.5,\ c=1.5,\ q=0.5,\ \gamma=0,\ \alpha=0.95,\ h=0.75$ were calculated as $\beta^{*}=0.6497$. The effect of the fractional order derivative parameter (α) and the discrete parameter (h) on the dynamic structure of the system can be seen.

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