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INDEPENDENCE NUMBER, MINIMUM DEGREE AND PATH-FACTORS IN GRAPHS

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Abstract. A path-factor in a graph G is a spanning subgraph of G whose components are paths. Let d and k be two nonnegative integers with $d \geq 2$. A $P_{\geq d}$-factor of a graph G is its spanning subgraph each of whose components is a path of order at least d. A graph G is called a $P_{\geq d}$-factor deleted graph if for any edge e of G, G admits a $P_{\geq d}$-factor excluding e. A graph G is called a $(P_{\geq d},k)$-factor critical deleted graph if for any $Q \subseteq V(G)$ with $|Q| = k$, the graph $G-Q$ is a $P_{\geq d}$-factor deleted graph. In other words, a graph G is called a $(P_{\geq d},k)$-factor critical deleted graph if for any $Q \subseteq V(G)$ with $|Q| = k$ and any $e \in E(G-Q)$, the graph $G-Q-e$ admits a $P_{\geq d}$-factor. In this paper, we prove that a $(k+2)$-connected graph G is a $(P_{\geq 3},k)$-factor critical deleted graph if G satisfies

$$\delta(G) > \frac{\alpha(G)+2k+2}{2}.$$

Furthermore, we show that the main result in this paper is best possible in some sense.

Key words: graph, minimum degree, independence number, $P_{\geq 3}$-factor, $P_{\geq 3}$-factor deleted graph, $(P_{\geq 3},k)$-factor critical deleted graph.

1. INTRODUCTION

The ruggedness and vulnerability of the network are the core issues of network security research, and it is also one of the key topics that researchers must consider during the network designing phase. Henceforth, we apply the term "graph" instead of "network". Vertices of the graph corresponds to nodes of the network and edges of the graph stand for links between the nodes of the network. In data transmission networks, the data transmission between two sites goes through a path between two corresponding vertices. Therefore, the availability of data transmission in the network is equivalent to the existence of path factor in the corresponding graph which is generated by the network. The existence of a path-factor critical deleted graph also plays an important role in transmitting data of networks. If a channel and some nodes of the network are damaged in the process of the data transmission at the moment, the possibility of data transmission between nodes is characterized by whether the corresponding graph of the network is a path-factor critical deleted graph or not. Hence, researches on the existence of path-factors and path-factor critical deleted graphs under specific network structures can help scientists design and construct networks with high data transmission rates. The minimum degree and independence number are often applied to measure the ruggedness and vulnerability of a network. Furthermore, we find that there is strong essential connection between the above two graphic parameters and the existence of path factors in graphs (or path-factor critical deleted graphs). Hence, investigations on minimum degree and independence number, which plays an irreplaceable role in the vulnerability of the network and the feasibility of data transmission, can yield theoretical guidance to meet data transmission and network security requirements.

In this work, the graphs discussed are finite, undirected and simple. We denote a graph by $G = (V(G), E(G))$, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let $d_G(x)$ denote the degree of a vertex x in G, and write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. We denote by $\alpha(G)$, $i(G)$ and $\omega(G)$ the independence number, the number of isolated vertices and the number of connected components in G, respectively. Let xy denote an edge joining vertices x and y. For a vertex subset X of G, we use $G[X]$ to denote the subgraph of G induced by X, and $G - X$ to denote the subgraph of G induced by $V(G) \setminus X$. A vertex subset X of G is called independent if $G[X]$ has no edges. For an edge subset E' of G, let $G - E'$ denote the subgraph derived from G by removing edges of E'. For two given graphs G_1 and G_2, let $G_1 \cup G_2$ denote the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and $G_1 \vee G_2$ denote the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{e = xy : x \in V(G_1), y \in V(G_2)\}$. The complete graph and the path of order n are denoted by K_n and P_n, respectively.

A path-factor in a graph G is a spanning subgraph of G whose components are paths. Let d be an integer with $d \geq 2$. A $P_{\geq d}$-factor of a graph G is its spanning subgraph each of whose components is a path of order at least d. A graph G is called a $P_{\geq d}$-factor deleted graph if for any edge e of G, G admits a $P_{\geq d}$-factor excluding e.

Egawa, Furuya and Ozeki [1], Johansson [2], Kelmans [3] studied the existence of path-factors in graphs. Bazgan et al. [4] verified that a 1-tough graph G of order at least 3 admits a $P_{\geq 3}$-factor. Kano, Lu and Yu [5] claimed that a graph G has a $P_{\geq 3}$-factor if G satisfies $i(G - X) \leq \frac{2}{3}|X|$ for each $X \subseteq V(G)$. Zhou [6], Zhou, Wu and Xu [7], Zhou, Sun and Yang [8], Zhou, Sun and Bian [9], Wang and Zhang [10], Gao, Chen and Wang [11], Gao and Wang [12], Hua [13] derived several results on $P_{\geq 3}$-factors of graphs with given properties. Kano, Lee and Suzuki [14] proved that every connected cubic bipartite graph of order at least 8 has a $P_{\geq 8}$-factor. Zhou, Sun and Liu [15] showed toughness and isolated toughness conditions for $P_{\geq 3}$-factor deleted graphs. Zhou [16] presented a binding number condition for graphs to be $P_{\geq 3}$-factor deleted graphs. Zhou, Liu and Xu [17], Zhou [18], Zhou, Wu and Bian [19], Zhou and Liu [20], Wang and Zhang [21,22] established some relationships between minimum degree and graph factors. Zhou, Wu and Liu [23], Kouider and Lonc [24] investigated the relationships between independence number and graph factors. Some other results on graph factors see [25–27].

A graph H is factor-critical if for every $x \in V(H)$, there is a perfect matching in $H - x$. Assume that a graph H with $V(H) = \{x_1, x_2, \cdots, x_n\}$ is a factor-critical graph. To characterize a graph with a $P_{\geq 3}$-factor, the concept of a sun was introduced by Kaneko [28]. A graph R is said to be a sun if R is derived from H by adding new vertices $y_1, y_2, \cdots, y_n$ together with new edges $x_1y_1, x_2y_2, \cdots, x_ny_n$ to H. By virtue of Kaneko, K_1 and K_2 are also regard as two suns. Usually, the suns other that K_1 and K_2 are called big suns. A sun component of G is a component isomorphic to a sun in G. The number of sun components in G is denoted by $\mathrm{sun}(G)$. Kaneko [28] posed a characterization for a graph to admit a $P_{\geq 3}$-factor.

THEOREM 1.1 ([28]). *A graph G has a $P_{\geq 3}$-factor if and only if*

$$\mathrm{sun}(G - X) \leq 2|X|$$

for every $X \subseteq V(G)$.

Theorem 1.1 will be applied in the proof of our main result.

2. MAIN RESULT AND ITS PROOF

Let d and k be two nonnegative integers with $d \geq 2$. A graph G is called a $(P_{\geq d}, k)$-factor critical deleted graph if for any $Q \subseteq V(G)$ with $|Q| = k$, the graph $G - Q$ is a $P_{\geq d}$-factor deleted graph.

Zhou, Bian and Pan [29] derived a binding number condition for $(P_{\geq 3}, k)$-factor critical deleted graphs. In this section, we proceed to study the $(P_{\geq 3}, k)$-factor critical deleted graph, and get a new sufficient condition for the existence of $(P_{\geq 3}, k)$-factor critical deleted graphs.

THEOREM 2.1. *Let k be a nonnegative integer. Then a $(k+2)$-connected graph G is $(P_{\geq 3},k)$-factor critical deleted if G satisfies*

$$\delta(G) > \frac{\alpha(G)+2k+2}{2}.$$

Proof. Let $H = G-Q-e$ for any $Q \subseteq V(G)$ with $|Q| = k$ and any $e = xy \in E(G-Q)$. To justify Theorem 2.1, it suffices to verify that H has a $P_{\geq 3}$-factor. By virtue of contrary, we assume that H has no $P_{\geq 3}$-factor. Then it follows from Theorem 1.1 that

$$\text{sun}(H-X) \geq 2|X|+1 \tag{1}$$

for some subset $X \subseteq V(H)$.

CLAIM 1. $|X| \geq 2$.

Proof. Assume that $|X| \leq 1$. If $|X| = 0$, then by (1), $\text{sun}(H) \geq 1$. Note that G is $(k+2)$-connected. Hence, $H = G-Q-e$ is connected, which implies $\omega(H) = 1$. Thus, we derive $1 \leq \text{sun}(H) \leq \omega(H) = 1$, and so $\text{sun}(H) = 1$. Since G is $(k+2)$-connected, we have $|V(G)| \geq k+3$. Thus, we infer $|V(H)| \geq 3$. Combining this with $\text{sun}(H) = 1$ and the definition of big sun, we know that H is a big sun. Then there exist at least three vertices x_1, x_2, x_3 with degree 1 in H. Without loss of generality, let $x_1 \notin \{x,y\}$. Then $d_G(x_1) \leq d_{G-Q}(x_1)+|Q| = d_{G-Q}(x_1)+k = d_{G-Q-e}(x_1)+k = d_H(x_1)+k = k+1$, which contradicts that G is $(k+2)$-connected.

If $|X| = 1$, then from (1), $\text{sun}(H-X) \geq 2|X|+1 = 3$. Since G is $(k+2)$-connected, $G-Q-X$ is connected. Hence, $\omega(G-Q-X) = 1$. Thus, we deduce

$$\begin{aligned} 3 &\leq \text{sun}(H-X) \leq \omega(H-X) = \omega(G-Q-e-X) \\ &\leq \omega(G-Q-X)+1 = 2, \end{aligned}$$

which is a contradiction. Hence, $|X| \geq 2$. This completes the proof of Claim 1. □

Suppose that there exist a isolated vertices, b K_2's and c big sun components $R_1, R_2, \cdots, R_c$, where $|V(R_i)| \geq 6$ for $1 \leq i \leq c$, in $H-X$. And so

$$\text{sun}(H-X) = a+b+c. \tag{2}$$

By means of (1), (2) and Claim 1,

$$a+b+c = \text{sun}(H-X) \geq 2|X|+1 \geq 5. \tag{3}$$

From (3) and $H = G-Q-e$, we get

$$\begin{aligned} \text{sun}(G-Q-X) &\geq \text{sun}(G-Q-e-X)-2 = \text{sun}(H-X)-2 \\ &\geq 5-2 = 3, \end{aligned}$$

which implies that $G-Q-X$ has a vertex v with $d_{G-Q-X}(v) \leq 1$. Thus, we derive

$$\delta(G) \leq d_G(v) \leq d_{G-Q-X}(v)+|Q|+|X| \leq |X|+k+1. \tag{4}$$

Note that $\text{sun}(H-X) = \text{sun}(G-Q-X-e) \leq \text{sun}(G-Q-X)+2$. In what follows, we proceed to verify Theorem 2.1 by considering the following two cases.

Case 1. $\text{sun}(G-Q-X-e) \leq \text{sun}(G-Q-X)+1$.

By virtue of (1) and $H = G-Q-e$, we admit

$$\begin{aligned} \alpha(G) &\geq \alpha(G-Q-X) \geq \text{sun}(G-Q-X) \\ &\geq \text{sun}(G-Q-X-e)-1 \\ &= \text{sun}(H-X)-1 \geq 2|X|. \end{aligned}$$

Combining this with (4) and $\delta(G) > \frac{\alpha(G)+2k+2}{2}$, we deduce

$$2|X| \leq \alpha(G) < 2\delta(G) - 2k - 2 \leq 2(|X| + k + 1) - 2k - 2 = 2|X|,$$

which is a contradiction.

Case 2. $\mathrm{sun}(G - Q - X - e) = \mathrm{sun}(G - Q - X) + 2$.

In this case, we may assume that $e = xy$ joins two sun components D_1 and D_2 of $G - Q - X - e$, where $x \in V(D_1)$ and $y \in V(D_2)$. We easily see that $D_1 \neq K_1$ or $D_2 \neq K_1$ (otherwise, $D_1 = K_1$ and $D_2 = K_1$, then $D_1 \cup D_2 \cup \{e\} = K_2$ is a sun component of $G - Q - X$, and so $\mathrm{sun}(G - Q - X - e) = \mathrm{sun}(G - Q - X) + 1$, which contradicts that $\mathrm{sun}(G - Q - X - e) = \mathrm{sun}(G - Q - X) + 2$). Thus, we infer

$$\alpha(D_1 \cup D_2 \cup \{e\}) \geq 2. \tag{5}$$

Note that $G - Q - X - e = H - X$ has $(a + b + c)$ sun components. Then $G - Q - X$ admits $(a + b + c - 2)$ sun components and a component $D_1 \cup D_2 \cup \{e\}$, and so

$$\alpha(G) \geq \alpha(G - Q - X) \geq (a + b + c - 2) + \alpha(D_1 \cup D_2 \cup \{e\}). \tag{6}$$

It follows from (1), (2), (4), (5), (6) and $\delta(G) > \frac{\alpha(G)+2k+2}{2}$ that

$$\begin{aligned}
\alpha(G) &\geq (a + b + c - 2) + \alpha(D_1 \cup D_2 \cup \{e\}) \\
&\geq (a + b + c - 2) + 2 = a + b + c \\
&= \mathrm{sun}(H - X) \geq 2|X| + 1 \\
&\geq 2(\delta(G) - k - 1) + 1 = 2\delta(G) - 2k - 1 \\
&> 2\left(\frac{\alpha(G) + 2k + 2}{2}\right) - 2k - 1 \\
&= \alpha(G) + 1,
\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 2.1. □

If $k = 0$ in Theorem 2.1, then we obtain the following corollary.

COROLLARY 2.2. *A 2-connected graph G is a $P_{\geq 3}$-factor deleted graph if G satisfies*

$$\delta(G) > \frac{\alpha(G) + 2}{2}.$$

3. REMARK

Remark 3.1. In what follows, we claim that the condition

$$\delta(G) > \frac{\alpha(G) + 2k + 2}{2}$$

in Theorem 2.1 is sharp, that is, it cannot be replaced by

$$\delta(G) \geq \frac{\alpha(G) + 2k + 2}{2}.$$

In order to show this, we construct a graph $G = K_{k+r} \vee (2rK_2)$, where k and r are two nonnegative integers with

$r \geq 2$. Obviously, G is a $(k+r)$-connected graph with $\delta(G) = k+r+1$ and $\alpha(G) = 2r$. Thus, we deduce

$$\delta(G) = \frac{\alpha(G)+2k+2}{2}.$$

For any $Q \subseteq V(K_{k+r})$ with $|Q| = k$ and any $e \in E(2rK_2) \subseteq E(G-Q)$, let $H = G-Q-e = K_r \vee ((2r-1)K_2 \cup (2K_1))$. Choose $X = V(K_r) \subseteq V(H)$. Then we admit $|X| = r$ and $\mathrm{sun}(H-X) = 2r+1$, and so

$$\mathrm{sun}(H-X) = 2r+1 > 2r = 2|X|.$$

In terms of Theorem 1.1, H has no $P_{\geq 3}$-factor. Combining this with the definition of $(P_{\geq 3}, k)$-factor critical deleted graph, G is not a $(P_{\geq 3}, k)$-factor critical deleted graph.

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