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AN OVERVIEW OF FUNCTIONALLY GRADED MATERIAL MODELS

Vasile NĂSTĂSESCU, Silvia MARZAVAN

Military Technical Academy "Ferdinand I", Bd. George Coșbuc 39-49, Bucharest 050141, Romania
Corresponding author: Vasile NASTASESCU, E-mail: nastasescuv@gmail.ro

Abstract. This paper presents an overview of functionally graded material models performed by the authors regarding functionally graded materials. Functionally Graded Materials (FGMs) represent a new material class belonging to the composite materials. We say that FGMs are composite materials because in different point of its thickness (beams, plates), the values of the elastic properties are varying according to a model. Our paper presents such a material models in a comparative way, by quantitative determinations and graphical representations. The analysis of these models of elastic properties allows us to obtain the properties we want (we need for a part or a structure). Conclusions upon the influence of an elastic property model are useful both in design, in manufacturing and in using of functionally graded materials.

Key words*:* FGMs, material model, model power coefficient.

1. INTRODUCTION

About 20 years ago, a new class of materials appeared, and they are in continuous development and research. It is about Functionally Graded Materials (FGMs), a category of materials that can be called composites, but whose elastic and physical properties of the material, such as Young's modulus, shear modulus, density, Poisson ratio, thermal or electrical conductivity, etc. are described [1] by adopted models.

The idea of functionally graded materials (FGMs) appeared in Japan during the mid-1980s [1] as a solution for avoiding the local stress concentrations initiated by sudden changes in material properties as it happens in multilayer composites.

FGMs are built on the basis of two materials [2] with different properties, even very different, so that on the thickness of the material (between its faces), the properties vary between the values corresponding to the respective materials in homogeneous and isotropic state.

Usually, one of the materials is a high resistance material (ceramics) and other is a common material (aluminum, steel etc.) having low resistance. Intuitively, a FGM can be represented like in the Figure 1.

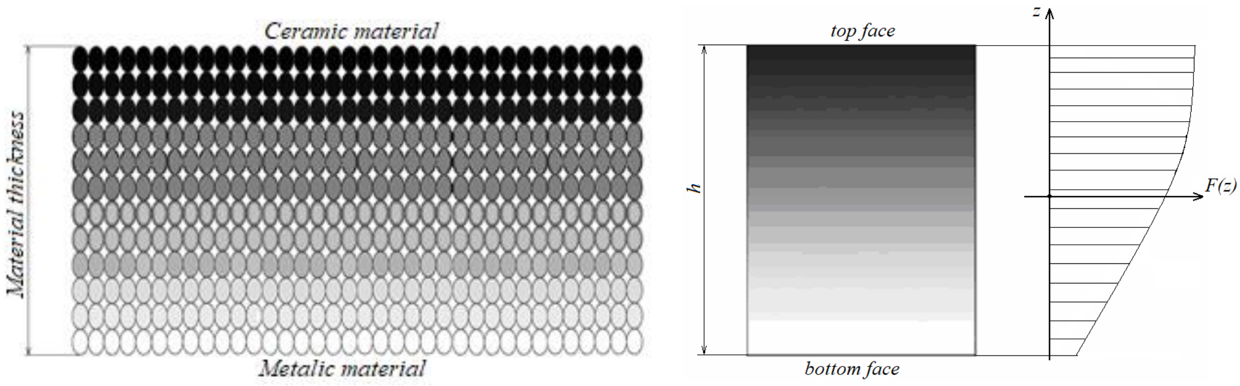
Of course, there are many other material pairs [2, 3] such as: metal-ceramic, polymer-elastomer, organic-inorganic glasses etc. Therefore, they are practically highly heterogeneous materials in terms of composition and therefore properties. Both practically and theoretically, at any point, defined by the coordinate that describes the thickness of the material (let's say the z coordinate), the properties mentioned above have different values.

By a specific technology, the functionally graded material (of a beam or a plate etc.) is a material with a continuous structure but at the same time with a continuous variation (by a model) of properties (elastic material constants, mechanical resistance, thermal conductivity and others).

For the study of this category of materials with a heterogeneous microstructure (Figure 1), it is necessary a mathematical modelling, an idealization [3, 4, 5] at the level of the material macrostructure, through which the variation of the properties to be described by known or adopted rules (models). In order to be operational in calculations, this modelling is supposed to be continuous and smooth, described by appropriate mathematical functions [5, 6].

This category of materials is based on two materials, with very different characteristics, whose the volume fractions in the thickness direction varies continuously, according to a certain model $F(z)$ - Figure 1.

This function can be for any of the material properties: Young's modulus, shear modulus, Poisson's ratio, density, etc. On the extreme faces (of a plate or beam) we find the two materials in total proportions, with the specific characteristics of those two materials. In this paper we present and discuss some material models, to highlight the special qualities of this new category of materials called FGMs.



a) Particle representation b) Multilayer representation

Fig. 1 – A changing composition model of a FGMs.

As FGMs are concerned, two aspects, two researching ways appear: one is the design and technology of such material and other is the engineering calculus. For both aspects, the knowing (or adopting) of the material properties model is very important, because this model determines the functional behavior of the structure.

For this reason, we often talk about functionally graded beams (FGBs) or functionally graded plates (FGPs). Such structures are made by a functionally graded material.

2. MATHEMATICAL IDEALIZATIONS. PROPERTY VARIATION MODELS

The material property variation models, which are going to be presented below, are formulated in a coordinate system as that used in the Figure 2.

These models are valid for any of the material properties [7] (modulus of elasticity E, shear modulus G, Poisson ratio ν, density ρ, coefficient of thermal expansion α, etc.) that underlie the realization of FGM.

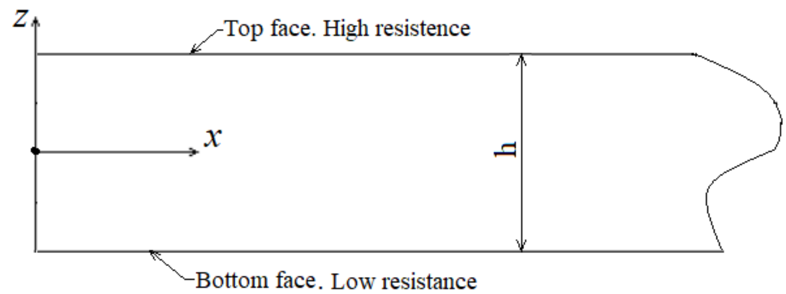


Fig. 2 – Coordinate system of a FGMs.

Material functions are continuous functions of a variable – z coordinate – and refer to the thickness of the material (Fig. 2). The indices *"t"* and *"b"*, used in the bellow relations, mean the top face or side of the material (material with higher properties) and the bottom face or side of the material (material with low properties) of FGM.

2.1. Reuss Material Model

According to this variation of the material model, its author Reuss [8] assumes the stress to be uniformly through the material. Any property, like Young's modulus E, is a function $E(z)$ of coordinate z, which is represented by the next relationship [9].

$$E(z) = \frac{E_t E_b}{E_t\left(1 - V_t\right) + E_b V_t}\,, \tag{1}$$

where V_t is the volume fraction of the top material (let's say ceramic) and it is calculated by the relation (2).

$$V_t = \left(0.5 + z/h\right)^k. \tag{2}$$

In the relations (1) and (2), z is the coordinate that describes the position of a point on the thickness of the material, h is its thickness (Fig. 2) and k is a coefficient (model power coefficient), which may have different values, less than or greater than 1.

2.2. Local Representative Volume Element (LRVE) Model

This model [9] is based on a "mesoscopic" length scale, which is much larger than the characteristic length scale of the component particles, but smaller than the characteristic length scale of a macroscopic specimen.

$$E(z) = E_b\left(1 + \frac{V_t}{FE - \sqrt[3]{V_t}}\right) \tag{3}$$

$$FE = \frac{1}{1 - E_t / E_b}. \tag{4}$$

The LRVE model is based on the hypothesis according to which the microstructure of the heterogeneous material is known. The main input parameter is represented by the average volume or the overall average of the microstructure descriptors.

Relations (3) and (4) represent the mathematical expression of the LRVE model.

2.3. Voigt Material Model

This model is used in many FGMs analysis. With the same notations, the Voigt model [10] is represented by the relation (5).

$$E(z) = E_t V_t + E_b V_b\,, \tag{5}$$

where V_t and V_b are volume fractions of the top material and bottom material, respectively. So,

$$(V_t + V_b) = 1\,. \tag{6}$$

If the volume fraction V_t is assumed to be described by the relationship,

$$V_t = \left(0.5 + z/h\right)^k, \tag{7}$$

only for any positive value of power coefficient k, the relation (5) becomes:

$$E(z) = E_t\left(0.5 + z/h\right)^k + E_b\left(1 - V_t\right) \tag{8}$$

$$E(z) = E_t\left(0.5 + z/h\right)^k + E_b - E_b\left(0.5 + z/h\right)^k \tag{9}$$

$$E(z) = E_b + \left(E_t - E_b\right)\left[1 - \left(0.5 + z/h\right)^k\right]. \tag{10}$$

The relationship (10) is presented in the current literature as representing the power model of material, as presented in subchapter 2.7 of this paper. The Voigt material model describes the material properties in terms of volume fractions. So, this material model has a degree of generalization provided by the relation (5), which may have different expressions.

2.4. Tamura Material Model

Using a linear rule of mixtures, this material model, [11] introduces an empirical fitting parameter named 'stress-to-strain transfer' [12]:

$$q = \left(\sigma_1 - \sigma_2\right) / \left(\varepsilon_1 - \varepsilon_2\right) \tag{11}$$

$$E(z) = \frac{(1-V_t)E_b(q-E_t) + V_t E_t (q-E_b)}{(1-V_t)(q-E_t) + V_t E_t (q-E_b)} . \tag{12}$$

This material model can become the Reuss model or the Voigt model, for certain values of the q .

2.5. Mori-Tanaka Material Model

In this model [12], the heterogeneous material of FGM is considered a composite with two materials, one being the material consolidated by spherical particles (the other material) distributed randomly

$$E(z) = E_b + \left(E_t - E_b\right) \cdot \frac{V_t}{1 + \left(1 - V_t\right) \cdot \left(E_t / E_b - 1\right) \cdot (1+\nu) / \left[3(1-\nu)\right]} . \tag{13}$$

In general, the variation of the Poisson's ratio ν is small [6] and as a result its influence on FGM behavior is small [13]. Consequently, in this model the Poisson's ratio is considered constant.

2.6. Exponential Material Model

Relation (14) expresses the exponential model [14] used in the construction of FGMs. The notations used are those described above in this paper.

$$E(z) = E_t \cdot e^{\left[-\lambda\left(1 - \frac{2z}{h}\right)\right]} \tag{14}$$

$$\lambda = 0.5 \ln\left(E_t / E_b\right) . \tag{15}$$

2.7. Power Material Model

This model [15, 16] is the most used both for the description and obtaining of the properties and in the development of the calculations regarding the deformations and tensions.

$$E(z) = E_b + \left(E_t - E_b\right)\left(0.5 + z / h\right)^k . \tag{16}$$

2.8. Sigmoid Material Model

This model [17, 18] consists of two model functions [19], according to the relation (17).

$$\begin{aligned} E(z) &= E_b + \left(E_t - E_b\right) \cdot \left[1 - 0.5 \cdot \left(z/h + 0.5\right)^k\right] \qquad -h/2 \le z \le 0 \\ E(z) &= E_b + 0.5 \cdot \left(E_t - E_b\right) \cdot \left(z/h - 0.5\right)^k \qquad 0 \le z \le h/2. \end{aligned} \tag{17}$$

The Sigmoid model has two different expressions for those two parts of a beam or plate, for which the z coordinate has two domains of variation $\left(-h/2 ... 0;\ 0 ... h/2\right)$.

3. GRAPHICAL REPRESENTATIONS OF THE FGMS MODELS

In Figs. 3–10, some of the models of the elastic properties, presented above, which are used in functionally graded materials, are presented in a graphical form and in a comparative way, for different values (0.1 … 8.0) of the model power coefficient.

For the graphical representations below, we chose one of the most important material properties – Young's modulus – as we did for the mathematical expression of the respective models. The graphical representations are referring to a functionally graded plate with a thickness of 0.004 m, made of two materials: a ceramic material of Al_2O_3 with $E_{ceramic} = 3.8 \cdot 10^5$ MPa and aluminum with $E_{aluminum} = 7 \cdot 10^4$ MPa .

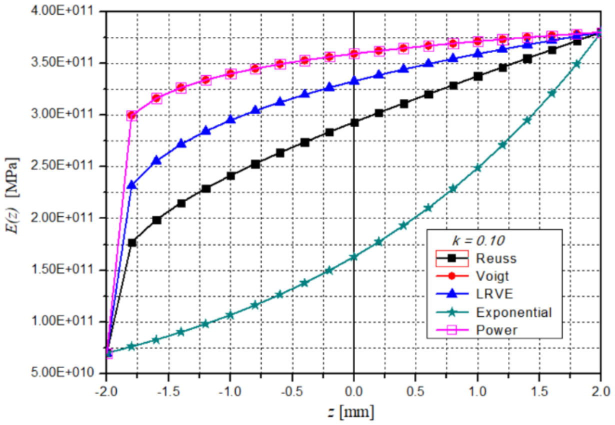


Fig. 3 – Material models for power coefficient k = 0.10.

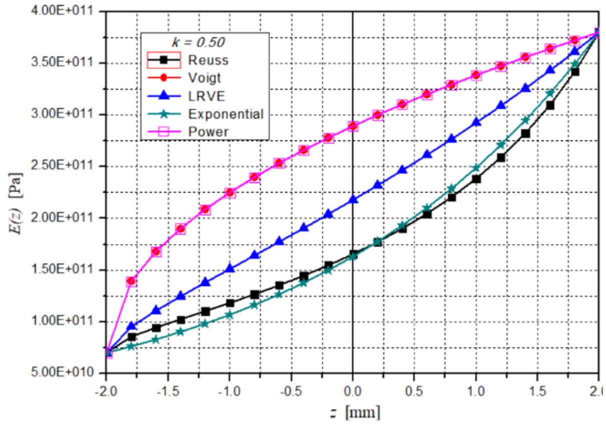


Fig. 4 – Material models for power coefficient k = 0.50.

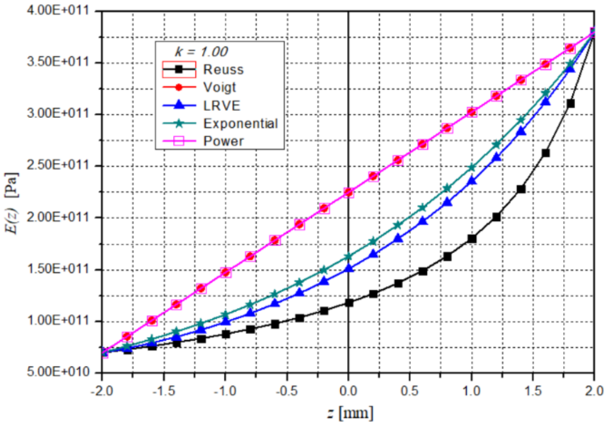


Fig. 5 – Material models for power coefficient k = 1.00.

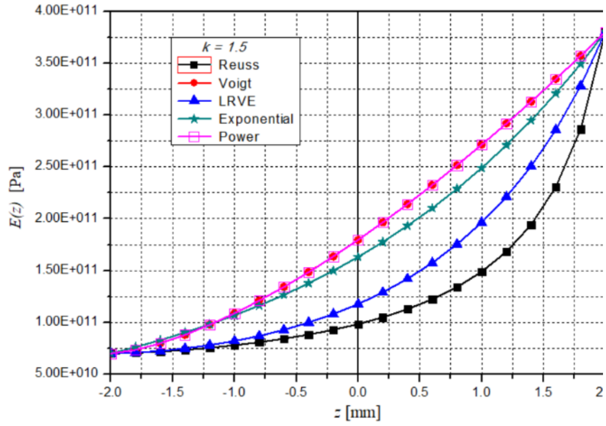


Fig. 6 – Material models for power coefficient k = 1.50.

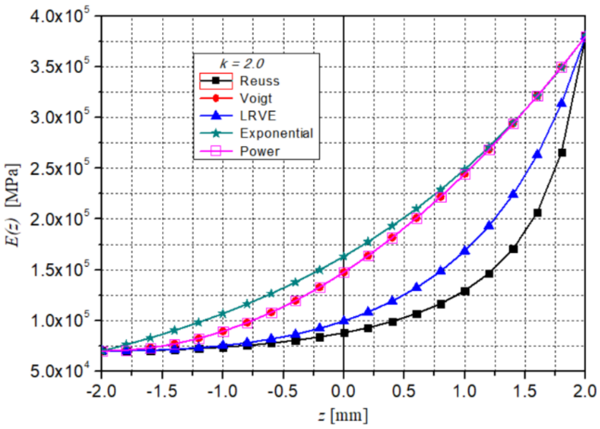


Fig. 7 – Material models for power coefficient k = 2.00.

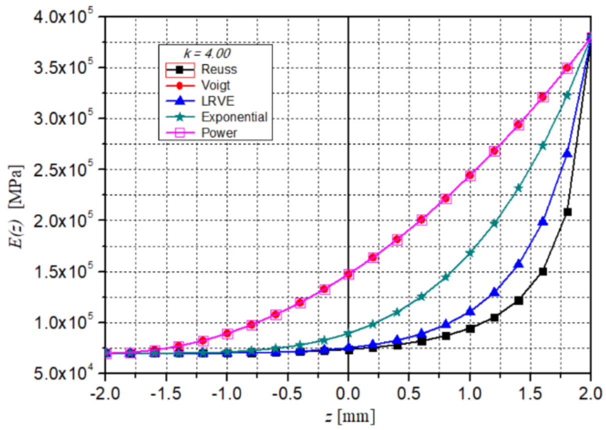


Fig. 8 – Material models for power coefficient k = 4.00.

From the analysis of graphical representations, we find that Power model and Voigt model practically coincide. The same thing results from their mathematical expressions. Thus, relation (3) can also be written:

$$E(z) = E_b + \left(E_t - E_b\right)V_b. \tag{18}$$

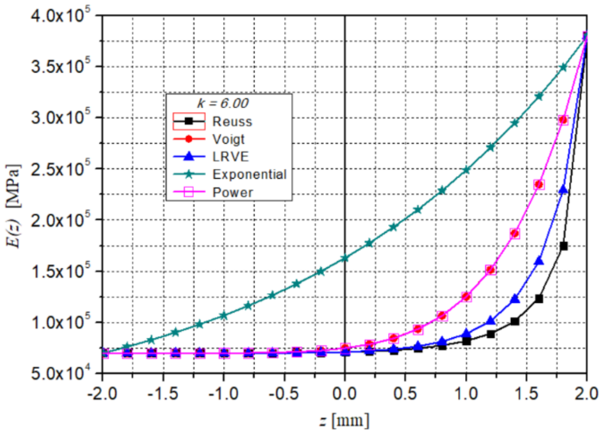


Fig. 9 – Material models for power coefficient k = 6.00.

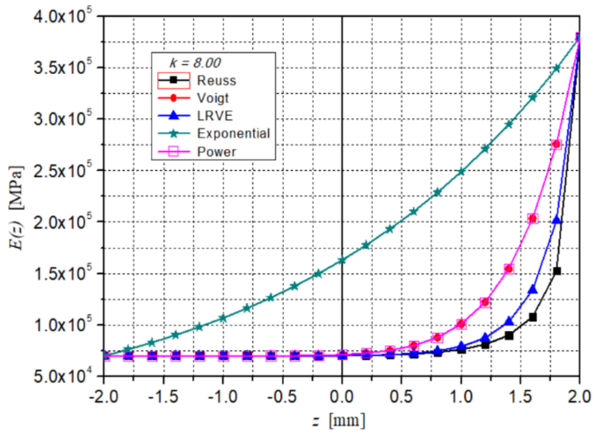


Fig. 10 – Material models for power coefficient k = 8.00.

We can immediately see that relationships (8) and (18) are different forms of the same parameter, $E(z)$ which explains the coincidence of the graphical representations of these two material models. Looking at the graphical representations in Figs. 3 to 10, we find that the values of the power coefficient k have a great influence on the parameter described by the model; for some models (Reuss, Voigt, LRVE and Power) its values even lead to the change of the position of the center of curvature of the respective curves.

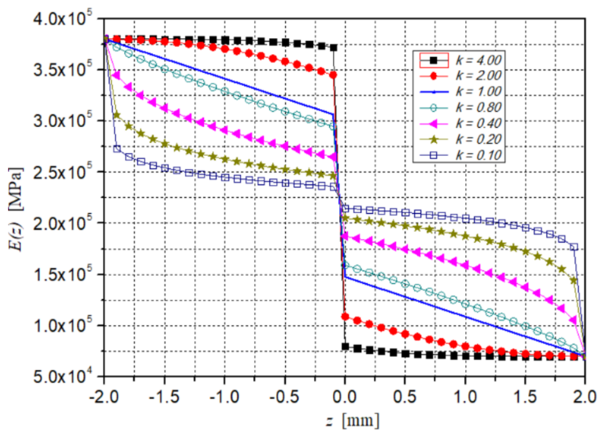


Fig. 11 – Sigmoid model for different k values.

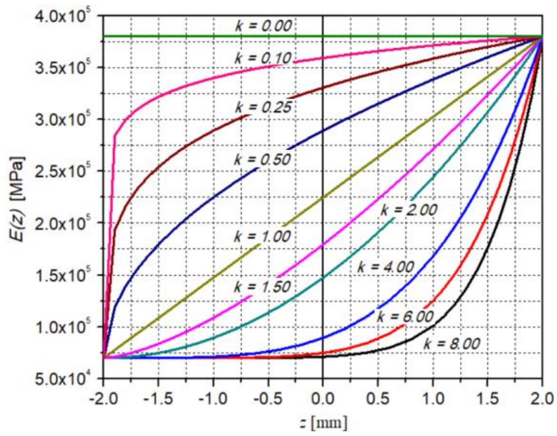


Fig. 12 – Power model for different k values.

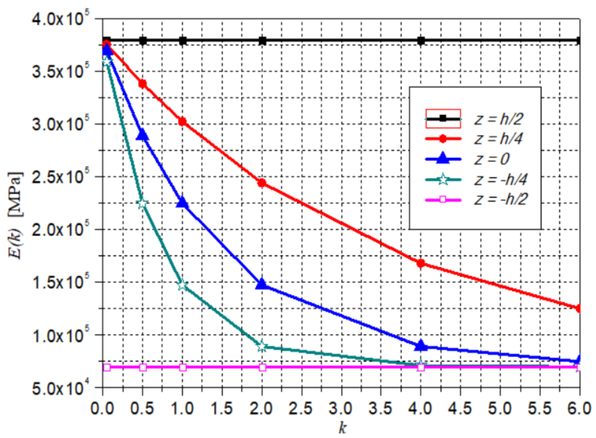


Fig. 13 – $E(k)$ for some constant z values.

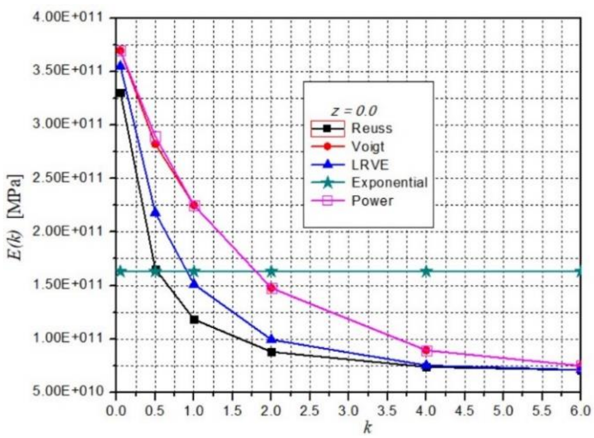


Fig. 14 – $E(k)$ by some material models at z = 0.

It is also found that the lower values of k, the higher properties material are favored in FGMs. This aspect is very important in the process of designing the overall properties of FGMs.

Also, this aspect is demonstrated, with the power of quantitative arguments, by the graphical representations in Figs. 12 to 16.

One of the most used material models is the Power model – relation (11). For this reason, this material model is subject to the graphical representations in Figs. 12 to 16.

Figure 11 shows the variation of Young's modulus on the thickness of the material, according to Sigmoid model. As we can see in Figure 11, also using relation (12), on the upper layer ($z = h/2$) appears the characteristic of the material with low properties, and on the lower layer ($z = -h/2$) the characteristic of the material with superior properties. For this representation the direction of the z-axis is downwards (vice versa compared to Figure 2).

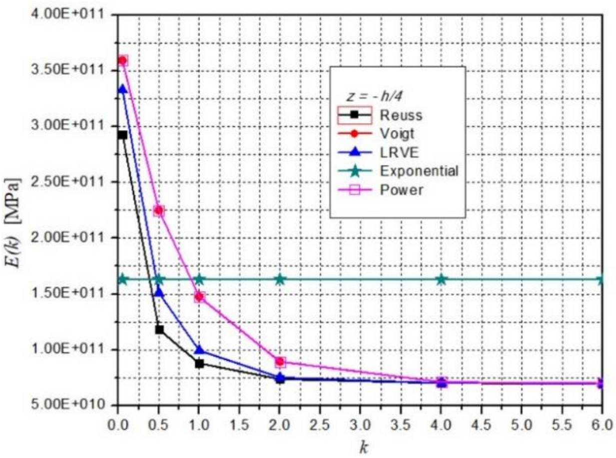


Fig. 15 – $E(k)$ by some material models at $z = -h/4$.

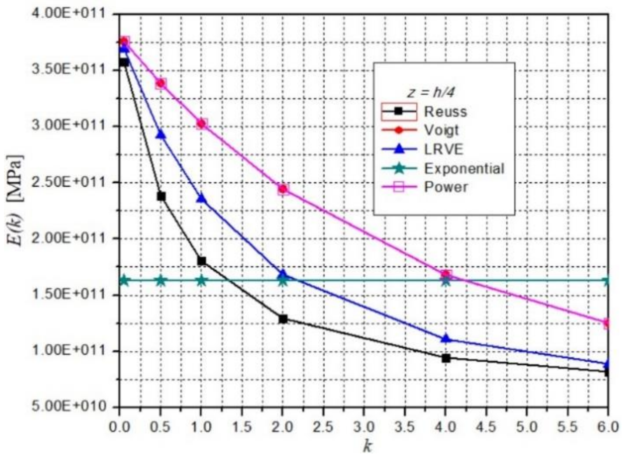


Fig. 16 – $E(k)$ by some material models at $z = h/4$.

At the point half of the material thickness ($z = 0$) there is a convergence and a divergence of the variation curves of the respective parameter (E) for the different values of the coefficient k.

As for any of the material models, the values of the power coefficient k strongly influence the variation of the respective parameter (E); in Fig. 12, this aspect is demonstrated in the case of the power model, where for $k = 1$ the variation is linear, and for values smaller than 1, the material with superior characteristics is favoured and for values greater than 1, the material with inferior characteristics is favoured.

This observation is all the more accentuated, as the values of power coefficient k take extreme values. For $k = 0$ we find the material characteristic of the material with superior characteristics.

Figure 13 shows the variation of the material characteristic (E), for a constant value of the power coefficient ($k = 0.05$) at the level of the different values of the z coordinate, on the thickness h of the material. It is observed that for any values of k, for z values closed to the upper layer ($z = h/4$), the characteristics of the material (E) are superior to the case when z is closer to the lower layer ($z = -h/4$).

Figures 14 to 16 show the variation of the material property *(E)* for a range of values of the power coefficient k, at several z levels of the FGM thickness.

Since the exponential model is not described by the variable z, the value of the material characteristic (E) is constant for the whole range of values of the power coefficient k.

As in the middle of the FGM thickness ($z = 0$), the value of the material characteristic is determined by the adopted material model. The Voigt and Power models lead to the highest values, and the Reuss model leads to the lowest values.

The situation is also confirmed by the graphical representations in Figs. 15 and 16. Besides this aspect, one of the previous observations is also confirmed, according to which, the low values of the power coefficient k led to the favouring of the material with superior characteristics within FGMs.

4. CONCLUSIONS

Functionally Graded Materials are the newest material class, belonging to composite materials. In the concept of realization of FGMs, the model of variation of physical and elastic properties has a special importance. The content of this paper strongly emphasizes this aspect.

The use of a certain function of varying the properties of the material along the thickness is the basis for the implementation of the concept of material design. The model of the material must correspond as closely as possible to the purpose for which the material is used.

Certainly, the material models presented in this paper do not exhaust the subject. The paper presents the best known models of material, contained in many papers on functionally graded materials.

The concept of designing materials involves making new appropriate material models. Their analysis, through numerical simulations and graphical representations is necessary. This paper provides a useful model from this point of view.

In order to ensure an applicative character of the work and an easier understanding, the graphic representations were made for a functional graded material, which is made of two materials much invoked in many applications. The purpose of this paper, which the authors consider fulfilled, is to systematize the most well-known material models used in the realization of FGMs and to analyse these models.

Knowing the influence of material models, the influence of parameters such as the power coefficient k and the z coordinate of a point on the material thickness is of great theoretical and practical importance, as convincingly shown in Figs. 3 to 16. All these aspects must be taken into account by those who use or design functionally graded materials.

The recent appearance (about 15–20 years ago) of this category of materials (FGMs) has led to intense research in both manufacturing and computing. Both areas are of great complexity. The authors consider that both fields are still under development, so that their own research continues, the main effort being dedicated to the use of current numerical methods.

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