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ISOLATED TOUGHNESS FOR FRACTIONAL $(2,b,k)$–CRITICAL COVERED GRAPHS

Sizhong ZHOU[1], Quanru PAN[1], Lan XU[2]

[1] Jiangsu University of Science and Technology, School of Science,
Zhenjiang, Jiangsu 212100, China
[2] Changji University, Department of Mathematics, Changji, Xinjiang 831100, China
Sizhong ZHOU, E-mail: zsz_cumt@163.com
Corresponding author: Quanru PAN, E-mail: qrpana@163.com
Lan XU, E-mail: xulan6400@163.com

Abstract. A graph G is called a fractional (a,b,k)-critical covered graph if for any $Q \subseteq V(G)$ with $|Q| = k$, $G-Q$ is a fractional $[a,b]$-covered graph. In particular, a fractional (a,b,k)-critical covered graph is a fractional $(2,b,k)$-critical covered graph if $a = 2$. In this work, we investigate the problem of a fractional $(2,b,k)$-critical covered graph, and demonstrate that a graph G with $\delta(G) \geq 3+k$ is fractional $(2,b,k)$-critical covered if its isolated toughness $I(G) \geq 1+\frac{k+2}{b-1}$, where b and k are nonnegative integers satisfying $b \geq 2+\frac{k}{2}$.

Key words: graph, isolated toughness, fractional $[a,b]$-factor, fractional $[a,b]$-covered graph, fractional (a,b,k)-critical covered graph.
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1. INTRODUCTION

We discuss only finite undirected simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, we let $d_G(x)$ denote the degree of x in G and $N_G(x)$ denote the set of vertices adjacent to x in G. Let X be a vertex subset of G. We write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$, $d_G(X) = \sum_{x\in X} d_G(x)$ and $N_G(X) = \bigcup_{x\in X} N_G(x)$. We use $G[X]$ to denote the subgraph of G induced by X, and use $G-X$ to denote the subgraph derived from G by removing vertices in X together with the edges adjacent to vertices in X. We use $i(G)$ to denote the number of isolated vertices of G, and define

$$p_j(G) = |\{x : x \in V(G), d_G(x) = j\}|.$$

Obviously, $p_0(G) = i(G)$. We denote by K_n the complete graph of order n. Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer with $\lfloor r \rfloor \leq r$.

Yang, Ma and Liu [23] first introduced the definition of isolated toughness, denoted by

$$I(G) = \min\left\{\frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) > 1\right\}$$

if G is not a complete graph; otherwise, $I(G) = +\infty$.

Let a,b and k be three nonnegative integers satisfying $1 \leq a \leq b$. A spanning subgraph F of G is called an $[a,b]$-factor if every vertex of F admits the degree between a and b. In particular, if $a = b = r$, then an $[a,b]$-factor is an r-factor, which is an r-regular spanning subgraph.

Let $h(e) \in [0,1]$ be a function defined on $E(G)$ and $d_G^h(x) = \sum_{e \in E_x} h(e)$, where $E_x = \{e : e = xy \in E(G)\}$. Then we call $d_G^h(x)$ the fractional degree of x in G, and call h an indicator function if $a \le d_G^h(x) \le b$ holds for every $x \in V(G)$. Let $E^h = \{e : e \in E(G), h(e) > 0\}$ and G_h be a spanning subgraph of G with $E(G_h) = E^h$. Then we call G_h a fractional $[a,b]$-factor. In particular, if $a = b = r$, then a fractional $[a,b]$-factor is a fractional r-factor. A graph G is called a fractional $[a,b]$-covered graph if for every $e \in E(G)$, G has a fractional $[a,b]$-factor G_h satisfying $h(e) = 1$. In particular, a fractional $[a,b]$-covered graph is called a fractional r-covered graph if $a = b = r$. A graph G is called a fractional (a,b,k)-critical covered graph if for any $Q \subseteq V(G)$ with $|Q| = k$, $G - Q$ is a fractional $[a,b]$-covered graph. In particular, a fractional (a,b,k)-critical covered graph is a fractional (r,k)-critical covered graph if $a = b = r$.

Kawarabayashi and Ozeki [8] discussed some problems on 2-factors in graphs. Chen [1] and Matsuda [15] studied the existence of $[2,b]$-factors in graphs. Wang and Zhang [18, 19], Wu [22], Zhou, Bian and Pan [32], Zhou, Sun and Liu [36], Zhou and Bian [31] derived some sufficient conditions for graphs to admit $[1,2]$-factors. Kouider and Lonc [9] investigated the relationship between stability number and $[a,b]$-factors of graphs. For some recent advances on the problem of factors in graphs, we refer to Nenadov [16], Chiba [2], Zhou and Liu [33], Zhou [29], Sun and Zhou [17], Katerinis [7], Zhou and Liu [34], Zhou, Wu and Bian [37], Zhou, Wu and Xu [39], Wang and Zhang [20]. Ma and Liu [14] presented an isolated toughness condition for graphs having fractional 2-factors. Katerinis [6] showed some results on fractional r-factors in regular graphs. Liu and Zhang [12] investigated the existence of fractional r-factors in graphs. More recently related results on fractional factors in graphs can be referred to Zhou [27,30], Liu, Yu and Zhang [11], Gao, Guirao and Chen [3], Gao, Wang and Dimitrov [5], Gao, Liang and Chen [4], Wang and Zhang [21], Zhou, Liu and Xu [35]. Yuan and Hao [24] got a degree condition for a graph to be a fractional $[a,b]$-covered graph. Yuan and Hao [25] put forward two sufficient conditions for the existence of fractional $[a,b]$-covered graphs. Lv [13] presented a degree condition for the existence of fractional (a,b,k)-critical covered graphs. Zhou, Wu and Liu [38] derived an independence number and connectivity condition for the existence of fractional (a,b,k)-critical covered graphs. Zhou [26, 28] claimed two neighborhood conditions for a graph being a fractional (a,b,k)-critical covered graph.

Motivated by above results, we derive an isolated toughness condition for a graph being a fractional $(2,b,k)$-critical covered graph, which will be shown in Section 2.

2. MAIN RESULT AND ITS PROOF

In order to verify our main result in this paper, we first present the following lemmas.

LEMMA 2.1 ([10]). *Let $0 \le a \le b$ be two integers. Then a graph G is a fractional $[a,b]$-covered graph if and only if*

$$a|Y| - d_{G-X}(Y) \le b|X| - \varepsilon(X,Y)$$

for every $X \subseteq V(G)$, where $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \le a\}$ and $\varepsilon(X,Y)$ is defined by

$$\varepsilon(X,Y) = \begin{cases} 2, & \text{if } X \text{ is not independent}, \\ 1, & \text{if } X \text{ is independent and there is an edge joining} \\ & X \text{ and } V(G) \setminus (X \cup Y), \text{ or there is an edge } e = xy \\ & \text{joining } X \text{ and } Y \text{ such that } d_{G-X}(y) = a \text{ for } y \in Y, \\ 0, & \text{otherwise}. \end{cases}$$

The following lemma is equivalent to Lemma 2.1.

LEMMA 2.2. *Let $0 \le a \le b$ be two integers. Then a graph G is a fractional $[a,b]$-covered graph if and only if*

$$\sum_{j=0}^{a-1} (a-j) p_j(G-X) \le b|X| - \varepsilon(X,Y)$$

for every $X \subseteq V(G)$, where $Y = \{x : x \in V(G) \setminus X, d_{G-X}(x) \leq a\}$ and $\varepsilon(X,Y)$ is same as that of Lemma 2.1.

Next, we show our main result in this paper.

THEOREM 2.1. *Let b and k be nonnegative integers such that $b \geq 2 + \frac{k}{2}$, and let G be a graph. If $\delta(G) \geq 3 + k$ and*

$$I(G) \geq 1 + \frac{k+2}{b-1},$$

then G is a fractional $(2,b,k)$-critical covered graph.

Proof. Theorem 2.1 clearly holds for a complete graph. In what follows, we consider the case when G is not a complete graph.

Let $W \subseteq V(G)$ with $|W| = k$. We write $H = G - W$. In order to verify Theorem 2.1, it suffices to show that H is a fractional $[2,b]$-covered graph. To the contrary, we assume that H is not a fractional $[2,b]$-covered graph. Then it follows from Lemma 2.2 that

$$2p_0(H-X) + p_1(H-X) > b|X| - \varepsilon(X,Y) \tag{1}$$

for some $X \subseteq V(H)$, where $Y = \{x : x \in V(H) \setminus X, d_{H-X}(x) \leq 2\}$.

If $|X| \leq 1$, then by (1) and $\varepsilon(X,Y) \leq |X|$, we have

$$2p_0(H-X) + p_1(H-X) > b|X| - \varepsilon(X,Y) \geq b|X| - |X| = (b-1)|X| \geq 0. \tag{2}$$

Moreover, it follows from $\delta(G) \geq 3+k$, $H = G - W$ with $|W| = k$, and $|X| \leq 1$ that

$$p_0(H-X) = p_1(H-X) = 0,$$

which contradicts (2). Henceforth, we shall consider the case when $|X| \geq 2$.

Note that $\varepsilon(X,Y) \leq 2$. From (1), we get

$$2p_0(H-X) + p_1(H-X) > b|X| - \varepsilon(X,Y) \geq b|X| - 2. \tag{3}$$

CLAIM 1. $\frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} < 1 + \frac{k+2}{b-1}$.

Proof. According to (3), $|X| \geq 2$ and $b \geq 2 + \frac{k}{2}$, we have

$$\begin{aligned}
\frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} &= 1 + \frac{k+|X|-p_0(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} \\
&\leq 1 + \frac{k+|X|}{p_0(H-X)+\frac{1}{2}p_1(H-X)} \\
&< 1 + \frac{k+|X|}{\frac{1}{2}b|X|-1} \\
&= 1 + \frac{2}{b} + \frac{2k+\frac{4}{b}}{b|X|-2} \\
&\leq 1 + \frac{2}{b} + \frac{2k+\frac{4}{b}}{2b-2} \\
&= 1 + \frac{k+2}{b-1}.
\end{aligned}$$

Claim 1 is proved. □

Let $Q = \{x : x \in V(H) \setminus X, d_{H-X}(x) = 1\}$. Then $|Q| = p_1(H-X)$. Further, we write

$$E(Q) = \{e = xy : x, y \in Q\},$$

$$E(Q,N_{H-X}(Q)) = \{e = xy : x \in Q, y \in N_{H-X}(Q) \setminus Q\},$$
$$D = \{x : x \in Q \cap N_{H-X}(Q)\}.$$

Let $M(Q) = \{x_1y_1, x_2y_2, \cdots, x_ty_t\}$ be a maximum matching in $G[Q]$. Put

$$Q_{\frac{1}{2}} = \{x_i : 1 \le i \le t\}.$$

The following proof will be divided into four cases.

Case 1. $E(Q) = \emptyset$ and $E(Q,N_{H-X}(Q)) = \emptyset$.

In this case, it is obvious that $Q = \emptyset$. Hence, we derive $p_1(H-X) = 0$. Combining this with (3) and $|X| \ge 2$, we possess

$$i(H-X) = p_0(H-X) > \frac{1}{2}b|X| - 1 \ge b-1 \ge 1 \tag{4}$$

Note that $H = G - W$ with $|W| = k$. Thus, $i(G-W\cup X) = i(H-X) > 1$. Using (4), $|X| \ge 2$ and $I(G) \ge 1+\frac{k+2}{b-1}$, we have

$$\begin{aligned}
1+\frac{k+2}{b-1} &\le I(G) \le \frac{|W\cup X|}{i(G-W\cup X)} = \frac{k+|X|}{i(H-X)} \\
&< \frac{k+|X|}{\frac{1}{2}b|X|-1} = \frac{2}{b} + \frac{2k+\frac{4}{b}}{b|X|-2} \\
&\le \frac{2}{b} + \frac{2k+\frac{4}{b}}{2b-2} = \frac{k+2}{b-1} \\
&< 1+\frac{k+2}{b-1},
\end{aligned}$$

which is a contradiction.

Case 2. $E(Q) = \emptyset$ and $E(Q,N_{H-X}(Q)) \ne \emptyset$.

Obviously, $|N_{H-X}(Q)| \le p_1(H-X)$. Then using (3) and $|X| \ge 2$, we get

$$\begin{aligned}
i(G-W\cup X\cup N_{H-X}(Q)) &= i(H-X\cup N_{H-X}(Q)) \ge i(H-X)+p_1(H-X) \\
&\ge \frac{1}{2}(2i(H-X)+p_1(H-X)) = \frac{1}{2}(2p_0(H-X)+p_1(H-X)) \\
&> \frac{1}{2}b|X|-1 \ge b-1 \ge 1.
\end{aligned} \tag{5}$$

It follows from (3), (5), $|X| \ge 2$, $b \ge 2+\frac{k}{2}$ and $I(G) \ge 1+\frac{k+2}{b-1}$ that

$$\begin{aligned}
1+\frac{k+2}{b-1} &\le I(G) \le \frac{|W\cup X\cup N_{H-X}(Q)|}{i(G-W\cup X\cup N_{H-X}(Q))} \\
&= \frac{k+|X|+|N_{H-X}(Q)|}{i(G-W\cup X\cup N_{H-X}(Q))} \\
&\le \frac{k+|X|+p_1(H-X)}{i(H-X)+p_1(H-X)} \\
&= \frac{k+|X|+p_1(H-X)}{p_0(H-X)+p_1(H-X)} \\
&= 1+\frac{k+|X|-p_0(H-X)}{p_0(H-X)+p_1(H-X)} \\
&= 1+\frac{k+|X|-p_0(H-X)}{2p_0(H-X)+p_1(H-X)-p_0(H-X)}
\end{aligned}$$

$$
\begin{aligned}
&\leq 1+\frac{k+|X|}{2p_0(H-X)+p_1(H-X)}\\
&< 1+\frac{k+|X|}{b|X|-2}\\
&= 1+\frac{1}{b}+\frac{k+\frac{2}{b}}{b|X|-2}\\
&\leq 1+\frac{1}{b}+\frac{k+\frac{2}{b}}{2b-2}\\
&= 1+\frac{1}{b}+\frac{bk+2}{2b(b-1)}\\
&< 1+\frac{2}{b}+\frac{bk+2}{b(b-1)}\\
&= 1+\frac{k+2}{b-1},
\end{aligned}
$$

which is a contradiction.

Case 3. $E(Q)\neq\emptyset$ and $E(Q,N_{H-X}(Q))=\emptyset$.

In this case, we easily see that $|Q_{\frac{1}{2}}|=\frac{1}{2}p_1(H-X)$ and

$$
\begin{aligned}
i(G-W\cup X\cup Q_{\frac{1}{2}}) &= i(H-X\cup Q_{\frac{1}{2}})\geq i(H-X)+\frac{1}{2}p_1(H-X)\\
&= p_0(H-X)+\frac{1}{2}p_1(H-X)>\frac{1}{2}b|X|-1\geq b-1\geq 1 \qquad (6)
\end{aligned}
$$

by (3), $|X|\geq 2$ and $b\geq 2+\frac{k}{2}\geq 2$.

According to (6), Claim 1 and $I(G)\geq 1+\frac{k+2}{b-1}$, we obtain

$$
\begin{aligned}
1+\frac{k+2}{b-1} &\leq I(G)\leq\frac{|W\cup X\cup Q_{\frac{1}{2}}|}{i(G-W\cup X\cup Q_{\frac{1}{2}})}\\
&= \frac{k+|X|+|Q_{\frac{1}{2}}|}{i(G-W\cup X\cup Q_{\frac{1}{2}})}\\
&\leq \frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)}\\
&< 1+\frac{k+2}{b-1},
\end{aligned}
$$

which is a contradiction.

Case 4. $E(Q)\neq\emptyset$ and $E(Q,N_{H-X}(Q))\neq\emptyset$.

Subcase 4.1. $|E(Q)|>|E(Q,N_{H-X}(Q))|$.

Let $N=(N_{H-X}(Q)\setminus D)\cup Q'$, where $Q'\subseteq Q_{\frac{1}{2}}$. Then there exists $Q'\subseteq Q_{\frac{1}{2}}$ such that $|N|=\lfloor\frac{1}{2}p_1(H-X)\rfloor$ and $i(H-X\cup N)\geq i(H-X)+\frac{1}{2}p_1(H-X)=p_0(H-X)+\frac{1}{2}p_1(H-X)$. Thus, we have

$$
\begin{aligned}
i(G-W\cup X\cup N) &= i(H-X\cup N)\geq p_0(H-X)+\frac{1}{2}p_1(H-X)\\
&> \frac{1}{2}b|X|-1\geq b-1\geq 1 \qquad (7)
\end{aligned}
$$

by (3), $|X| \geq 2$ and $b \geq 2+\frac{k}{2} \geq 2$. Using (7), Claim 1 and $I(G) \geq 1+\frac{k+2}{b-1}$, we get

$$\begin{aligned} 1+\frac{k+2}{b-1} &\leq I(G) \leq \frac{|W \cup X \cup N|}{i(G-W \cup X \cup N)} \\ &= \frac{k+|X|+\lfloor \frac{1}{2}p_1(H-X)\rfloor}{i(G-W\cup X\cup N)} \\ &\leq \frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} \\ &< 1+\frac{k+2}{b-1}, \end{aligned}$$

which is a contradiction.

Subcase 4.2. $|E(Q)| \leq |E(Q, N_{H-X}(Q))|$.

Let $N = Q_{\frac{1}{2}} \cup Q'$, where $Q' \subseteq N_{H-X}(Q) \setminus D$. Then there exists $Q' \subseteq N_{H-X}(Q) \setminus D$ such that $|N| = \lfloor \frac{1}{2}p_1(H-X)\rfloor$ and $i(H-X\cup N) \geq i(H-X)+\frac{1}{2}p_1(H-X) = p_0(H-X)+\frac{1}{2}p_1(H-X)$. Thus, we derive

$$\begin{aligned} i(G-W\cup X\cup N) = i(H-X\cup N) &\geq p_0(H-X)+\frac{1}{2}p_1(H-X) \\ &> \frac{1}{2}b|X|-1 \geq b-1 \geq 1 \end{aligned} \tag{8}$$

by (3), $|X| \geq 2$ and $b \geq 2+\frac{k}{2} \geq 2$. It follows from (8), Claim 1 and $I(G) \geq 1+\frac{k+2}{b-1}$ that

$$\begin{aligned} 1+\frac{k+2}{b-1} &\leq I(G) \leq \frac{|W \cup X \cup N|}{i(G-W \cup X \cup N)} \\ &= \frac{k+|X|+\lfloor \frac{1}{2}p_1(H-X)\rfloor}{i(G-W\cup X\cup N)} \\ &\leq \frac{k+|X|+\frac{1}{2}p_1(H-X)}{p_0(H-X)+\frac{1}{2}p_1(H-X)} \\ &< 1+\frac{k+2}{b-1}, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 2.1. □

If $k = 0$ in Theorem 2.1, then we get the following corollary.

COROLLARY 2.1. *Let $b \geq 2$ be an integer, and let G be a graph. If $\delta(G) \geq 3$ and*

$$I(G) \geq 1+\frac{2}{b-1},$$

then G is a fractional $[2,b]$-covered graph.

If $b = 2$ in Corollary 2.1, then we get the following corollary.

COROLLARY 2.2. *Let G be a graph. If $\delta(G) \geq 3$ and $I(G) \geq 3$, then G is a fractional 2-covered graph.*

3. REMARK

Next, we show that the condition $I(G) \geq 1+\frac{k+2}{b-1}$ in Theorem 2.1 is best possible in some sense, namely, it cannot be replaced by $I(G) \geq 1+\frac{k+2}{2b-1}$. To check this, we consider a graph G constructed from K_{k+2}, $(2b-1)K_1$

and K_{2b-1} as follows: letting $V((2b-1)K_1)=\{x_1,x_2,\cdots,x_{2b-1}\}$ and $V(K_{2b-1})=\{y_1,y_2,\cdots,y_{2b-1}\}$, where $k\geq 0$ and $b\geq 2+\frac{k}{2}$ are two integers. We first join every vertex x_i to the vertex y_i with the same subscript i, and then join every vertex x_i to all the vertices of K_{k+2}. Then we easily see

$$I(G)=\frac{|Q|}{i(G-Q)}=\frac{k+2+2b-1}{2b-1}=1+\frac{k+2}{2b-1},$$

where $Q=V(K_{k+2})\cup V(K_{2b-1})$.

Set $D=V(K_k)\subseteq V(K_{k+2})$, $G'=G-D$ and $X=V(K_{k+2})\setminus V(K_k)$. Then $\varepsilon(X,Y)=2$ since X is not an independent set, where $Y=\{x:x\in V(G')\setminus X, d_{G'-X}(x)\leq 2\}$. Hence, we deduce

$$2p_0(G'-X)+p_1(G'-X)=2b-1>2b-2=b|X|-\varepsilon(X,Y).$$

In light of Lemma 2.2, G' is not fractional $[2,b]$-covered, that is, G is not fractional $(2,b,k)$-critical covered.

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