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EXISTENCE RESULTS FOR ANISOTROPIC NONLINEAR WEIGHTED ELLIPTIC EQUATIONS WITH VARIABLE EXPONENTS AND L^1 DATA

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Abstract. In this paper we prove existence of distributional solutions for a class of anisotropic nonlinear weighted elliptic equations with variable exponents, where the right-hand side f is in $L^1(\Omega)$ and the weight function $W(\cdot) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $W(\cdot) > 0$.

Key words: weighted elliptic equation, anisotropic Sobolev space, variable exponent, distributional solution, L^1data.
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1. INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. Our aim is to prove the existence at least one distributional solution to the anisotropic nonlinear weighted elliptic equations of the form

$$\begin{aligned} -\sum_{i=1}^{N} D_i\big(W(x)\Theta_i(x,D_iu)\big) + a(x)\sum_{i=1}^{N}|u|^{p_i(x)-2}u = f, \quad &\text{in } \Omega, \\ u = 0, \quad &\text{on } \partial\Omega, \end{aligned} \tag{1}$$

where:
•) $\Theta_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,N$, are Carathéodory functions such that; a.e. $x \in \Omega$ and for all $\eta, \eta' \in \mathbb{R}$ $(\eta \neq \eta')$:

$$\Theta_i(x,\eta)\eta \geq c_1|\eta|^{p_i(x)}, \tag{2}$$

$$|\Theta_i(x,\eta)| \leq c_2\left(\sum_{j=1}^{N}|\eta|^{p_j(x)} + |h|\right)^{1-\frac{1}{p_i(x)}}, \quad h \in L^1(\Omega) \tag{3}$$

$$\big(\Theta_i(x,\eta) - \Theta_i(x,\eta')\big)\left(\eta - \eta'\right) \geq \begin{cases} c_3|\eta-\eta'|^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\ c_4\frac{|\eta-\eta'|^2}{(|\eta|+|\eta'|)^{2-p_i(x)}}, & \text{if } 1 < p_i(x) < 2 \end{cases} \tag{4}$$

where c_l, $l = 1,\dots,4$ are positive constants.
•) f and $a(\cdot)$ are in $L^1(\Omega)$, $W(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, such that

$$\exists\, \alpha > 0 : |f(x)| \leq \alpha a(x), \tag{5}$$

$$W(x) \geq \beta, \quad \text{for some } \beta \in \mathbb{R}_+^*. \tag{6}$$

As prototype example, we consider for $f,a \in L^1(\Omega)$, $W(\cdot)$ and $p_i(\cdot)$ are restricted as in Theorem 1, the model:

$$
\begin{aligned}
-\sum_{i=1}^{N} D_i\left(W(x)|D_iu|^{p_i(x)-2}D_iu\right)+a(x)\sum_{i=1}^{N}|u|^{p_i(x)-2}u=f, &\quad \text{in } \Omega,\\
u=0, &\quad \text{on } \partial\Omega.
\end{aligned}
\tag{7}
$$

Anisotropic equations and systems with variable exponents has many applications in applied science, for that see [15–17]. From the theoretical side it has been studied, for example, but not limited to, in the works [5–11]. The finite Morse index solutions of weighted elliptic equations and the critical exponents were proved in [1], also in [2] the further study of a weighted elliptic equation has been processed.

In this paper we prove existence results of distributional solutions for a class of anisotropic nonlinear weighted elliptic equations with variable exponents (1), where the right-hand side f is in $L^1(\Omega)$ under the condition (5), and the weight function $W(\cdot)$ is in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and strictly positive. The advantage of this method, which depends on the fact that weight function belongs to the anisotropic Sobolev space with variable exponents and zero boundary $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, which can be reused in many other cases leading to solutions for other different equations. The existence results for (1) are proven in the isotropic scalar case in [3], and in [4] the linear case $p=2$ it was studied.

The proof requires a priori estimates for a sequence of suitable approximate solutions (u_n), which in turn is proving its existence by Leray-Schauder's fixed point Theorem. So in Lemma 5 we've turned the approximate problems into a new problems with no weight function at its left-hand side. After that we prove the strong convergence, then we pass to the limit in the weak formulation.

We need a bounded Lipschitz domain in this work to arrive at the correct formulation of boundary conditions, it must therefore impose on $\partial\Omega$ to have sufficient regularity (i.e. the domain Ω with Lipschitz boundary $\partial\Omega$), and within this condition we can apply Green Riemann's theorem.

Section 2 is dedicated to mathematical preliminaries, where we talked about $p(x)-$Lebesgue-Sobolev spaces, then some embedding theorems. The main existence result and proof is in section 3.

2. MATHEMATICAL PRELIMINARIES

In this section we're going to try to recall the $p(x)-$Lebesgue-Sobolev spaces (see [12–14]).

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$\mathscr{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot):\overline{\Omega}\longmapsto\mathbb{R} \quad \text{such that} \quad 1<p^-\leq p^+<\infty\},$$

where, $p^+=\max_{x\in\overline{\Omega}} p(x)$ and $p^-=\min_{x\in\overline{\Omega}} p(x)$. We define the Lebesgue space with variable exponent

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u:\Omega\mapsto\mathbb{R}; \rho_{p(\cdot)}(u)<\infty\} \text{ where, } \rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}dx.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm; $\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda>0 \mid \rho_{p(\cdot)}(f/\lambda)\leq 1\right\}$ becomes a Banach space. Moreover, is reflexive if $p^->1$.

The Hölder type inequality: $|\int_\Omega uv\,dx| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$, holds true, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

We define also the Banach space $W_0^{1,p(\cdot)}(\Omega) := \left\{f\in L^{p(\cdot)}(\Omega): |Df|\in L^{p(\cdot)}(\Omega) \text{ and } f=0 \text{ on } \partial\Omega\right\}$ endowed with the norm $\|f\|_{W_0^{1,p(\cdot)}(\Omega)} := \|Df\|_{p(\cdot)}$. Moreover, is reflexive and separable if $p(\cdot)\in\mathscr{C}_+(\overline{\Omega})$.

For $u\in W_0^{1,p(\cdot)}(\Omega)$ with $p\in C(\overline{\Omega},[1,+\infty))$, the Poincaré inequality; $\|u\|_{p(\cdot)}\leq C\|Du\|_{p(\cdot)}$, holds (see [13]) for some constant C which depends on Ω and the function $p(x)$. The following Lemma will be used later.

LEMMA 1 ([13, 14]). *If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold*
(i) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,
(ii) $\min\left(\|u\|^{p^-}_{p(\cdot)}, \|u\|^{p^+}_{p(\cdot)}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|^{p^-}_{p(\cdot)}, \|u\|^{p^+}_{p(\cdot)}\right)$, (iii) $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$.

Now, we present the anisotropic Sobolev space with variable exponent.
First of all, let $p_i(\cdot) : \overline{\Omega} \to [1, +\infty)$ for all $i = 1, \ldots, N$ be a continuous functions, we set

$$\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)) \text{ and } p_+(x) = \max_{1 \leq i \leq N} p_i(x),\ p_-(x) = \min_{1 \leq i \leq N} p_i(x),\ \forall x \in \overline{\Omega}\,.$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega) := \left\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\right\}$, which is Banach space with respect to the norm, $\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} := \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}$.

We define: $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}$, $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) := W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$.

Remark 1 ([11]). If Ω has a Lipschitz boundary $\partial\Omega$, then $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ u_{|\partial\Omega} = 0\right\}$, where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$.

We set $\forall x \in \overline{\Omega} : \overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}}$, $p_+^+ = \max_{x \in \overline{\Omega}} p_+(x)$, $p_-^- = \min_{x \in \overline{\Omega}} p_-(x)$, $\overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N - \overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \geq N. \end{cases}$

We have the following embedding results.

LEMMA 2 ([11]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathscr{C}_+(\overline{\Omega}))^N$. If $r \in \mathscr{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact.} \tag{8}$$

LEMMA 3 ([11]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathscr{C}_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{9}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{10}$$

where C is a positive constant independent of u. Thus $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

3. STATEMENT OF RESULTS

Definition 1. We say that u is a distributional solution for problem (1) if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \sum_{i=1}^{N} W(x)\Theta_i(x, D_i u) D_i \varphi\, dx + \int_\Omega \sum_{i=1}^{N} a(x)|u|^{p_i(x)-2} u \varphi\, dx = \int_\Omega f(x)\varphi\, dx.$$

Our main result is the following.

THEOREM 1. *Let $p_i(\cdot) > 1$, $i = 1, \ldots, N$, are continuous functions on Ω where $\overline{p} < N$, and let f and $a(\cdot)$ are in $L^1(\Omega)$ such that* (5) *and* (9) *holds; let $W(\cdot)$ be such that* (6) *holds. Then the problem* (1) *has at least one solution $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense of distributions.*

3.1. APPROXIMATE SOLUTIONS

We are going to prove the existence of solution to problem (1). We define

$$f_n(x) = \frac{f(x)}{1+\frac{|f(x)|}{n}}, \quad a_n(x) = \frac{a(x)}{1+\frac{\alpha}{n}a(x)}, \quad W_n(x) = \kappa_n(W(x)), \quad n \in \mathbb{N}^* \tag{11}$$

where $\kappa_n(x) = x/(1+\frac{x}{n})$. Since κ_n is increasing for the positive real variable x, we deduce by (5) that

$$|f_n(x)| \leq \frac{\alpha a(x)}{1+\frac{\alpha}{n}a(x)} = \alpha a_n(x). \tag{12}$$

Also, thanks to (6), we have for all $x \in \overline{\Omega}$

$$\frac{\beta}{1+\beta} \leq W_n(x) \leq n. \tag{13}$$

LEMMA 4. *Let $p_i(\cdot) > 1$, $i = 1,\dots,N$, are continuous functions on Ω such that $\overline{p} < N$, and* (9) *holds, and let f, $a(\cdot)$ are in $L^1(\Omega)$, such that* (5) *holds; let $W(\cdot)$ be a function in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that* (6) *holds.*
Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems

$$\begin{aligned} -\sum_{i=1}^{N} D_i\big(W_n(x)\Theta_i(x,D_iu_n)\big) + a_n(x)u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2} = f_n, \quad &\textit{in } \Omega,\\ u_n = 0, \quad &\textit{on } \partial\Omega, \end{aligned} \tag{14}$$

in the sense that; for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$

$$\sum_{i=1}^{N}\int_\Omega W_n(x)\Theta_i(x,D_iu_n)D_i\varphi\,dx + \int_\Omega\sum_{i=1}^{N}a_n(x)|u_n|^{p_i(x)-2}u_n\varphi\,dx = \int_\Omega f_n\varphi\,dx, \tag{15}$$

$$\textit{Moreover, } \sum_{i=1}^{N}|u_n|^{p_i(x)-1} \leq \alpha. \tag{16}$$

Proof. We consider for $X = L^{p_+(\cdot)}(\Omega)$ the operator

$$\begin{aligned} \psi : X \times [0,1] &\longrightarrow X\\ (v_n,\sigma) &\longmapsto u_n = \psi(v_n,\sigma), \end{aligned}$$

where u_n is the only weak solution of the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} D_i\left(W_n(x)\Theta_i(x,D_iu_n)\right) = \sigma\left(f_n - v_n\sum\limits_{i=1}^{N}a_n(x)|v_n|^{p_i(x)-2}\right) & \text{in } \Omega,\\ u_n = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{17}$$

The existence of the weak solution u of the problem (17) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is directly produced by the main Theorem on pseudo-monotone operators, and the uniqueness of this solution which is a clear consequence of the uniqueness for the homogeneous problem $(=0)$.

It is clear that $\psi(v_n,0) = 0$ for all $v_n \in X$, because $u_n = 0 \in L^{p_+}(\Omega)$ is the only weak solition of the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} D_i\left(W_n(x)\Theta_i(x,D_iu_n)\right) = 0 & \text{in } \Omega,\\ u_n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

As the solution to the problem (17) verify, for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\sum_{i=1}^{N}\int_{\Omega} W_n(x)\Theta_i(x,D_iu_n)D_i\varphi\,dx = \sigma\int_{\Omega}\big(f_n - \sum_{i=1}^{N} a_n(x)|v_n|^{p_i(x)-2}v_n\big)\varphi\,dx. \tag{18}$$

Taking $\varphi = u_n$ as test function, and using (2), (12), (5), (13), (10), Lemma 1, and Hölder inequality, we have

$$\begin{aligned}
\frac{c_1\beta}{1+\beta}\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\,dx &\le \sigma\int_{\Omega}|a_n(x)|\big(\alpha+\sum_{i=1}^{N}|v_n|^{p_i(x)-1}\big)|u_n|\,dx\\
&\le n\int_{\Omega}\left(\alpha+\sum_{i=1}^{N}|v_n|^{p_i(x)-1}\right)|u_n|\,dx\\
&\le cn\big\|\big(1+\sum_{i=1}^{N}|v_n|^{p_i(x)-1}\big)\big\|_{p_i'(\cdot)}\|u_n\|_{p_i(\cdot)}\\
&\le c'n\left(1+\sum_{i=1}^{N}\big\||v_n|^{p_i(x)-1}\big\|_{p_i'(\cdot)}\right)\|u_n\|_{p_+(\cdot)}\\
&\le c'n\left(1+N+\sum_{i=1}^{N}\rho_{p_i(\cdot)}(v_n)\right)\|u_n\|_{p_+(\cdot)}\\
&\le c''n\big(1+N+N|\Omega|+N\rho_{p_+(\cdot)}(v_n)\big)\|u_n\|_{\overrightarrow{p}(\cdot)}.
\end{aligned} \tag{19}$$

On the other hand, we have $\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\,dx \ge \sum_{i=1}^{N}\min\{\|D_iu_n\|_{p_i(x)}^{p_i^-},\|D_iu_n\|_{p_i(x)}^{p_i^+}\}$.

We define for all $i=1,\ldots,N$; $\xi_i = \begin{cases} p_+^+, & \text{si } \|D_iu_n\|_{p_i(\cdot)} < 1\\ p_-^-, & \text{si } \|D_iu_n\|_{p_i(\cdot)} \ge 1\end{cases}$, we obtain

$$\begin{aligned}
\sum_{i=1}^{N}\min\{\|D_iu_n\|_{p_i(.)}^{p_i^-},\|D_iu_n\|_{p_i(.)}^{p_i^+}\} &\ge \sum_{i=1}^{N}\|D_iu_n\|_{p_i(.)}^{\xi_i}\\
&\ge \sum_{i=1}^{N}\|D_iu_n\|_{p_i(.)}^{p_-^-} - \sum_{\{i,\xi_i=p_+^+\}}\big(\|D_iu_n\|_{p_i(.)}^{p_-^-}-\|D_iu_n\|_{p_i(.)}^{p_+^+}\big)\\
&\ge \sum_{i=1}^{N}\|D_iu_n\|_{p_i(.)}^{p_-^-} - \sum_{\{i,\xi_i=p_+^+\}}\|D_iu_n\|_{p_i(.)}^{p_-^-} \ge \big(\frac{1}{N}\sum_{i=1}^{N}\|D_iu_n\|_{p_i(.)}\big)^{p_-^-}-N.
\end{aligned}$$

Then, we get

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\,dx \ge \left(\frac{1}{N}\|u_n\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-}-N. \tag{20}$$

From (19) and (20), we conclude

$$\frac{c_1\beta}{(1+\beta)N^{p_-^-}}\|u_n\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \le c''n\big(1+N+N|\Omega|+N\rho_{p_+(\cdot)}(v_n)\big)\|u_n\|_{\overrightarrow{p}(\cdot)}+C'. \tag{21}$$

Si $\|u_n\|_{\overrightarrow{p}(\cdot)} \le 1$, we have; $\|u_n\|_{\overrightarrow{p}(\cdot)} \le 1$, and si $\|u_n\|_{\overrightarrow{p}(\cdot)} > 1$, from (21) we have; $\|u_n\|_{\overrightarrow{p}}^{p_-^- - 1} \le C''(n)$.

Then, there exists $C(n) > 0$ such that

$$\|u_n\|_{\overrightarrow{p}(\cdot)} \le C(n). \tag{22}$$

Compactness of ψ: Let $\tilde{B}$ be a bounded of $L^{p_+(\cdot)}(\Omega)\times[0,1]$. Thus $\tilde{B}$ is contained in a product of the type $B\times[0,1]$ with B a bounded of $L^{p_+(\cdot)}(\Omega)$, which can be assumed to be a ball of center O and of radius $r>0$.

For $u \in \psi(\tilde{B})$, we have, thanks to (22): $\|u\|_{\overrightarrow{p}(\cdot)} \le \rho$.

For $u = \psi(v,\sigma)$ with $(v,\sigma) \in B \times [0,1]$ ($\|v\|_{p_+(\cdot)} \le r$). This proves that ψ applies $\tilde{B}$ in the closed ball of center O and radius ρ (ρ depend on n and r due (19)) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)(\hookrightarrow L^{p_+(\cdot)}(\Omega))$ compactly due (9) and (8).

Let u_n be a sequence of elements of $\psi(\tilde{B})$, therefore $u_n = \psi(v_n,\sigma_n)$ with $(v_n,\sigma_n) \in \tilde{B}$. Since u_n remains in a bounded of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, it is possible to extract a sub-sequence which converges strongly to an element u of $L^{p_+(\cdot)}(\Omega)$. This proves that $\overline{\psi(\tilde{B})}^{L^{p_+(\cdot)}(\Omega)}$ is compact. So ψ is compact. Now, let's prove that; $\exists M > 0$,

$$\forall (v_n,\sigma) \in X \times [0,1] : v_n = \psi(v_n,\sigma) \Rightarrow \|v_n\|_X \le M.$$

For that, we give the estimate of elements of $L^{p_+(\cdot)}(\Omega)$ such that $v_n = \psi(v_n,\sigma)$, then we have,

$$\sum_{i=1}^{N} \int_\Omega W_n(x)\Theta_i(x,D_i v_n) D_i\varphi \, dx = \sigma \int_\Omega \Big(f_n - \sum_{i=1}^{N} a_n(x)|v_n|^{p_i(x)-2} v_n \Big) \varphi \, dx, \text{ for all } \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{23}$$

Choosing $\varphi = v_n$ in (23), we have

$$\sum_{i=1}^{N} \int_\Omega W_n(x)\Theta_i(x,D_i v_n) D_i v_n \, dx + \sigma \int_\Omega \sum_{i=1}^{N} a_n(x)|v_n|^{p_i(x)} \, dx = \sigma \int_\Omega f_n v_n \, dx. \tag{24}$$

After dropping the nonnegative term in (24) due (12), and using (13), (2), Young's inequality, and the fact that

$$1 + |D_i v_n|^{p_i(\cdot)} \ge |D_i v_n|^{p_-^-}, i = 1,\ldots,N,$$

we have

$$\begin{aligned} \frac{c_1\beta}{1+\beta} \sum_{i=1}^{N} \int_\Omega |D_i v_n|^{p_i(x)} \, dx &\le n \int_\Omega |v_n| \, dx \\ &\le n \left(C(\varepsilon) + \varepsilon \int_\Omega |D_i v_n|^{p_-^-} \, dx \right) \\ &\le n \left(C(\varepsilon) + \varepsilon|\Omega| + \varepsilon \int_\Omega |D_i v_n|^{p_i(\cdot)} \, dx \right) \\ &\le n \left(C(\varepsilon) + \varepsilon|\Omega| + \varepsilon \sum_{i=1}^{N} \int_\Omega |D_i v_n|^{p_i(\cdot)} \, dx \right). \end{aligned} \tag{25}$$

Choosing $\varepsilon = \frac{c_1\beta}{2n(1+\beta)}$, then using the fact that (see (20)); $\sum_{i=1}^{N} \int_\Omega |D_i v_n|^{p_i(x)} \, dx \ge \left(\frac{1}{N} \|v_n\|_{\overrightarrow{p}(\cdot)} \right)^{p_-^-} - N$, we obtain

$$\|v_n\|_{\overrightarrow{p}(\cdot)} \le C(n). \tag{26}$$

It then follows from the Leray-Schauder's Theorem that the operator $\psi_1 : X \longrightarrow X$ defined by $\psi_1(u) = \psi(u,1)$ has a fixed point, which shows the existence of a solution of (14) in the sense of (15).

In order to prove (16), we cosider the following function defined for $t \in \mathbb{R}$ by

$$G_k(t) = \begin{cases} 0, & \text{if } |t| \le k, \\ t-k, & \text{if } t > k, \quad k > 0 \\ t+k, & \text{if } t < -k. \end{cases}$$

The use of $G_\alpha(u_n)$ as a test function in (15) gives, thanks to (2), (6) and (12),

$$\frac{c_1\beta}{1+\beta} \sum_{i=1}^{N} \int_\Omega |D_i(G_\alpha(u_n))|^{p_i(x)} \, dx + \int_\Omega |a_n(x)| \left(\sum_{i=1}^{N} |u_n|^{p_i(x)-1} - \alpha \right) |G_\alpha(u_n)| \, dx \le 0, \tag{27}$$

which implies (16). □

Remark 2. The fact that $1+|u_n|^{p_i(x)-1} \geq |u_n|^{p_-^- -1}$ and (16), gives us $|u_n| \leq \left(\frac{\alpha}{N}+1\right)^{\frac{1}{p_-^- -1}}$, so

$$(u_n) \text{ is bounded in } L^\infty(\Omega). \tag{28}$$

3.2. A PRIORI ESTIMATES

LEMMA 5. *Let f, a, W and $p_i, i=1,\ldots,N$ be restricted as in Theorem 1. Then*

$$u_n \textit{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{29}$$

where u_n the weak solution to the problem (14). *And, we have*

$$\sum_{i=1}^{N} \int_\Omega \Theta_i(x, D_i u_n) D_i \varphi \, dx = \int_\Omega G_n \varphi \, dx, \tag{30}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, where

$$\{G_n\} \textit{ is bounded in } L^{p_i'}(\Omega),\ i=1,\ldots,N. \tag{31}$$

Proof. After choosing $\varphi = u_n$ in the weak formulation (15), and dropping the nonnegative term, and the same technique as in the proof of (26) we can get

$$\|u_n\|_{\overrightarrow{p}(\cdot)} \leq C(n). \tag{32}$$

Since, for all $x \in \overline{\Omega}$

$$D_i W_n(x) = \frac{D_i W(x)}{\left(1+\frac{W(x)}{n}\right)^2},\ i=1,\ldots,N,$$

we have that $|D_i W_n(x)| \leq |D_i W(x)|$, and therefore $W_n(\cdot) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, due to $0 \leq W_n(x) \leq W(x)$, we get

$$W_n \text{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{33}$$

$$W_n \text{ strongly converges to } W \text{ in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{34}$$

Now, by the boundedness of $\frac{1}{w_n(x)}$ since (13), we find that; $\frac{1}{w_n(x)}\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, so we can chosen as test function in the weak formulation (15). We obtain

$$\sum_{i=1}^{N} \int_\Omega \Theta_i(x, D_i u_n) D_i \varphi \, dx = \int_\Omega G_n(x) \varphi \, dx,$$

where G_n defined by

$$G_n(x) = \frac{1}{W_n(x)} \left(f_n(x) - a_n(x) u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} + \sum_{i=1}^{N} \Theta_i(x, D_i u_n) D_i W_n(x) \right). \tag{35}$$

Now, for all $i=1,\ldots,N$ we have

$$\int_\Omega |a_n|^{p_i'(x)} \, dx \leq |\Omega| n^{p_+'^+}, \tag{36}$$

then (36) implies that

$$(a_n) \text{ is bounded in } L^{p_i'}(\Omega). \tag{37}$$

And in the same way that we find that

$$(f_n) \text{ is bounded in } L^{p_i'}(\Omega). \tag{38}$$

Also, thanks to (16), we have

$$a_n(x)\sum_{i=1}^{N}|u_n|^{p_i(x)-1} \le \alpha a_n(x), \tag{39}$$

so, from (39) and (37) we get

$$\Big(a_n(x)u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\Big) \text{ is bounded in } L^{p_i'}(\Omega),\ i=1,\dots,N. \tag{40}$$

Now, from (3) and (32), we obtain for all $i=1,\dots,N$

$$\begin{aligned}\int_\Omega |\Theta_i(x,D_iu_n)|^{p_i'(\cdot)}\,dx \le& (1+c_2^{p_+'^{+}})\int_\Omega\left(\sum_{j=1}^{N}|D_iu_n|^{p_j(x)}+|h|\right)dx\\ \le& (1+c_2^{p_+'^{+}})\int_\Omega\left(N\sum_{j=1}^{N}|D_ju_n|^{p_j(x)}+|h|\right)dx \le C\big\|u_n\big\|_{\overrightarrow{p}(\cdot)}^{p_+^+}+C' \le C''.\end{aligned}$$

And therefore

$$\Theta_i(x,D_iu_n) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega), \quad i=1,\dots,N. \tag{41}$$

Using (13), (33), (37), (38), (40), (41), and the boundedness of $\frac{1}{w_n(x)}$, we obtain (31). □

LEMMA 6. *There exists a subsequence (still denoted (u_n)) such that, for all $i=1,\dots,N$*

$$D_iu_n \longrightarrow D_iu \quad \textit{strongly in } L^{p_i(x)} \textit{ and a.e. in } \overline{\Omega}, \tag{42}$$

where u is the weak limit of the sequence (u_n) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. From (32) the sequence (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. So, there exists a function $u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega, \tag{43}$$

$$\text{and } D_iu_n \rightharpoonup D_iu \text{ in } L^{p_i(x)}, \quad i=1,\dots,N. \tag{44}$$

First, let's prove that, for all $i=1,\dots,N$

$$\lim_{n\longrightarrow+\infty} I_{i,n}=0, \tag{45}$$

where, for all $i=1,\dots,N$, $\quad I_{i,n}=\int_\Omega(\Theta_i(x,D_iu_n)-\Theta_i(x,D_iu))(D_iu_n-D_iu)\,dx$.

Note that, for all $i=1,\dots,N$, $\quad I_{i,n}=\int_\Omega\Theta_i(x,D_iu_n)(D_iu_n-D_iu)\,dx-\int_\Omega\Theta_i(x,D_iu)(D_iu_n-D_iu)\,dx$. As, (44) and (41) we get , for all $i=1,\dots,N$,

$$\lim_{n\longrightarrow+\infty}\int_\Omega\Theta_i(x,D_iu)(D_iu_n-D_iu)\,dx=0.$$

So, let's prove that, for all $i=1,\dots,N$

$$\lim_{n\longrightarrow+\infty}\int_\Omega\Theta_i(x,D_iu_n)(D_iu_n-D_iu)\,dx=0. \tag{46}$$

Choosing $\varphi = (u_n - u)$ in (30) as a test function, we get

$$\sum_{i=1}^{N} \int_{\Omega} \Theta_i(x, D_i u_n)(D_i u_n - D_i u)\, dx = \int_{\Omega} G_n(x)(u_n - u)\, dx.$$

By (8) in Lemma 2 we get $u_n \longrightarrow u$ strongly in $L^{p_i(x)}$ since (9), then from this and (31) we obtain (46). So, (45) has been proven.

Right now, we put : $\Omega_i^1 = \{x \in \Omega, p_i(x) \geq 2\}$, and $\Omega_i^2 = \{x \in \Omega, 1 < p_i(x) < 2\}, \quad i = 1, \dots, N$ then, By (4) we have, for all $i = 1, \dots, N$

$$I_{i,n} \geq c_3 \int_{\Omega_i^1} |D_i(u_n - u)|^{p_i(x)}. \tag{47}$$

On the other hand, by Hölder inequality, (4), and Lemma 1, we have

$$\begin{aligned}
\int_{\Omega_i^2} |D_i(u_n - u)|^{p_i(x)}\, dx &\leq 2 \left\| \frac{|D_i(u_n - u)|^{p_i(x)}}{(|D_i u_n| + |D_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}}} \right\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i^2)} \times \left\| (|D_i u_n| + |D_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}} \right\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega_i^2)} \\
&\leq 2 \max \left\{ \left(\int_{\Omega_i^2} \frac{|D_i(u_n - u)|^2}{(|D_i u_n| + |D_i u|)^{2-p_i(x)}}\, dx \right)^{\frac{p_i^-}{2}}, \left(\int_{\Omega_i^2} \frac{|D_i(u_n - u)|^2}{(|D_i u_n| + |D_i u|)^{2-p_i(x)}}\, dx \right)^{\frac{p_i^+}{2}} \right\} \\
&\quad \times \max \left\{ \left(\int_{\Omega} (|D_i u_n| + |D_i u|)^{p_i(x)}\, dx \right)^{\frac{2-p_i^+}{2}}, \left(\int_{\Omega} (|D_i u_n| + |D_i u|)^{p_i(x)}\, dx \right)^{\frac{2-p_i^-}{2}} \right\} \\
&\leq 2c \max \left\{ \left(I_{i,n} \right)^{\frac{p_i^-}{2}}, \left(I_{i,n} \right)^{\frac{p_i^+}{2}} \right\} \left((1 + \rho_{p_i}(|D_i u_n| + |D_i u|))^{\frac{2-p_-^-}{2}} \right).
\end{aligned} \tag{48}$$

From (32), (45), and after letting $n \longrightarrow +\infty$ in (47) and in (48), we obtain

$$\lim_{n \longrightarrow +\infty} \int_{\Omega} |D_i u_n - D_i u|^{p_i(x)}\, dx = 0, \text{ for all } i = 1, \dots, N.$$

Which implies (42). □

3.3. PROOF OF THE THEOREM 1 :

By (42) we have, for all $i = 1, \dots, N$

$$\Theta_i(x, D_i u_n) \rightharpoonup \Theta_i(x, D_i u) \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega),\ p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot) - 1}. \tag{49}$$

From (34) we conclude that, for all $i = 1, \dots, N$

$$W_n(\cdot) \longrightarrow W(\cdot) \text{ strongly in } L^{p_i(\cdot)}(\Omega). \tag{50}$$

Furthermore, as we have a_n is in $L^1(\Omega)$, and from (28), we obtain

$$a_n(x) u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} \longrightarrow a(x) u \sum_{i=1}^{N} |u|^{p_i(x)-2} \text{ strongly in } L^1(\Omega). \tag{51}$$

Then, through this, we can pass to the limit in the weak formulation (15). This proves Theorem 1.

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