ISOLATED TOUGHNESS FOR FRACTIONAL \((2,b,k)\)-CRITICAL COVERED GRAPHS

Sizhong ZHOU\(^1\), Quanru PAN\(^1\), Lan XU\(^2\)

\(^1\) Jiangsu University of Science and Technology, School of Science, Zhenjiang, Jiangsu 212100, China
\(^2\) Changji University, Department of Mathematics, Changji, Xinjiang 831100, China
Sizhong ZHOU, E-mail: zsz_cumont@163.com
Corresponding author: Quanru PAN, E-mail: qrpana@163.com
Lan XU, E-mail: xulan6400@163.com

Abstract. A graph \(G\) is called a fractional \((a,b,k)\)-critical covered graph if for any \(Q \subseteq V(G)\) with \(|Q| = k\), \(G - Q\) is a fractional \([a,b]\)-covered graph. In particular, a fractional \((a,b,k)\)-critical covered graph is a fractional \((2,b,k)\)-critical covered graph if \(a = 2\). In this work, we investigate the problem of a fractional \((2,b,k)\)-critical covered graph, and demonstrate that a graph \(G\) with \(\delta(G) \geq 3 + k\) is fractional \((2,b,k)\)-critical covered if its isolated toughness \(I(G) \geq 1 + \frac{k+2}{b+1}\), where \(b\) and \(k\) are nonnegative integers satisfying \(b \geq 2 + \frac{k}{2}\).

Key words: graph, isolated toughness, fractional \([a,b]\)-factor, fractional \([a,b]\)-covered graph, fractional \((a,b,k)\)-critical covered graph.
Mathematics Subject Classification (MSC2020): 05C70.

1. INTRODUCTION

We discuss only finite undirected simple graphs \(G\) with vertex set \(V(G)\) and edge set \(E(G)\). For any \(x \in V(G)\), we let \(d_G(x)\) denote the degree of \(x\) in \(G\) and \(N_G(x)\) denote the set of vertices adjacent to \(x\) in \(G\). Let \(X\) be a vertex subset of \(G\). We write \(\delta(G) = \min\{d_G(x) : x \in V(G)\}\), \(d_G(X) = \sum_{x \in X} d_G(x)\) and \(N_G(X) = \bigcup_{x \in X} N_G(x)\).

We use \(G[X]\) to denote the subgraph of \(G\) induced by \(X\), and use \(G - X\) to denote the subgraph derived from \(G\) by removing vertices in \(X\) together with the edges adjacent to vertices in \(X\). We use \(i(G)\) to denote the number of isolated vertices of \(G\), and define \(p_j(G) = |\{x : x \in V(G), d_G(x) = j\}|\).

Obviously, \(p_0(G) = i(G)\). We denote by \(K_n\) the complete graph of order \(n\). Let \(r\) be a real number. Recall that \([r]\) is the greatest integer with \([r] \leq r\).

Yang, Ma and Liu [23] first introduced the definition of isolated toughness, denoted by

\[I(G) = \min\left\{ \frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) > 1 \right\}\]

if \(G\) is not a complete graph; otherwise, \(I(G) = +\infty\).

Let \(a, b\) and \(k\) be three nonnegative integers satisfying \(1 \leq a \leq b\). A spanning subgraph \(F\) of \(G\) is called an \([a,b]\)-factor if every vertex of \(F\) admits the degree between \(a\) and \(b\). In particular, if \(a = b = r\), then an \([a,b]\)-factor is an \(r\)-factor, which is an \(r\)-regular spanning subgraph.
Let \( h(e) \in [0, 1] \) be a function defined on \( E(G) \) and \( d_{G}^{h}(x) = \sum_{e \in E_{x}} h(e) \), where \( E_{x} = \{ e : e = xy \in E(G) \} \).

Then we call \( d_{G}^{h}(x) \) the fractional degree of \( x \) in \( G \), and call \( h \) an indicator function if \( a \leq d_{G}^{h}(x) \leq b \) holds for every \( x \in V(G) \). Let \( E^{h} = \{ e : e \in E(G), h(e) > 0 \} \) and \( G_{h} \) be a spanning subgraph of \( G \) with \( E(G_{h}) = E^{h} \).

Then we call \( G_{h} \) a fractional \([a, b]\)-factor. In particular, if \( a = b = r \), then a fractional \([a, b]\)-factor is a fractional \( r \)-factor. A graph \( G \) is called a fractional \([a, b]\)-covered graph if for every \( e \in E(G) \), \( G \) has a fractional \([a, b]\)-factor \( G_{h} \) satisfying \( h(e) = 1 \). In particular, a fractional \([a, b]\)-covered graph is called a fractional \( r \)-covered graph if \( a = b = r \). A graph \( G \) is called a fractional \((a, b, k)\)-critical covered graph if for any \( Q \subseteq V(G) \) with \( |Q| = k \), \( G - Q \) is a fractional \([a, b]\)-covered graph. In particular, a fractional \((a, b, k)\)-critical covered graph is a fractional \((r, k)\)-critical covered graph if \( a = b = r \).


For some recent advances on the problem of factors in graphs, we refer to Nenadov [16], Chiba [2], Zhou and Kouider and Lonc [9] investigated the relationship between stability number and \([a, b]\)-factors of graphs having fractional 2-factors. Katerinis [6] showed some results on fractional \( r \)-factors in regular graphs.

Liu and Zhang [12] investigated the existence of fractional \( r \)-factors in graphs. More recently related results on fractional factors in graphs can be referred to Zhou [27, 30], Liu, Yu and Zhang [11], Gao, Guirao and Chen [3], Gao, Wang and Dimitrov [5], Gao, Liang and Chen [4], Wang and Zhang [21], Zhou, Liu and Xu [33]. Yuan and Hao [24] got a degree condition for a graph to be a fractional \([a, b]\)-covered graph.

Yuan and Hao [25] put forward two sufficient conditions for the existence of fractional \([a, b]\)-covered graphs. Lv [13] presented a degree condition for the existence of fractional \((a, b, k)\)-critical covered graphs. Zhou, Wu and Liu [38] derived an independence number and connectivity condition for the existence of fractional \((a, b, k)\)-critical covered graphs. Zhou [26, 28] claimed two neighborhood conditions for a graph being a fractional \((a, b, k)\)-critical covered graph.

Motivated by above results, we derive an isolated toughness condition for a graph being a fractional \((2, b, k)\)-critical covered graph, which will be shown in Section 2.

### 2. MAIN RESULT AND ITS PROOF

In order to verify our main result in this paper, we first present the following lemmas.

**LEMMA 2.1** ([10]). Let \( 0 \leq a \leq b \) be two integers. Then a graph \( G \) is a fractional \([a, b]\)-covered graph if and only if

\[
|Y| - d_{G - X}(Y) \leq b|X| - \varepsilon(X, Y)
\]

for every \( X \subseteq V(G) \), where \( Y = \{ x : x \in V(G) \setminus X, d_{G - X}(x) \leq a \} \) and \( \varepsilon(X, Y) \) is defined by

\[
\varepsilon(X, Y) = \begin{cases} 
2, & \text{if } X \text{ is not independent}, \\
1, & \text{if } X \text{ is independent and there is an edge joining } X \text{ and } V(G) \setminus (X \cup Y), \text{ or there is an edge } x = y \text{ joining } X \text{ and } Y \text{ such that } d_{G - X}(y) = a \text{ for } y \in Y, \\
0, & \text{otherwise.}
\end{cases}
\]

The following lemma is equivalent to Lemma 2.1.

**LEMMA 2.2.** Let \( 0 \leq a \leq b \) be two integers. Then a graph \( G \) is a fractional \([a, b]\)-covered graph if and only if

\[
\sum_{j=0}^{a-1} (a - j)p_{j}(G - X) \leq b|X| - \varepsilon(X, Y)
\]
for every $X \subseteq V(G)$, where $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \leq a\}$ and $\varepsilon(X,Y)$ is same as that of Lemma 2.1.

Next, we show our main result in this paper.

**Theorem 2.1.** Let $b$ and $k$ be nonnegative integers such that $b \geq 2 + \frac{k}{2}$, and let $G$ be a graph. If $\delta(G) \geq 3 + k$ and

$$I(G) \geq 1 + \frac{k+2}{b-1},$$

then $G$ is a fractional $(2,b,k)$-critical covered graph.

**Proof.** Theorem 2.1 clearly holds for a complete graph. In what follows, we consider the case when $G$ is not a complete graph.

Let $W \subseteq V(G)$ with $|W| = k$. We write $H = G - W$. In order to verify Theorem 2.1, it suffices to show that $H$ is a fractional $[2,b]$-covered graph. To the contrary, we assume that $H$ is not a fractional $[2,b]$-covered graph. Then it follows from Lemma 2.2 that

$$2p_0(H - X) + p_1(H - X) > b|X| - \varepsilon(X,Y)$$

for some $X \subseteq V(H)$, where $Y = \{x : x \in V(H) \setminus X, d_{H-X}(x) \leq 2\}$.

If $|X| \leq 1$, then by (1) and $\varepsilon(X,Y) \leq |X|$, we have

$$2p_0(H - X) + p_1(H - X) > b|X| - \varepsilon(X,Y) \geq b|X| - |X| = (b - 1)|X| \geq 0.$$  \hspace{1cm} (2)

Moreover, it follows from $\delta(G) \geq 3 + k$, $H = G - W$ with $|W| = k$, and $|X| \leq 1$ that

$$p_0(H - X) = p_1(H - X) = 0,$$

which contradicts (2). Henceforth, we shall consider the case when $|X| \geq 2$.

Note that $\varepsilon(X,Y) \leq 2$. From (1), we get

$$2p_0(H - X) + p_1(H - X) > b|X| - \varepsilon(X,Y) \geq b|X| - 2.$$ \hspace{1cm} (3)

**Claim 1.** $\frac{k+|X|+\frac{k}{2}p_1(H - X)}{p_0(H - X)+\frac{k}{2}p_1(H - X)} < 1 + \frac{k+2}{b-1}.$

**Proof.** According to (3), $|X| \geq 2$ and $b \geq 2 + \frac{k}{2}$, we have

$$\frac{k+|X|+\frac{k}{2}p_1(H - X)}{p_0(H - X)+\frac{k}{2}p_1(H - X)} = 1 + \frac{k+|X|-p_0(H - X)}{p_0(H - X)+\frac{k}{2}p_1(H - X)}$$

$$\leq 1 + \frac{k+|X|}{p_0(H - X)+\frac{k}{2}p_1(H - X)}$$

$$< 1 + \frac{k+|X|}{\frac{b}{2}b|X| - 1}$$

$$= 1 + \frac{\frac{2}{b} + \frac{2k+4}{b}}{b|X| - 2}$$

$$\leq 1 + \frac{\frac{2}{b} + \frac{2k+4}{b}}{2b - 2}$$

$$= 1 + \frac{k+2}{b-1}.$$  \hspace{1cm} \Box

Claim 1 is proved.

Let $Q = \{x : x \in V(H) \setminus X, d_{H-X}(x) = 1\}$. Then $|Q| = p_1(H - X)$. Further, we write

$$E(Q) = \{e = xy : x, y \in Q\},$$
Let $M(Q) = \{x_1y_1, x_2y_2, \cdots, x_ly_l\}$ be a maximum matching in $G[Q]$. Put

$$Q_+ = \{x_i : 1 \leq i \leq t\}.$$ 

The following proof will be divided into four cases.

Case 1. $E(Q) = \emptyset$ and $E(Q, N_{H-X}(Q)) = \emptyset$.

In this case, it is obvious that $Q = \emptyset$. Hence, we derive $p_1(H - X) = 0$. Combining this with (3) and $|X| \geq 2$, we possess

$$i(H - X) = p_0(H - X) > \frac{1}{2}b|X| - 1 \geq b - 1 \geq 1 \quad (4)$$

Note that $H = G - W$ with $|W| = k$. Thus, $i(G - W \cup X) = i(H - X) > 1$. Using (4), $|X| \geq 2$ and $I(G) \geq 1 + \frac{k+2}{b-1}$, we have

$$1 + \frac{k+2}{b-1} \leq I(G) \leq \frac{|W \cup X|}{i(G - W \cup X)} = \frac{k + |X|}{i(H - X)}$$

$$< \frac{k + |X|}{\frac{1}{2}b|X| - 1} = \frac{2 + \frac{2k}{b}}{b} \leq \frac{2k + 4}{b} \leq \frac{b}{b-1} = \frac{k + 2}{b-1},$$

which is a contradiction.

Case 2. $E(Q) = \emptyset$ and $E(Q, N_{H-X}(Q)) \neq \emptyset$.

Obviously, $|N_{H-X}(Q)| \leq p_1(H - X)$. Then using (3) and $|X| \geq 2$, we get

$$i(G - W \cup X \cup N_{H-X}(Q)) = i(H - X \cup N_{H-X}(Q)) \geq i(H - X) + p_1(H - X) \geq \frac{1}{2}(2i(H - X) + p_1(H - X)) = \frac{1}{2}(2p_0(H - X) + p_1(H - X))$$

$$> \frac{1}{2}b|X| - 1 \geq b - 1 \geq 1. \quad (5)$$

It follows from (3), (5), $|X| \geq 2$, $b \geq 2 + \frac{k}{2}$ and $I(G) \geq 1 + \frac{k+2}{b-1}$ that

$$1 + \frac{k+2}{b-1} \leq I(G) \leq \frac{|W \cup X \cup N_{H-X}(Q)|}{i(G - W \cup X \cup N_{H-X}(Q))}$$

$$= \frac{k + |X| + |N_{H-X}(Q)|}{i(G - W \cup X \cup N_{H-X}(Q))} \leq \frac{k + |X| + p_1(H - X)}{i(H - X) + p_1(H - X)} \leq \frac{k + |X| + p_1(H - X)}{p_0(H - X) + p_1(H - X)}$$

$$= \frac{1 + \frac{k + |X| - p_0(H - X)}{p_0(H - X) + p_1(H - X)}}{1 + \frac{2p_0(H - X) + p_1(H - X) - p_0(H - X)}{k + |X| - p_0(H - X)}}$$
\[ \leq 1 + \frac{k + |X|}{2p_0(H - X) + p_1(H - X)} \]
\[ < 1 + \frac{k + |X|}{b|X| - 2} \]
\[ = 1 + \frac{1}{b} + \frac{k + 2}{b|X| - 2} \]
\[ \leq 1 + \frac{1}{b} + \frac{k + \frac{1}{2}}{2b - 2} \]
\[ = 1 + \frac{1}{b} + \frac{bk + 2}{2b(b - 1)} \]
\[ < 1 + \frac{2}{b} + \frac{bk + 2}{b(b - 1)} \]
\[ = 1 + \frac{k + 2}{b - 1} \]

which is a contradiction.

**Case 3.** \( E(Q) \neq \emptyset \) and \( E(Q, N_{H - X}(Q)) = \emptyset \).

In this case, we easily see that \( |Q_1| = \frac{1}{2} p_1(H - X) \) and
\[ i(G - W \cup X \cup Q_1) = i(H - X \cup Q_1) \geq i(H - X) + \frac{1}{2} p_1(H - X) \]
\[ = p_0(H - X) + \frac{1}{2} p_1(H - X) > \frac{1}{2} b|X| - 1 \geq b - 1 \geq 1 \quad (6) \]

by (3), \(|X| \geq 2\) and \(b \geq 2 + \frac{4}{b} \geq 2\).

According to (6), Claim 1 and \( I(G) \geq 1 + \frac{k + \frac{1}{2}}{b - 1} \), we obtain
\[ 1 + \frac{k + 2}{b - 1} \leq I(G) \leq \frac{|W \cup X \cup Q_1|}{i(G - W \cup X \cup Q_1)} \]
\[ = \frac{k + |X| + |Q_1|}{i(G - W \cup X \cup Q_1)} \]
\[ \leq \frac{k + |X| + \frac{1}{2} p_1(H - X)}{p_0(H - X) + \frac{1}{2} p_1(H - X)} \]
\[ < 1 + \frac{k + 2}{b - 1} \]

which is a contradiction.

**Case 4.** \( E(Q) \neq \emptyset \) and \( E(Q, N_{H - X}(Q)) \neq \emptyset \).

**Subcase 4.1.** \(|E(Q)| > |E(Q, N_{H - X}(Q))|\).

Let \( N = (N_{H - X}(Q) \setminus D) \cup Q' \), where \( Q' \subseteq Q_2 \). Then there exists \( Q' \subseteq Q_2 \) such that \(|N| = \left\lfloor \frac{1}{2} p_1(H - X) \right\rfloor \) and \( i(H - X \cup N) \geq i(H - X) + \frac{1}{2} p_1(H - X) = p_0(H - X) + \frac{1}{2} p_1(H - X) \). Thus, we have
\[ i(G - W \cup X \cup N) = i(H - X \cup N) \geq p_0(H - X) + \frac{1}{2} p_1(H - X) \]
\[ > \frac{1}{2} b|X| - 1 \geq b - 1 \geq 1 \quad (7) \]
by (3), \(|X| \geq 2 \) and \( b \geq 2 + \frac{k}{2} \geq 2 \). Using (7), Claim 1 and \( I(G) \geq 1 + \frac{k+2}{b-1} \), we get

\[
1 + \frac{k+2}{b-1} \leq I(G) \leq \frac{|W \cup X \cup N|}{i(G-W \cup X \cup N)} = \frac{k+|X|+\frac{1}{2}p_1(H-X)}{i(G-W \cup X \cup N)} \leq \frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} < 1 + \frac{k+2}{b-1},
\]

which is a contradiction.

\textit{Subcase 4.2.} \(|E(Q)| \leq |E(Q,N_{H-X}(Q))|\).

Let \( N = Q + Q' \), where \( Q' \subseteq N_{H-X}(Q) \setminus D \). Then there exists \( Q' \subseteq N_{H-X}(Q) \setminus D \) such that \(|N| = \frac{1}{2}p_1(H-X)\) and \(i(H-X \cup N) \geq i(H-X)+\frac{1}{2}p_1(H-X) = p_0(H-X)+\frac{1}{2}p_1(H-X)\). Thus, we derive

\[
i(G-W \cup X \cup N) = i(H-X \cup N) \geq p_0(H-X)+\frac{1}{2}p_1(H-X) > \frac{1}{2}b|X| - 1 \geq b-1 \geq 1
\]

by (3), \(|X| \geq 2 \) and \( b \geq 2 + \frac{k}{2} \geq 2 \). It follows from (8), Claim 1 and \( I(G) \geq 1 + \frac{k+2}{b-1} \) that

\[
1 + \frac{k+2}{b-1} \leq I(G) \leq \frac{|W \cup X \cup N|}{i(G-W \cup X \cup N)} = \frac{k+|X|+\frac{1}{2}p_1(H-X)}{i(G-W \cup X \cup N)} \leq \frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} < 1 + \frac{k+2}{b-1},
\]

which is a contradiction. This completes the proof of Theorem 2.1.

If \( k = 0 \) in Theorem 2.1, then we get the following corollary.

\textbf{COROLLARY 2.1.} Let \( b \geq 2 \) be an integer, and let \( G \) be a graph. If \( 1(G) \geq 3 \) and

\[
I(G) \geq 1 + \frac{2}{b-1},
\]

then \( G \) is a fractional \([2, b]\)-covered graph.

If \( b = 2 \) in Corollary 2.1, then we get the following corollary.

\textbf{COROLLARY 2.2.} Let \( G \) be a graph. If \( 1(G) \geq 3 \) and \( I(G) \geq 3 \), then \( G \) is a fractional 2-covered graph.

\section{3. REMARK}

Next, we show that the condition \( I(G) \geq 1 + \frac{k+2}{b-1} \) in Theorem 2.1 is best possible in some sense, namely, it cannot be replaced by \( I(G) \geq 1 + \frac{k+2}{2b-1} \). To check this, we consider a graph \( G \) constructed from \( K_{k+2}, (2b-1)K_1 \)
and $K_{2b-1}$ as follows: letting $V((2b-1)K_1) = \{x_1,x_2,\cdots,x_{2b-1}\}$ and $V(K_{2b-1}) = \{y_1,y_2,\cdots,y_{2b-1}\}$, where $k \geq 0$ and $b \geq 2 + \frac{k}{2}$ are two integers. We first join every vertex $x_i$ to the vertex $y_i$ with the same subscript $i$, and then join every vertex $x_i$ to all the vertices of $K_{k+2}$. Then we easily see

$$I(G) = \frac{|Q|}{i(G-Q)} = \frac{k+2+2b-1}{2b-1} = 1 + \frac{k+2}{2b-1},$$

where $Q = V(K_{k+2}) \cup V(K_{2b-1})$.

Set $D = V(K_k) \subseteq V(K_{k+2})$, $G' = G - D$ and $X = V(K_{k+2}) \setminus V(K_k)$. Then $\varepsilon(X, Y) = 2$ since $X$ is not an independent set, where $Y = \{x : x \in V(G') \setminus X, d_{G'\setminus X}(x) \leq 2\}$. Hence, we deduce

$$2p_0(G' - X) + p_1(G' - X) = 2b - 1 > 2b - 2 = b|X| - \varepsilon(X, Y).$$

In light of Lemma 2.2, $G'$ is not fractional $[2, b]$-covered, that is, $G$ is not fractional $(2, b, k)$-critical covered.

ACKNOWLEDGEMENTS

The authors take this opportunity to thank the anonymous referees for their careful reading of the manuscript and suggestions which have immensely helped us in getting the paper to its present form.

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