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# EXPLICIT EIGENVALUE INTERVALS FOR THE DIRICHLET PROBLEM OF A SINGULAR $k$-HESSIAN EQUATION

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**Abstract**. We study the Dirichlet problem of a singular $k$-Hessian equation with an eigenvalue parameter $\lambda$. We prove that the problem has at least one nontrivial radial solution for each $\lambda$ in an explicit eigenvalue interval. Some results in the literature are generalized and improved.

***Key words**:* nontrivial radial solutions, $k$-Hessian equations, Dirichlet problem, explicit eigenvalue interval.
*Mathematics Subject Classification (MSC2020):* 35A16, 35B09.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following eigenvalue problem of the $k$-Hessian equation

$$\begin{cases} S_k(D^2u) = \lambda f(|x|, -u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $\lambda$ is a positive parameter, $f : [0,1] \times [0,1) \to [0,+\infty)$ is a continuous function and

$$\lim_{v \to 1^-} f(r,v) = +\infty, \quad \text{uniformly for } r \in [0,1]. \tag{2}$$

For $k \in \{1,2,\cdots,n\}$, $S_k(D^2u)$ is the $k$-Hessian operator, which denotes the $k$-th elementary symmetric function of the eigenvalues for $D^2u$, i.e.,

$$S_k(D^2u) = P_k(\Lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

where $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the eigenvalues of the Hessian matrix $D^2u$.

The $k$-Hessian equations arise from fluid mechanics, geometric problems and other applied subjects. For instance, when $k = n$, the $k$-Hessian equations can describe the Weingarten curvature and the reflector shape design. Recently, the radial solutions for the Dirichlet problems of the $k$-Hessian equations have been discussed by many scholars, and some excellent results have been obtained. See [3–7, 12–20] and the references therein. For example, in [18], the existence and uniqueness of nontrivial radial solutions to the following $k$-Hessian problem

$$\begin{cases} S_k(D^2u) = \lambda f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by the fixed point index in a cone, where the author assumed that $\lambda$ is a large parameter, $f$ is a continuous function and may have $k$-superlinear growth at 0. Later, in [19], the same author continued to consider the following $k$-Hessian problem

$$\begin{cases} S_k(D^2u) = \lambda H(|x|)f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $f$ is a continuous function and may be singular at 0 with possible $k$-superlinear growth at $\infty$. She proved that there exists an interval $\mathbf{S} \subset (0,+\infty)$, such that the problem (3) has at least two nontrivial radial solutions for any $\lambda \in \mathbf{S}$. Recently, Zhang, Xu and Wu in [20] considered the following eigenvalue problem of $k$-Hessian equation

$$\begin{cases} (-1)^k S_k^{\frac{1}{k}}(D^2u) = \lambda f(|x|,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $k \le n < 2k$ and $f : C(B_1 \times \mathbb{R}/\{0\} \to (0,+\infty))$ has a singularity at $u = 0$. By constructing the upper and lower solutions and using the Schauder's fixed point theorem, they proved that there exist two positive constants $\lambda_1 < \lambda^*$ such that the problem (4) has at least one radial solution for any $\lambda \in (\lambda_1, \lambda^*)$.

Notice that the eigenvalue intervals obtained in the above references are not explicit intervals. Motivated by this, in this paper, we continue to study the eigenvalue problem of $k$-Hessian problem (1). The main result of this paper is the following theorem.

THEOREM 1. *Assume that* (2) *holds and there exists a continuous function* $h : [0,1] \to [0,\infty)$ *such that*

$$\limsup_{v\to 0^+} \frac{f(r,v)}{v^k} = h(r), \quad \textit{uniformly for } r \in [0,1]. \tag{5}$$

*Then the problem* (1) *has at least one nontrivial radial solution for every* $\lambda \in E$*, where*

$$E = \begin{cases} \left(0, \frac{2^k n C_{n-1}^{k-1}}{kh^*}\right], & \textit{if } h^* > 0, \\ (0,+\infty), & \textit{if } h^* = 0 \end{cases}$$

*and* $h^* = \max_{r\in[0,1]} h(r)$.

Significantly, the eigenvalue interval we obtain is an explicit interval and the nonlinear term we deal with is more general than those in some known results, because the nonlinear term $f$ includes not only the case of $k$-superlinear at $v = 0$ ($h(r) = 0$) but also some other interesting situations ($h^* > 0$). Moreover, the nonlinear term has a singularity at $v = 1$.

The proof of Theorem 1 will be presented in Section 2. In Section 3, we give an example to illustrate our result.

## 2. PROOF OF THEOREM 1

In this section, we will give the proof of Theorem 1 by the following fixed point theorem in cones.

LEMMA 1 [9]. *Let $K$ be a cone in the Banach space $X$. Suppose that $A$ and $B$ are open bounded subsets of $X$ with $\overline{A}_K \subset B_K$, $A_K \neq \emptyset$, where $A_K = A \cap K$ and $B_K = B \cap K$. Let $T : \overline{B}_K \to K$ be a completely continuous operator such that*

($H_1$) $\|Tv\| \le \|v\|$ *for* $v \in \partial_K A = (\partial A) \cap K$,

($H_2$) *there exists* $\theta \in K \setminus \{0\}$ *such that* $v \neq Tv + \gamma\theta$ *for* $v \in \partial_K B = (\partial B) \cap K$ *and* $\gamma > 0$.

*Then $T$ has at least one fixed point in $\overline{B}_K \setminus A_K$.*

Such method was used in [1, 2, 11] to study the periodic problem of differential equations. The proof of Theorem 1 will be divided into a sequence of lemmas.

Let $u(x) = -v(r)$, where $r = |x|$, then the problem (1) is transformed to the problem

$$\begin{cases} C_{n-1}^{k-1}(r^{n-k}(-v')^k)' = kr^{n-1}\lambda f(r,v), & r\in(0,1), \\ v'(0)=0, \ \ v(1)=0. \end{cases} \tag{6}$$

Let $X = C[0,1]$ with the norm $\|v\| = \sup\limits_{r\in[0,1]}|v(r)|$. Define $K$ to be a cone in $X$ by

$$K = \{v\in X : v(r)\geq 0,\ \ r\in[0,1] \ \text{ and } \min_{r\in[\sigma,1-\sigma]} v(r)\geq \sigma\|v\|\},$$

where $\sigma$ is a positive constant with $0<\sigma<\frac{1}{2}$. Define

$$B^a = \{v\in X : \|v\|<a\},\ \ \Omega^b = \{v\in X : \min_{r\in[\sigma,1-\sigma]} v(r) < \sigma b\}.$$

Since the sets $\Omega^b$ are unbounded for each $b>0$, we can not use Lemma 1 to $\Omega^b$. However, we will be able to apply Lemma 1, taking into account that for each $c>b$, the following relations hold:

LEMMA 2. $\Omega^b_K = (\Omega^b\cap B^c)_K$ *and* $\overline{\Omega^b}_K = (\overline{\Omega^b\cap B^c})_K$.

*Proof.* According to [8, Lemma 2.4] or [10, Lemma 2.5], we have the following properties

($p_1$) $\Omega^b_K$ and $B^b_K$ are open relative to $K$; ($p_2$) $B^{\sigma b}_K \subset \Omega^b_K \subset B^b_K$;

($p_3$) $v\in\partial_K\Omega^b$ if and only if $\min\limits_{r\in[\sigma,1-\sigma]} v(r) = \sigma b$;

($p_4$) if $v\in\partial_K\Omega^b$, then $b\geq v(r)\geq\sigma b,\ r\in[\sigma,1-\sigma]$.

By ($p_2$), the first equality can be obtained directly. Now we prove that the second equality holds. On the one hand, notice that $(\overline{\Omega^b\cap B^c})_K \subseteq \overline{\Omega^b}_K$. On the other hand, by ($p_3$), for any $v\in\overline{\Omega^b}_K$, we have the following inequality

$$\sigma\|v\| \leq \min_{r\in[\sigma,1-\sigma]} v(r) \leq \sigma b < \sigma c,$$

which means that $v\in(\overline{\Omega^b}\cap B^c)_K \subset (\overline{\Omega^b\cap B^c})_K$. Thus $\overline{\Omega^b}_K \subseteq (\overline{\Omega^b\cap B^c})_K$. Taken together, we get the second equality. □

Define

$$(Tv)(r) = \int_r^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n}\int_0^t s^{n-1}\lambda f(s,v(s))\mathrm{d}s\Big)^{\frac{1}{k}}\mathrm{d}t, \quad v\in\overline{\Omega^b}_K,\ \ r\in[0,1],$$

where $0<b<1$. Notice that the fixed point $v\in\overline{\Omega^b}_K$ of $T$ corresponds to a positive solution $v$ of the problem (6).

LEMMA 3. $T : \overline{\Omega^b}_K \to K$ *is a completely continuous operator.*

*Proof.* For any $v\in\overline{\Omega^b}_K$, similar to [6, Lemma 2.2], we can verify that

$$(Tv)(r)\geq 0,\ \ (Tv)'(r)\leq 0,\ \ (Tv)''(r)\leq 0,\ \ \min_{r\in[\sigma,1-\sigma]} Tv(r)\geq\sigma\|Tv\|,$$

which implies that $T(\overline{\Omega^b}_K)\subset K$. Moreover, similar to the analysis in the proof of [7, Theorem 1], we can prove that $T$ is a completely continuous operator. □

LEMMA 4. *There exist positive constants a and b with* $0 < a < \sigma b < b < 1$ *such that*

($C_1$) $\|Tv\| \le \|v\|$, *for* $v \in \partial_K B^a$;

($C_2$) *there exists* $\theta \in K \setminus \{0\}$ *such that* $v \neq Tv + \gamma\theta$, *for* $v \in \partial_K \Omega^b$ *and* $\gamma > 0$;

($C_3$) $\overline{B^a}_K \subset \Omega^b_K$.

*Proof.* By (5), for every $\lambda \in E$, we get that there exists a constant $a$ with $0 < a < \sigma < 1$ such that

$$\lambda f(r,v) \le \lambda h(r) v^k \le \frac{2^k n C_{n-1}^{k-1}}{k} v^k, \ \text{ for } (r,v) \in [0,1] \times [0,a].$$

Then, for any $v \in \partial_K B^a$, we have

$$\begin{aligned}
\|Tv\| &= \sup_{r\in[0,1]} \int_r^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \lambda f(s,v(s))\mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t = \int_0^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \lambda f(s,v(s))\mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t \\
&\le \int_0^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \frac{2^k n C_{n-1}^{k-1}}{k} v^k \mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t \\
&\le \int_0^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t \frac{2^k n C_{n-1}^{k-1} a^k}{k} s^{n-1} \mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t = \int_0^1 2at \ \mathrm{d}t = a = \|v\|,
\end{aligned}$$

that is, ($C_1$) is established.

Let $\theta \equiv 1 \in K \setminus \{0\}$, we claim that

$$v \neq Tv + \gamma, \quad \forall v \in \partial_K \Omega^b \ \text{ and } \ \gamma > 0.$$

By contradiction, assume that there exist $v_0 \in \partial_K \Omega^b$ and $\gamma_0 > 0$ such that $v_0 = Tv_0 + \gamma_0$. It follows from the property ($p_4$) that $v_0$ satisfies

$$\sigma b = \sigma \|v_0\| \le v_0(r) \le b, \ \ r \in [\sigma, 1-\sigma].$$

By (2), for every $\lambda \in E$, we get that there exists a positive constant $b \in (\frac{a}{\sigma}, 1)$ such that

$$\lambda f(r,v) \ge \frac{2^k n C_{n-1}^{k-1}}{k(2\sigma - \sigma^2)^k} v^k, \ \text{ for all } (r,v) \in [0,1] \times [\sigma b, b].$$

Then, for any $r \in [\sigma, 1-\sigma]$, it can be known that

$$\begin{aligned}
v_0(r) &= Tv_0(r) + \gamma_0 \\
&= \int_r^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \lambda f(s,v_0(s))\mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t + \gamma_0 \\
&\ge \int_{1-\sigma}^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \lambda f(s,v_0(s))\mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t + \gamma_0 \\
&\ge \int_{1-\sigma}^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t s^{n-1} \frac{2^k n C_{n-1}^{k-1} v_0^k}{k(2\sigma - \sigma^2)^k} \mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t + \gamma_0 \\
&\ge \int_{1-\sigma}^1 \Big(\frac{k}{C_{n-1}^{k-1}} t^{k-n} \int_0^t \frac{2^k n C_{n-1}^{k-1} b^k}{k(2-\sigma)^k} s^{n-1} \mathrm{d}s\Big)^{\frac{1}{k}} \mathrm{d}t + \gamma_0 \\
&= \int_{1-\sigma}^1 \frac{2b}{2-\sigma} t \ \mathrm{d}t + \gamma_0 = \sigma b + \gamma_0 > \sigma b,
\end{aligned}$$

which contradicts ($p_3$). Thus, ($C_2$) is satisfied.

According to the property ($p_2$), we have

$$\overline{B^a}_K \subset B_K^{\sigma b} \subset \Omega_K^b,$$

that is, ($C_3$) holds. □

Finally, based on the above analysis and Lemma 1, we get that $T$ has at least one positive fixed point $v \in \overline{\Omega^b}_K \setminus B_K^a$, which satisfies

$$\sigma b \geq \min_{r \in [\sigma, 1-\sigma]} v(r) \geq \sigma \|v\| \geq \sigma a,$$

that is, the problem (6) has at least one positive solution $v$ satisfying

$$b \geq \|v\| \geq a \text{ and } \sigma b \geq \min_{r \in [\sigma, 1-\sigma]} v(r) \geq \sigma a.$$

The proof of Theorem 1 is finished.

## 3. EXAMPLE

In this section, we present an example to illustrate our main result.

*Example* 1. Consider the following Dirichlet problem

$$\begin{cases} S_k(D^2u) = \lambda a(|x|) \frac{(-u)^{2p}}{1-(-u)^{3q}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $\lambda$ is a positive parameter, $p$ and $q$ are positive constants, and $a : [0,1] \to [0,\infty)$ is a continuous function.

COROLLARY 1. *The problem* (7) *has at least one nontrivial radial solution for every* $\lambda \in E$, *where*

$$E = \begin{cases} \left(0, \frac{2^k n C_{n-1}^{k-1}}{k a^*}\right], & \text{if } 2p = k, \\ (0, +\infty), & \text{if } 2p > k, \end{cases}$$

*where* $a^* = \max_{r \in [0,1]} a(r) > 0$.

*Proof.* The problem (7) can be regarded as a special form of (1), where

$$f(r,v) = a(r) \frac{v^{2p}}{1 - v^{3q}}.$$

Obviously, we have

$$\lim_{v \to 1^-} f(r,v) = +\infty, \quad \text{uniformly for } r \in [0,1],$$

and

$$\limsup_{v \to 0^+} \frac{f(r,v)}{v^k} = h(r) = \begin{cases} 0 & \text{if } 2p > k, \\ a(r) & \text{if } 2p = k, \end{cases} \quad \text{uniformly for } r \in [0,1],$$

which imply that (2) and (5) are satisfied. Then, Theorem 1 guarantees that the results in Corollary 1 hold. □

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