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FERMI CONVOLUTION AND VARIANCE FUNCTION

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Abstract. In this paper, we determine the effect of the Fermi convolution power on the variance function of a Cauchy-Stieltjes Kernel (CSK) family. We then use the criteria of convergence for a sequence of variance functions to give an approximation of elements of the CSK family generated by the Fermi Poisson distribution.

***Key words**:* variance function, Cauchy kernel, Fermi convolution, fermionic Poisson law.
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1. INTRODUCTION

It is well known that classical, free and boolean convolution is related to all partitions, non-crossing partitions and interval partitions, respectively. A new convolution is investigated in [11], which is called "Fermi convolution". It is related to the set of those non-crossing partitions in which all the inner blocks are singletons, i.e. the partitions corresponding to the fermionic creation and annihilation operators. The definition of this operation is simple that at the first sight it might suggest that it is not a new type of convolution at all. Namely, the Fermi convolution of two probability measures μ and ν with mean λ_1 and λ_2 respectively, is the shift of the boolean convolution μ^0 and ν^0 by $\lambda_1+\lambda_2$ where μ^0 and ν^0 are the zero-mean shifts of μ and ν. As for probability measures with zero mean, their Fermi and boolean convolutions coincide, many properties of the boolean convolution remain valid in the Fermi case. One important consequence of this fact is that the fermionic central limit theorem follows immediately from the boolean one. However, for measures with nonzero mean there are some important differences between the Fermi and the boolean cases. One of these is that the boolean convolution of a measure μ with δ_a (the Dirac measure in a) is not necessarily the shift of μ by the amount of a (unlike in the classical, free and fermionic cases). The limit distributions of the corresponding Poisson limit theorems are also different. In the Fermi case one obtains the "fermionic Poisson-law" given in [13].

On the other hand, in the setting of noncommutative probability theory and in analogy with the theory of natural exponential families (NEFs), a theory of Cauchy-Stieltjes Kernel (CSK) families has been recently introduced, it is based on the Cauchy-Stieltjes kernel $1/(1-\theta x)$. Bryc [1] studied CSK families for compactly supported probability measures ν. It was shown that in the neighborhood of $\theta=0$, such families can be parameterized by the mean and under this parametrization, the family (and measure ν) is uniquely determined by the variance function $V(m)$ and the mean m_0 of ν. Bryc and Hassairi [2] continued the study of CSK families by extending the results in [1] to allow probability measures ν with unbounded support, providing the method to determine the domain of means, introducing the "pseudo-variance" function that has no direct probabilistic interpretation but has similar properties to the variance function and is equal to the variance function of the CSK family generated by a probability measure ν of mean zero. Other properties and characterizations of CSK families are also given in [3], [4], [5], [8], [9], [10] and [12].

In this paper, we deal with Fermi convolution from a point of view related to CSK families. We determine the formula for variance function under Fermi convolution power. Also, we give an approximation of the elements of the CSK family generated by the Fermi Poisson distribution. The rest of this section will describe some facts regarding CSK families as a background for the reader. Section 2 will give the effect of Fermi convolution power on CSK families. In section 3, we give an approximation of elements of the CSK family generated by the Fermi Poisson distribution.

Our notations are the ones used in [6]. Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta) = \int \frac{1}{1-\theta x}\nu(\mathrm{d}x) \tag{1}$$

is defined for all $\theta \in [0,\theta_+)$ with $1/\theta_+ = \max\{0, \sup\operatorname{supp}(\nu)\}$.

For $\theta \in [0,\theta_+)$, we set

$$P_{(\theta,\nu)}(\mathrm{d}x) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(\mathrm{d}x).$$

The set

$$\mathscr{K}_+(\nu) = \{P_{(\theta,\nu)}(\mathrm{d}x); \theta \in (0,\theta_+)\}$$

is called the one-sided CSK family generated by ν.

Let $k_\nu(\theta) = \int x P_{(\theta,\nu)}(\mathrm{d}x)$ denote the mean of $P_{(\theta,\nu)}$. According to [2, pages 579–580], the map $\theta \mapsto k_\nu(\theta)$ is strictly increasing on $(0,\theta_+)$, it is given by the formula

$$k_\nu(\theta) = \frac{M_\nu(\theta)-1}{\theta M_\nu(\theta)}. \tag{2}$$

The image of the interval $(0,\theta_+)$ by the function $k_\nu(.)$ is called the (one sided) domain of means of the family $\mathscr{K}_+(\nu)$, it is denoted $(m_0(\nu), m_+(\nu))$. This leads to a parametrization of the family $\mathscr{K}_+(\nu)$ by the mean. In fact, denoting by ψ_ν the reciprocal of k_ν, and writing for $m \in (m_0(\nu), m_+(\nu))$, $Q_{(m,\nu)}(\mathrm{d}x) = P_{(\psi_\nu(m),\nu)}(\mathrm{d}x)$, we have that

$$\mathscr{K}_+(\nu) = \{Q_{(m,\nu)}(\mathrm{d}x); m \in (m_0(\nu), m_+(\nu))\}.$$

Now let

$$B = B(\nu) = \max\{0, \sup\operatorname{supp}(\nu)\} = 1/\theta_+ \in [0,\infty). \tag{3}$$

Then it is shown in [2] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu), m_+(\nu))$ are given by

$$m_0(\nu) = \lim_{\theta\to 0^+} k_\nu(\theta) \qquad \text{and} \qquad m_+(\nu) = B - \lim_{z\to B^+} \frac{1}{G_\nu(z)},$$

with $G_\nu(z)$ is the Cauchy transform of ν given by

$$G_\nu(z) = \int \frac{1}{z-x}\nu(\mathrm{d}x). \tag{4}$$

It is worth mentioning here that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted $\mathscr{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/b(\nu)$ or $-\infty$ with $b = b(\nu) = \min\{0, \inf\operatorname{supp}(\nu)\}$. The domain of means for $\mathscr{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = b - 1/G_\nu(b)$.

If ν has compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathscr{K}(\nu) = \mathscr{K}_+(\nu) \cup \mathscr{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.

We now come to the notions of variance and pseudo-variance functions. The variance function

$$m \mapsto V_\nu(m) = \int (x-m)^2 Q_{(m,\nu)}(\mathrm{d}x) \tag{5}$$

is a fundamental concept in the theory of CSK families as presented in [1]. Unfortunately, if ν doesn't have a first moment which is for example the case for free 1/2-stable laws, all the distributions in the CSK family generated by ν have infinite variance. This fact has led the authors in [2] to introduce a notion of pseudo-variance function $\mathbb{V}_\nu(m)$ defined by

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right). \tag{6}$$

If $m_0(\nu) = \int x d\nu$ is finite, then from [2] the pseudo-variance function is related to the variance function by

$$\mathbb{V}_\nu(m) = \frac{m}{m-m_0} V_\nu(m). \tag{7}$$

In particular, $\mathbb{V}_\nu = V_\nu$ when $m_0(\nu) = 0$.

Another interesting fact is that the pseudo-variance function $\mathbb{V}_\nu(.)$ characterizes the CSK family, that is, the generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu(.)$: if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{8}$$

then the Cauchy transform satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}. \tag{9}$$

Also the distribution $Q_{(m,\nu)}(\mathrm{d}x)$ may be written as $Q_{(m,\nu)}(\mathrm{d}x) = f_\nu(x,m)\nu(\mathrm{d}x)$ with

$$f_\nu(x,m) := \begin{cases} \frac{\mathbb{V}_\nu(m)}{\mathbb{V}_\nu(m)+m(m-x)}, & m \neq 0 \quad ; \\ 1, & m = 0,\ \mathbb{V}_\nu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\nu(0)}{\mathbb{V}'_\nu(0)-x}, & m = 0,\ \mathbb{V}_\nu(0) = 0 \quad . \end{cases} \tag{10}$$

We now recall the effect on a CSK family of applying an affine transformation to the generating measure. Consider the affine transformation $\varphi : x \longmapsto (x-\lambda)/\beta$ where $\beta \neq 0$ and $\lambda \in \mathbb{R}$. Let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X-\lambda)/\beta$, or $\varphi(\nu) = D_{1/\beta}(\nu \boxplus \delta_{-\lambda})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0-\lambda)/\beta$. In particular, if $\beta < 0$ the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathscr{K}_-(\varphi(\nu))$. For m close enough to $(m_0-\lambda)/\beta$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\beta(m\beta+\lambda)} \mathbb{V}_\nu(\beta m + \lambda). \tag{11}$$

In particular, if the variance function exists, then $V_{\varphi(\nu)}(m) = \frac{1}{\beta^2} V_\nu(\beta m + \lambda)$.

Note that using the special case where φ is the reflection $\varphi(x) = -x$, one can transform a right-sided CSK family to a left-sided family. If ν has support bounded from above and its right-sided CSK family $\mathscr{K}_+(\nu)$ has domain of means (m_0, m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathscr{K}_-(\varphi(\nu))$ with domain of means $(-m_+, -m_0)$ and pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = \mathbb{V}_\nu(-m)$.

To close this section, we state the following result, due to Bryc [1], that will be used in Section 3.

PROPOSITION 1. *Let V_{ν_n} be a family of analytic functions which are variance functions of a sequence of CSK families $(\mathscr{K}(\nu_n))_{n\geq 1}$.*

If $V_{\nu_n} \xrightarrow{n\to+\infty} V$ uniformly in a (complex) neighborhood of $m_0 \in \mathbb{R}$ and if $V(m_0) > 0$, then there is $\varepsilon > 0$ such that V is the variance function of a CSK family $\mathscr{K}(\nu)$, generated by a probability measure ν parameterized by the mean $m \in (m_0-\varepsilon, m_0+\varepsilon)$.

Moreover, if a sequence of measures $\mu_n \in \mathscr{K}(\nu_n)$ such that $m_1 = \int x\mu_n(\mathrm{d}x) \in (m_0-\varepsilon, m_0+\varepsilon)$ does not depends on n, then $\mu_n \xrightarrow{n\to+\infty} \mu$ in distribution, where $\mu \in \mathscr{K}(\nu)$ has the same mean $\int x\mu(\mathrm{d}x) = m_1$.

2. FERMI CONVOLUTION

We will use the following notations. By $\mathscr{P}$ we denote the set of probability measures on $\mathbb{R}$, $\mathscr{P}^1$, $\mathscr{P}^2$ and $\mathscr{P}^\infty$ are the subsets of probability measures with finite mean, finite mean and variance, and with finite moments of all orders, respectively. A lower index 0 indicates vanishing mean, i.e. $\mathscr{P}_0^1$ ($\mathscr{P}_0^2$, $\mathscr{P}_0^\infty$) are the probability measures with zero mean (and finite variance, and finite moments of all orders, respectively). By

$$\mathbb{C}_+ = \{x+iy \in \mathbb{C};\ y>0\} \quad \text{and} \quad \mathbb{C}_- = \{x+iy \in \mathbb{C};\ y<0\}$$

we denote the upper and lower complex half planes, respectively.

Let $\nu \in \mathscr{P}$. The K-transform K_ν of ν is given by

$$K_\nu(z) = z - \frac{1}{G_\nu(z)}, \quad \text{for } z \in \mathbb{C}_+. \tag{12}$$

It is usually called self energy and it represent the analytic backbone of boolean additive convolution. The following result list some useful properties of the K-transform (see [7, Proposition 2.2]).

PROPOSITION 2. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathscr{K}_+(\nu)$ generated by a non degenerate probability measure ν with $A = \sup \operatorname{supp}(\nu) < \infty$. Then*

(i) *K_ν is strictly decreasing on $(A, +\infty)$.*

(ii) *For $m \in (m_0(\nu), m_+(\nu))$*
$$K_\nu\left(m + \mathbb{V}_\nu(m)/m\right) = m. \tag{13}$$

(iii) $\lim\limits_{z \longrightarrow B^+} K_\nu(z) = m_+(\nu)$*, with $B = B(\nu)$ given by* (3).

(iv) $\lim\limits_{z \longrightarrow +\infty} K_\nu(z) = m_0(\nu) \geq -\infty$.

The author in [11] has introduced the $\widetilde{B}$-transform for $\nu \in \mathscr{P}^2$ by

$$\widetilde{B}_\nu(z) = \lambda z + z K_{\nu^0}\left(\frac{1}{z}\right), \tag{14}$$

where λ is the mean of ν and ν^0 is the zero mean shift of ν. Since a measure $\nu \in \mathscr{P}^2$ is uniquely determined by its Cauchy transform G_ν, the same is true for $\widetilde{B}_\nu$.

Let $\nu_1,\ \nu_2 \in \mathscr{P}^2$. Let $\nu = \nu_1 \bullet \nu_2$ be the Fermi convolution of ν_1 and ν_2. According to [11, Theorem 3.1] we have,

$$\widetilde{B}_\nu(z) = \widetilde{B}_{\nu_1}(z) + \widetilde{B}_{\nu_2}(z). \tag{15}$$

Furthermore, $\nu \in \mathscr{P}^2$ and the mean of ν is the sum of the means of ν_1 and ν_2.

We say that the probability measure $\nu \in \mathscr{P}^2$ is infinitely divisible with respect to Fermi convolution if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathscr{P}^2$ such that

$$\nu = \underbrace{\nu_n \bullet \ldots.. \bullet \nu_n}_{n \text{ times}}.$$

According to [11, Remark 3.2], all probability measures $\nu \in \mathscr{P}^2$ are infinitely divisible in the Fermi sense.

For the clarity of our results in this paper, instead of considering the $\widetilde{B}$-transform, we consider the H-transform given by

$$H_\nu(z) = z\widetilde{B}_\nu\left(\frac{1}{z}\right) = m_0(\nu) + K_{\nu^0}(z) = m_0(\nu) + z - \frac{1}{G_{\nu^0}(z)}. \tag{16}$$

Our interest in the H-transform stems from its linear property to Fremi convolution power, that is for all $\alpha > 0$, $H_{\nu^{\bullet\alpha}}(z) = \alpha H_\nu(z)$.

The following result lists some properties of the H-transform that we need.

PROPOSITION 3. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathscr{K}_+(\nu)$ generated by a non degenerate probability measure $\nu \in \mathscr{P}^2$ with $A = \sup\operatorname{supp}(\nu) < \infty$. Then*

(i) *H_ν is strictly decreasing on $(A - m_0(\nu), +\infty)$.*

(ii) *For $m \in (m_0(\nu), m_+(\nu))$*

$$H_\nu\left(m + \frac{\mathbb{V}_\nu(m)}{m} - m_0(\nu)\right) = m. \tag{17}$$

(iii) $\lim\limits_{z \longrightarrow +\infty} H_\nu(z) = m_0(\nu)$.

Proof.
(i) The proof follows easily from (16) and Proposition 2(i).
(ii) For $m \in (m_0(\nu), m_+(\nu))$, using (16) and (13), we get

$$\begin{aligned} H_\nu\left(m + \frac{\mathbb{V}_\nu(m)}{m} - m_0(\nu)\right) &= m_0(\nu) + K_{\nu^0}\left(m + \frac{\mathbb{V}_\nu(m)}{m} - m_0(\nu)\right) \\ &= m_0(\nu) + K_{\nu^0}\left(m - m_0(\nu) + \frac{\mathbb{V}_{\nu^0}(m - m_0(\nu))}{m - m_0(\nu)}\right) = m. \end{aligned}$$

(iii) The proof follows easily from (16) and Proposition 2(iv). □

Next, we determine the formula for variance function under Fermi convolution power.

THEOREM 1. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathscr{K}_+(\nu)$ generated by a non degenerate probability measure $\nu \in \mathscr{P}^2$ with $A = \sup\operatorname{supp}(\nu) < +\infty$. For $\alpha > 0$, we have that:*

(i) *The support of $\nu^{\bullet\alpha}$ is bounded from above.*

(ii) *For m close enough to $m_0(\nu^{\bullet\alpha}) = \alpha m_0(\nu)$,*

$$\mathbb{V}_{\nu^{\bullet\alpha}}(m) = \alpha \mathbb{V}_\nu(m/\alpha) + m^2(1/\alpha - 1) + m_0(\nu)(\alpha - 1)m. \tag{18}$$

The variance functions of the CSK families generated by ν and $\nu^{\bullet\alpha}$ exists and

$$V_{\nu^{\bullet\alpha}}(m) = \alpha V_\nu(m/\alpha) + m(m - \alpha m_0(\nu))(1/\alpha - 1) + m_0(\nu)(\alpha - 1)(m - \alpha m_0(\nu)). \tag{19}$$

Proof.
(i) For measure $\nu \in \mathscr{P}^2$ with support bounded from above by $A < +\infty$ and finite mean $m_0(\nu)$, G_{ν^0} is analytic on the slit complex plane $\mathbb{C} \setminus (-\infty, A - m_0(\nu)]$. We have that $\{x \in (supp\ \nu^0)^c;\ G_{\nu^0}(x) \neq 0\} \subset (supp\ \nu^0)^c$ (see [14, Lemma 2.1]). This implies that $H_\nu(.)$ and so $H_{\nu^{\bullet\alpha}}(.)$ are well defined on a subset of $(supp\ \nu^0)^c$. So $G_{(\nu^{\bullet\alpha})^0}(z)$ is well defined and analytic on a subset of $(A - m_0(\nu), +\infty)$. Then the support of $(\nu^{\bullet\alpha})^0$ is bounded from above. This is the same for the support of $\nu^{\bullet\alpha}$.
(ii) One see that

$$m_0(\nu^{\bullet\alpha}) = \lim_{z \longrightarrow +\infty} H_{\nu^{\bullet\alpha}}(z) = \lim_{z \longrightarrow +\infty} \alpha H_\nu(z) = \alpha m_0(\nu).$$

For m close enough to $\alpha m_0(\nu)$ so that $m/\alpha \in (m_0(\nu), m_+(\nu))$ and $m + \mathbb{V}_{\nu^{\bullet\alpha}}(m)/m - m_0(\nu^{\bullet\alpha}) \in (A - m_0(\nu), +\infty)$, we can apply (17) and the additive property of the H-transform to see that

$$\begin{aligned} H_\nu\left(m + \frac{\mathbb{V}_{\nu^{\bullet\alpha}}(m)}{m} - m_0(\nu^{\bullet\alpha})\right) &= \frac{1}{\alpha} H_{\nu^{\bullet\alpha}}\left(m + \frac{\mathbb{V}_{\nu^{\bullet\alpha}}(m)}{m} - m_0(\nu^{\bullet\alpha})\right) \\ &= \frac{m}{\alpha} = H_\nu\left(m/\alpha + \frac{\mathbb{V}_\nu(m/\alpha)}{m/\alpha} - m_0(\nu)\right). \end{aligned}$$

Since H_ν is strictly decreasing on $(A-m_0(\nu),+\infty)$, this implies that

$$m+\frac{\mathbb{V}_{\nu^{\bullet\alpha}}(m)}{m}-\alpha m_0(\nu)=m/\alpha+\frac{\mathbb{V}_\nu(m/\alpha)}{m/\alpha}-m_0(\nu),$$

which is nothing but (18). Furthermore, the variance functions of the CSK families $\mathscr{K}_+(\nu)$ and $\mathscr{K}_+(\nu^{\bullet\alpha})$ exists and relation (19) follows from (7) and (18). $\square$

Remark 1. When $m_0(\nu)=0$, we have $H_\nu(z)=K_\nu(z)$ and the Fermi convolution $\bullet$ coincide with the boolean additive convolution $\uplus$. Let $\nu=\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution, its Cauchy transform and self energy are respectively

$$G_\nu(z)=\frac{z}{z^2-1} \qquad \text{and} \qquad K_\nu(z)=\frac{1}{z}.$$

With $B(\nu)=\max\{0,\ \sup supp(\nu)\}=1$, we have, from Proposition 2(iii), that $m_+(\nu)=\lim_{z\longrightarrow 1}K_\nu(z)=1$. Consider $\mu=\nu^{\bullet 2}$, then we have $K_\mu(z)=K_{\nu^{\bullet 2}}(z)=2K_\nu(z)=2/z$ and $G_\mu(z)=\frac{z}{z^2-2}$. So $\mu=\frac{1}{2}\delta_{-\sqrt{2}}+\frac{1}{2}\delta_{\sqrt{2}}$. With $B(\mu)=\max\{0,\ \sup supp(\mu)\}=\sqrt{2}$, we have that $m_+(\mu)=\lim_{z\longrightarrow\sqrt{2}}K_\mu(z)=\sqrt{2}$. This implies that $m_+(\nu^{\bullet 2})\neq 2m_+(\nu)$. So there is no “simple formula” for m_+ under Fermi convolution power. For this reason, in Theorem 1 we restrict ourselves to m close enough to $\alpha m_0(\nu)$.

3. APPROXIMATION OF FERMI-POISSON CSK FAMILY

According to [11], the Fermi-Poisson distribution is given by

$$\mu_\lambda=\left(\frac{1}{2}+\frac{1}{2\sqrt{4\lambda+1}}\right)\delta_{x_1}+\left(\frac{1}{2}-\frac{1}{2\sqrt{4\lambda+1}}\right)\delta_{x_2}, \tag{20}$$

where $\lambda=m_0(\mu)\geq 0$, $x_1=\lambda+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}}$ and $x_2=\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}$.

If $\lambda=0$, then $\mu_0=\delta_0$. Since in the theory of CSK families we deal with non degenerate probability measure, so we suppose that $\lambda>0$.

We have for all $\theta\in(\theta_-(\mu_\lambda),\theta_+(\mu_\lambda))=\left(-\infty,\frac{1}{\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}\right)$,

$$M_{\mu_\lambda}(\theta)=\frac{-1+\theta+\theta\lambda}{-1-\lambda^2\theta^2+\theta(1+2\lambda)}. \tag{21}$$

Also, the mean function is given by

$$k_{\mu_\lambda}(\theta)=\frac{\lambda(\lambda\theta-1)}{-1+\theta+\theta\lambda}. \tag{22}$$

One see that $-1+\theta+\theta\lambda=0$ for $\theta=\frac{1}{1+\lambda}>\frac{1}{\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}$. So equation (23) is well defined for all $\theta\in\left(-\infty,\frac{1}{\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}\right)$. Let $\psi_{\mu_\lambda}(.)$ the inverse of the function $k_{\mu_\lambda}(.)$. It is given by

$$\psi_{\mu_\lambda}(m)=\frac{m-\lambda}{\lambda(m-\lambda)+m}, \tag{23}$$

for all $m\in(m_-(\mu_\lambda),m_+(\mu_\lambda))=k_{\mu_\lambda}((\theta_-(\mu_\lambda),\theta_+(\mu_\lambda)))=\left(\frac{\lambda^2}{1+\lambda},\left[\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}\right]^2\right)$.

From (6), the pseudo-variance function of the family $\mathscr{K}(\mu_\lambda)$ is given, for $m \in \left(\frac{\lambda^2}{1+\lambda}, \left[\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}\right]^2\right)$, by

$$\mathbb{V}_{\mu_\lambda}(m) = \frac{m}{m-\lambda}(m-(m-\lambda)^2), \tag{24}$$

and from (7), the variance function of the family $\mathscr{K}(\mu_\lambda)$ is given by

$$V_{\mu_\lambda}(m) = m-(m-\lambda)^2, \tag{25}$$

The CSK family generated by μ_λ is given by

$$\mathscr{K}(\mu_\lambda) = \Bigg\{ Q_{(m,\mu_\lambda)}(\mathrm{d}x) = -\frac{(m^2+\lambda^2-m(1+2\lambda))(1+\sqrt{1+4\lambda})}{\lambda(-1-4\lambda+\sqrt{1+4\lambda})+m(1+4\lambda+\sqrt{1+4\lambda})}\delta_{x_1}$$
$$+\frac{(m^2+\lambda^2-m(1+2\lambda))(-1+\sqrt{1+4\lambda})}{\sqrt{1+4\lambda}(m+\lambda-m\sqrt{1+4\lambda}+\lambda\sqrt{1+4\lambda})}\delta_{x_2} : \ m \in \left(\lambda^2/(1+\lambda), \left[1/2+\sqrt{\lambda+1/4}\right]^2\right) \Bigg\}.$$

For $N \in \mathbb{N}^*$, and $0<\lambda<N$, consider

$$\nu_N = \left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1.$$

We have that for all $\theta \in (-\infty, 1)$,

$$M_{\nu_N}(\theta) = 1-\frac{\lambda}{N}+\frac{\lambda/N}{1-\theta} \qquad \text{and} \qquad k_{\nu_N}(\theta) = \frac{\lambda}{N-N\theta+\lambda\theta}.$$

As the inverse of the function $k_{\nu_N}(.)$, for all $m \in (0,1) = k_{\nu_N}((-\infty,1))$, we have $\psi_{\nu_N}(m) = \frac{\lambda-Nm}{m(\lambda-N)}$. Formula (6) implies that the pseudo-variance function of the two sided CSK family $\mathscr{K}(\nu_N)$ is $\mathbb{V}_{\nu_N}(m) = \frac{Nm^2(m-1)}{\lambda-Nm}$. With $m_0(\nu_N) = \lambda/N$, we see from (7) that the variance function of the two sided CSK family $\mathscr{K}(\nu_N)$ is $V_{\nu_N}(m) = m(1-m)$. The CSK family generated by ν_N is given by

$$\mathscr{K}(\nu_N) = \left\{ Q_{(m,\nu_N)}(\mathrm{d}x) = (1-m)\delta_0 + m\delta_1 : \ m \in (0,1) \right\}.$$

THEOREM 2. *For $N \in \mathbb{N}^*$ and $0<\lambda<N$, let*

$$\nu_N = \left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1,$$

and consider the CSK family generated by $\nu_N^{\bullet N}$, with mean $m_0(\nu_N^{\bullet N}) = \lambda$ and variance function $V_{\nu_N^{\bullet N}}(.)$. We have that

$$Q_{(m,\nu_N^{\bullet N})} \xrightarrow{N\to+\infty} Q_{(m,\mu_\lambda)}, \quad \textit{in distribution}.$$

for all m in a neighborhood of λ. In particular we get the Fermi Poisson limit theorem: $\nu_N^{\bullet N} \xrightarrow{N\to+\infty} \mu_\lambda$, in distribution.

Proof. We have that $m_0(\nu_N^{\bullet N}) = \lambda = m_0(\mu_\lambda)$.

There exists $\varepsilon>0$ such that $(m_-(\nu_N^{\bullet N}), m_+(\nu_N^{\bullet N})) \cap (m_-(\mu_\lambda), m_+(\mu_\lambda)) = (\lambda-\varepsilon, \lambda+\varepsilon)$. For all $m \in (\lambda-\varepsilon, \lambda+\varepsilon)$

$$\int xQ_{(m,\nu_N^{\bullet N})}(\mathrm{d}x) = m = \int xQ_{(m,\mu_\lambda)}(\mathrm{d}x).$$

Using variance functions and formula (19), we have for all $m \in (\lambda - \varepsilon, \lambda + \varepsilon)$

$$\begin{aligned} V_{\nu_N^{\bullet N}}(m) &= NV_{\nu_N}(m/N) + m(m - Nm_0(\nu_N))(1/N - 1) + m_0(\nu_N)(N-1)(m - Nm_0(\nu_N)) \\ &= m(1 - m/N) + m(m - \lambda)(1/N - 1) + \lambda(1 - 1/N)(m - \lambda) \\ &\xrightarrow{N \to +\infty} m - (m-\lambda)^2 = V_{\mu_\lambda}(m). \end{aligned}$$

This together with Proposition 1 applied to the sequence of measure $Q_{(m, \nu_N^{\bullet N})}$ gives that, for all $m \in (\lambda - \varepsilon, \lambda + \varepsilon)$, $Q_{(m,\nu_N^{\bullet N})} \xrightarrow{N \to +\infty} Q_{(m,\mu_\lambda)}$, in distribution. In particular for $m = \lambda$ we get the Fermi Poisson limit theorem, that is, $\nu_N^{\bullet N} \xrightarrow{N \to +\infty} \mu_\lambda$ in distribution.

$\square$

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