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TWO-DISTANCE VERTEX-DISTINGUISHING TOTAL COLORING OF SUBCUBIC GRAPHS

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Abstract. A 2-distance vertex-distinguishing total coloring of graph G is a proper total coloring of G such that any pair of vertices at distance of two have distinct sets of colors. The 2-distance vertex-distinguishing total chromatic number $\chi''_{d2}(G)$ of G is the minimum number of colors needed for a 2-distance vertex-distinguishing total coloring of G. In this paper, it's proved that if G is a subcubic graph, then $\chi''_{d2}(G) \leq 7$.

Key words*:* 2-distance vertex distinguishing total coloring, total coloring, subcubic graph.
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1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree and maximum degree of graph G, respectively. The *distance* between two vertices u and v, denoted by $d(u,v)$, is the length of the shortest path connecting them. The graph G is denoted as *Cubic* if it is a 3-regular graph, and *subcubic* if $\Delta(G) \leq 3$. Let C_n be a cycle whose length is n.

A *total-k-coloring* of graph G is a mapping ϕ: $V(G) \cup E(G) \rightarrow \{1,2,\cdots,k\}$ so that $\phi(x) \neq \phi(y)$ for any pair of adjacent or incident elements $x,y \in V(G) \cup E(G)$. The *total chromatic index* $\chi''(G)$ of graph G is defined as the smallest integer k to make sure a proper total- k-coloring exist in G. The total coloring of graph G was introduced by Behzad [1] and independently by Vizing [2], and each raised the following conjecture:

CONJECTURE 1. *Every simple graph G has $\chi''(G) \leq \Delta(G)+2$.*

So far Conjecture 1 remains open. A well-known upper bound of $\chi''(G)$ for simple graph G may be $\Delta(G)+10^{26}$, by Molloy and Reed [3].

For a total-k-coloring ϕ of G, we use $C_\phi(v) = \{\phi(v)\} \cup \{\phi(xv) | xv \in E(G)\}$ to denote the set of colors assigned to a vertex v and those edges incident to v. The *neighbor-distinguishing total chromatic index* $\chi''_a(G)$ of G is defined as the smallest integer k for which G can be totally-k-colored by using k colors so that $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v.

Zhang et al. [4] studied neighbor-distinguishing total coloring of cycles, wheels, trees, complete graphs and complete bipartite graphs, and proposed the following conjecture:

CONJECTURE 2. *Every graph G with $|V(G)| \geq 2$ has $\chi''_a(G) \leq \Delta(G)+3$.*

Wang [5] and Chen [6], independently, proved this Conjecture holds for graphs with $\Delta(G) \leq 3$. Lu et al. [7] proved this Conjecture holds for graphs with $\Delta(G) = 4$, and Papaioannou et al. [8] verified this Conjecture for 4-regular graphs. Applying a probabilistic analysis, Coker et al. [9] verified that $\chi''_a(G) \leq \Delta(G)+C$, where C

is a constant. Huang et al. [10] proved that $\chi_a''(G) \le 2\Delta(G)$ for any graph G with $\Delta(G) \ge 3$. The conjecture is still open for planar graphs. Furthermore, Chang et al. [11] proved that $\chi_a''(G) \le \Delta(G)+3$ for every planar graph G with $\Delta(G) \ge 8$.

The 2-*distance vertex-distinguishing total coloring* of graph G is a proper total coloring of G such that $C_\phi(u) \ne C_\phi(v)$ for any pair of vertices u and v with $d(u,v)=2$. The 2-*distance vertex-distinguishing total chromatic index* $\chi_{d2}''(G)$ of graph G is the smallest integer k such that G has a 2-distance vertex-distinguishing total coloring using k colors.

Hu et al. [12] studied 2-distance vertex-distinguishing total coloring of paths, cycles, wheels, trees, unicycle graphs, $P_m \times P_n$ and $C_m \times P_n$. Then they proposed the following conjecture:

CONJECTURE 3. *Every simple graph G has* $\chi_{d2}''(G) \le \Delta(G)+3$.

In this paper, we will prove $\chi_{d2}''(G) \le 7$ for any subcubic graphs.

2. MAIN RESULTS

Before showing our main result, we introduce a few of concepts and notation. $N_G(v)$ denotes the set of neighbors of the vertex v and $d_G(v)$ denotes the degree of the vertex v in G. A vertex of degree k is called k-vertex. Similarly, a vertex of degree at least k(at most k) is called k^+-vertex(k^--vertex). A 3-vertex v is called a 3_i-vertex if v is adjacent to exactly i 2-vertices for $0 \le i \le 3$. Let $\chi(G)$ denote the chromatic index of G, which is the least integer k for which G has a vertex coloring using k colors such that any two adjacent vertices get distinct colors.

In what follows, a 2-distance vertex-distinguishing total k-coloring of G is shortly written as a 2DVDT-k-coloring. Two vertices $u,v \in V(G)$ with $d(u,v)=2$ are called a conflict with respect to the coloring ϕ if $C_\phi(u)=C_\phi(v)$. Otherwise, they are called compatible. For a subgraph G' of G and a 2DVDT-coloring ϕ of G', we say that ϕ is a *legal coloring* of G' for short.

The proof of main result is based on the following facts:

LEMMA 1 ([12]). *Let G be a simple graph with* $\Delta(G) \le 2$, *then* $\chi_{d2}''(G) \le 5$.

LEMMA 2 ([13]). *If G is a connected graph and is neither an odd cycle nor a complete graph, then* $\chi(G) \le \Delta$.

LEMMA 3 ([14]). *Every connected cubic graph G without cut edges can be edge-partitioned into a perfect matching and a class of cycles.*

THEOREM 1. *If G be a subcubic graph, then* $\chi_{d2}''(G) \le 7$.

Proof. The proof is by contradiction. Let G be a minimum counterexample in the Theorem 1 to make its edges $E(G)$ as small as possible. Obviously, G is a connected graph and $\chi_{d2}''(G) > 7$. However, if $|E(H)| < |E(G)|$ for any graph H, then $\chi_{d2}''(H) \le 7$. Assume that $C=\{1,2,\cdots,7\}$ is a color set and ϕ is a 2DVDT-7-coloring. For the sake of simplicity in the following proof, we write $C_\phi(v)$ as $C(v)$ for a vertex $v \in V(G')$.

If $\Delta(G) \le 2$, then graph G is a cycle or a path. From Lemma 1 it is follows that $\chi_{d2}''(G) \le 5$. So, assume that $\Delta(G)=3$. To complete the proof, we need to establish a series of auxiliary claims.

CLAIM 1. $\delta(G) \ge 2$.

Proof. Assume to the contrary that G contains a 1-vertex v. Let u be the neighbor of v. Consider the subgraph $G'=G-\{v\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. We have two possibilities:

Case 1. $d_G(u) = 2$.

Let u_1 be the neighbor of u other than v. It suffices to color uv with color $a \in C \setminus \{\phi(u), \phi(uu_1)\}$ so that u is compatible with the neighbors of u_1, then color v with color in $C \setminus \{\phi(u), \phi(uu_1), a\}$.

Case 2. $d_G(u) = 3$.

Let u_1, u_2 be the neighbors of u other than v. If at least one of u_1 and u_2 is a 2^--vertex, then it suffices to color uv with color $b \in C \setminus \{\phi(u), \phi(uu_1), \phi(uu_2)\}$ so that u is compatible with the neighbors of u_1 and u_2, and color v with color in $C \setminus \{\phi(u), \phi(uu_1), \phi(uu_2), b\}$. Otherwise, $d_G(u_1) = d_G(u_2) = 3$. Let $N_G(u_1) = \{u, w_1, w_2\}$ and $N_G(u_2) = \{u, w_3, w_4\}$. Furthermore, assume that $\phi(uu_1) = 1, \phi(uu_2) = 2$ and $\phi(u) = 3$. If uv cannot not be legally colored, suppose without loss of generality that $C(w_i) = \{1,2,3,i+3\}$ for $i = 1,2,3,4$. It suffices to recolor u with color $a \in \{4,5,6,7\} \setminus \{\phi(u_1), \phi(u_2)\}$, color uv with color in $\{4,5,6,7\} \setminus \{a\}$, and color v with 1. Thus, G has a 2DVDT-7-coloring, a contradiction. □

CLAIM 2. *G does not contain a 3-cycle $v_1v_2v_3v_1$ satisfying one of the following:*
(1) $d_G(v_2) = d_G(v_3) = 2$ and $d_G(v_1) = 3$.
(2) $d_G(v_2) = d_G(v_3) = 3$ and $d_G(v_1) = 2$.

Proof. (1) Assume to the contrary that G contains a 3-cycle $v_1v_2v_3v_1$ satisfying $d_G(v_2) = d_G(v_3) = 2$ and $d_G(v_1) = 3$. Let u_1 be the neighbor of v_1 other than v_1 and v_3. Consider the subgraph $G' = G - \{v_2v_3\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. It suffices to color v_2v_3 with color in $C \setminus (C(v_2), C(v_3))$ such that v_2 and v_3 are compatible with u_1. Thus, G has a 2DVDT-7-coloring, a contradiction.

(2) Assume to the contrary that G contains a 3-cycle $v_1v_2v_3v_1$ satisfying $d_G(v_2) = d_G(v_3) = 3$ and $d_G(v_1) = 2$. Let u_i be the neighbor of v_i other than v_1. Consider the subgraph $G' = G - \{v_1v_3\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(v_1u_1) = 1, \phi(v_2v_2) = 2, \phi(v_2u_2) = 3$ and $\phi(v_2) = 4$. It is easy to notice that $\phi(v_3) \notin \{2,4\}$. Then we use color 2 firstly to recolor v_1. We have to handle the following two situations:

Case 1. $d_G(u_2) = 2$.

Firstly, assume that $\phi(v_3) \in \{1,3\}$. It suffices to color v_1v_3 with color in $\{4,5,6,7\} \setminus \{\phi(v_3u_3)\}$ such that G can be legally colored. Next, assume that $\phi(v_3) \in \{5,6,7\}$. Say $\phi(v_3) = 5$ by symmetry. If $\phi(v_3u_3) \in \{1,3\}$, it suffices to color v_1v_3 with color in $\{4,6,7\}$. Otherwise, $\phi(v_3u_3) \in \{4,6,7\}$. Say $\phi(v_3u_3) = 4$ by symmetry. If v_1v_3 cannot be legally colored, assume without loss of generality that $C(u_3') = \{2,4,5,6\}$ and $C(u_3'') = \{2,4,5,7\}$ with $N_G(u_3) = \{v_3, u_3', u_3''\}$. It suffices to color v_1v_3 with 3 and recolor v_1 with color 6 or 7.

Case 2. $d_G(u_2) = 3$.

Firstly, assume that $\phi(v_3) = 1$. It suffices to color v_1v_3 with color in $\{3,4,5,6,7\} \setminus \{\phi(v_3u_3)\}$ such that G can be legally colored. Next, assume that $\phi(v_3) \in \{3,5,6,7\}$. Say $\phi(v_3) = 3$ by symmetry. If $\phi(v_3u_3) = 1$, it suffices to color v_1v_3 with color in $\{4,5,6,7\}$. Otherwise, $\phi(v_3u_3) \in \{4,5,6,7\}$. Say $\phi(v_3u_3) = 4$ by symmetry. If v_1v_3 cannot be legally colored, assume without loss of generality that $C(u_3') = \{2,3,4,5\}, C(u_3'') = \{2,3,4,6\}$ and $C(u_2) = \{2,3,4,7\}$ with $N_G(u_3) = \{v_3, u_3', u_3''\}$. It suffices to recolor v_3 with color $a \in \{1,5,6,7\} \setminus \{\phi(u_3)\}$ and color v_1v_3 with color in $\{5,6,7\} \setminus \{a\}$. □

CLAIM 3. *G does not contain adjacent 2-vertices.*

Proof. Assume to the contrary, G contains adjacent 2-vertices u, v. Let $N_G(u) = \{v, u_1\}$ and $N_G(v) = \{u, v_1\}$. If $d_G(v_1) = 3$, then let $N_G(v_1) = \{v, v_1', v_1''\}$. By Claim 2 and $\Delta(G) = 3$, $vu_1 \notin E(G)$. Let $G' = G - \{u\} + \{u_1v\}$. Then G' has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. We have to handle the following two situations:

Case 1. $d_G(u_1) = 2$.

Let $N_G(u_1) = \{u, u_1'\}$. Based on ϕ, in G, color uu_1 with $\phi(vu_1)$. It suffices to color uv with color $b \in C \setminus \{\phi(v), \phi(vv_1), \phi(uu_1), \phi(u_1u_1')\}$, and color u with color in $C \setminus (C(u_1) \cup \{b\})$.

Case 2. $d_G(u_1) = 3$.

Let $N_G(u_1) = \{u, u_1', u_1''\}$. Assume without loss of generality that $\phi(u_1) = 4, \phi(u_1u_1') = 1, \phi(u_1u_1'') = 2$ and $\phi(vu_1) = 3$. It follows that $\phi(v) \notin \{3,4\}$ and $\phi(vv_1) \neq 3$. Based on ϕ, in G, color uu_1 with 3. We have to handle two possibilities:

• Assume that $\phi(v) \in \{1,2\}$. Say $\phi(v) = 1$ by symmetry. It suffices to color uv with color $b \in \{4,5,6,7\} \setminus \{\phi(vv_1)\}$ such that v is compatible with the neighbours of v_1, and color u with color in $\{5,6,7\} \setminus \{b\}$.

• Assume that $\phi(v) \in \{5,6,7\}$. Say $\phi(v) = 5$ by symmetry. If $\phi(vv_1) \notin \{4,6,7\}$, then it is similar to the former case of $\phi(v) \in \{1,2\}$. Otherwise, $\phi(vv_1) \in \{4,6,7\}$. Say $\phi(vv_1) = 4$ by symmetry. If $d_G(v_1) = 2$ or $d_G(v_1) = 3$ and $C(v_1^{'}) \neq \{4,5,i\}$ for $i = 6,7$. Then it suffices to color uv with $b \in \{6,7\}$, and color u with color in $\{6,7\} \setminus \{b\}$. If $d_G(v_1) = 3$ and $(C(v_1^{'}), C(v_1^{''})) = (\{4,5,6\},\{4,5,7\})$. It suffices to recolor v with color $a \in \{1,2\} \setminus \{\phi(v_1)\}$, and color uv and u with 6 and 7, respectively. □

CLAIM 4. *G does not contain a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_2) = d_G(v_4) = 2$.*

Proof. Assume to the contrary, G contains a 4-cycle $v_1v_2v_3v_4v_1$ such that $d_G(v_2) = d_G(v_4) = 2$. By Claim 2 and Claim 3, $d_G(v_1) = d_G(v_3) = 3$ and $v_1v_3 \notin E(G)$. Then let u_1, u_3 be the neighbors of v_1, v_3 other than v_2 and v_4, respectively. If $d_G(u_1) = d_G(u_3) = 3$, then let $N_G(u_1) = \{v_1, u_1^{'}, u_1^{''}\}$ and $N_G(u_3) = \{v_3, u_3^{'}, u_3^{''}\}$. There are three situations to be handled:

Case 1. $u_1 = u_3$.

Let $G^{'} = G - \{v_2, v_4\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(v_3u_3) = 1, \phi(v_1u_3) = 2, \phi(u_3u_3^{'}) = 4$ and $\phi(u_3) = 4(d_G(u_3) = 2$ is similar). Remove firstly the colors of v_1 and v_3, and use 4 to color v_2, v_4. Next, use 1, 3, 2, 3 to color $v_1, v_1v_4, v_4v_3, v_3v_2$, respectively. Finally, use $b \in \{5,6\}$ to color v_3 to make v_3 compatible with $u_3^{'}$, and color v_1v_2 with color in $\{5,6,7\} \setminus \{b\}$ such that v_1 is compatible with $u_3^{'}$.

Case 2. $u_1u_3 \notin E(G)$ and $u_1 \neq u_3$.

Let $G^{'} = G - \{v_1, v_2, v_3, v_4\} + \{u_1u_3\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(u_1u_3) = 1, \phi(u_3u_3^{'}) = 2, \phi(u_3u_3^{''}) = 3$ and $\phi(u_3) = 4(d_G(u_3) = 2$ is similar). Based on ϕ, in G, use 1 to color u_1v_1, u_3v_3. It is easy to notice that color sets of u_1 and u_3 don't change in G, and $\phi(u_1) \neq 4$. So it suffices to color $v_2, v_2v_3, v_3, v_3v_4, v_4, v_1v_4, v_1$ with 2, 4, 5, 6, 2, 3, 4, respectively. And color v_1v_2 with color in $\{5,6,7\}$ such that G can be legally colored.

Case 3. $u_1u_3 \in E(G)$ and $u_1 \neq u_3$.

Let $G^{'} = G - \{v_2, v_4\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(u_1u_3) = 2, \phi(u_3u_3^{'}) = 3, \phi(u_3v_3) = 1$ and $\phi(u_3) = 4(d_G(u_3) = 2$ is similar). It is easy to notice that $\phi(u_1) \neq 2$ and $\phi(u_1v_1) \neq 2$. Remove the colors of v_1 and v_3. It firstly suffices to color $v_1, v_4, v_3v_4, v_3, v_2$ with color 2, 4, 2, 3, 4, respectively. Nextly, we use $b \in \{5,6,7\}$ to color v_2v_3 such that v_3 is compatible with u_1 and $u_3^{'}$. Finally, we use $a \in \{1,3,5,6,7\} \setminus \{b, \phi(u_1v_1)\}$ to color v_2v_2, and color v_1v_4 with color in $\{3,5,6,7\} \setminus \{a, \phi(v_1u_1)\}$. Obviously, v_1 has at least 4 distinctive color sets. Since v_1 has at most three vertices of conflict, G has a 2DVDT-7-coloring. Thus, a contradiction. □

CLAIM 5. *G does not contain* 3_3*-vertex.*

Proof. Assume to the contrary, G contains a 3_3-vertex v. Let $N_G(v) = \{v_1, v_2, v_3\}$ and $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$. And let u_i be the neighbor of v_i other than v for $i = 1,2,3$. By Claim 3, $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. Let $N_G(u_i) = \{v_i, u_i^{'}, u_i^{''}\}$ for $i = 1,2,3$. By Claim 4, $u_1 \neq u_2$. Consider the subgraph $G^{'} = G - \{v_1\} - \{vv_2\} + \{u_1v_2\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(v_2u_1) = 1, \phi(u_1u_1^{'}) = 2, \phi(u_1u_1^{''}) = 3$ and $\phi(u_1) = 4$. Based on ϕ, in G, we color u_1v_1 with 1. Remove the color of v. There are two possibilities to be handled:

Case 1. $\phi(vv_3) \neq 1$.

We color vv_2 with 1. It is easy to notice that the color set of v_2 doesn't change in G. We have to handle three situations:

(1) $\phi(vv_3) \in \{2,3\}$. Say $\phi(vv_3) = 2$ by symmetry. Firstly, we use 5 to color v_1 and $b_1 \in \{4,6,7\}$ to color vv_1 such that v_1 is compatible with v_2. It follows that vv_1 can be colored with at least two colors. Next, we have two possibilities need to be handled according to the color of v_3:

• $\phi(v_3) \in \{1,4,5,6,7\}$. It suffices to color v with color in $\{3,4,6,7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v

has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

• $\phi(v_3) = 3$. We color v with $b_2 \in \{4,6,7\} \setminus \{\phi(v_2), b_1\}$. If $C(v_2) = \{1,5,i\}, \phi(v_2) = i$, and $C(u_3) = \{1,2,m,p\}$ for $i \in \{4,6,7\}$ and $m, p \in \{4,6,7\} \setminus \{i\}$. Then G cannot be legally colored. Thus, it suffices to recolor v_1, vv_1, v with 6, 4, 7, respectively. Otherwise, we are done.

(2) $\phi(vv_3) = 4$. We have three possibilities need to be handled according to the color of v_3:

• $\phi(v_3) \in \{5,6,7\}$. Let $\phi(v_3) = a$. It suffices to color v_1 with a, vv_1 with $b_1 \in \{5,6,7\} \setminus \{a\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{2,3,5,6,7\} \setminus \{\phi(v_2), a, b_1\}$.

• $\phi(v_3) = 1$. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{6,7\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{2,3,6,7\} \setminus \{\phi(v_2), b_1\}$.

• $\phi(v_3) \in \{2,3\}$. Say $\phi(v_3) = 2$ by symmetry. It suffices to color v_1 with 2, vv_1 with $b_1 \in \{5,6,7\} \setminus \{a\}$ such that v_1 is compatible with v_2 and $u_1^{'}$, and color v with $b_2 \in \{3,5,6,7\} \setminus \{\phi(v_2), b_1\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

(3) $\phi(vv_3) \in \{5,6,7\}$. Say $\phi(vv_3) = 5$ by symmetry. We have four possibilities need to be handled according to the color of v_3:

• $\phi(v_3) \in \{6,7\}$. Say $\phi(v_3) = 6$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{4,7\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{2,3,4,7\} \setminus \{\phi(v_2), b_1\}$.

• $\phi(v_3) = 1$. It suffices to color v_1 with 6, vv_1 with $b_1 \in \{4,7\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{2,3,4,7\} \setminus \{\phi(v_2), b_1\}$.

• $\phi(v_3) \in \{2,3\}$. Say $\phi(v_3) = 2$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{4,6,7\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{3,4,6,7\} \setminus \{\phi(v_2), b_1\}$.

• $\phi(v_3) = 4$. It suffices to color v_1 with 6, vv_1 with $b_1 \in \{4,7\}$ such that v_1 is compatible with v_2, and color v with $b_2 \in \{2,3,7\} \setminus \{\phi(v_2), b_1\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

Case 2. $\phi(vv_3) = 1$.

It is easy to notice that $\phi(v_2) \notin \{1,4\}$ and $\phi(v_2u_2) \neq 1$. Remove the color of v_2. We have three situations to handle according to the value of $\phi(v_3)$:

(1) $\phi(v_3) = 4$. There are two possibilities to be handled:

• $\phi(v_2u_2) \neq 4$. Firstly, assume that $\phi(u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 5, respectively. Then use $b_1 \in \{2,3,5,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2, $b_2 \in \{6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{2,3,6,7\} \setminus \{b_1, b_2\}$ to color v. Next, assume that $\phi(u_2) = 4$. It firstly suffices to color v_2, v_1 with 5, respectively. Then use color $b_1 \in \{2,3,4,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{2,3,6,7\} \setminus \{b_1, b_2\}$ to color v.

• $\phi(v_2u_2) = 4$. Firstly, assume that $\phi(u_2) \neq 2$. It firstly suffices to color v_2, v_1 with 2, 5, respectively. Then use color $b_1 \in \{3,5,6,7\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{3,6,7\} \setminus \{b_1, b_2\}$ to color v. Next, assume that $\phi(u_2) = 2$. It firstly suffices to color v_2, v_1 with 5, respectively. Then use color $b_1 \in \{2,3,6,7\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{2,3,6,7\} \setminus \{b_1, b_2\}$ to color v.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

(2) $\phi(v_3) \in \{2,3\}$. Say $\phi(v_3) = 2$ by symmetry. There are two possibilities to be handled:

• $\phi(u_2) \neq 1$. By $\phi(v_2u_2) \neq 1$, it firstly suffices to color v_2, v_1 with 1, 5, respectively. Then use color $b_1 \in \{2,3,4,5,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}, u_2^{''}$ and v_3, $b_2 \in \{4,6,7\} \setminus \{b_1\}$ to color vv_1 such that v_1 is compatible with v_2, and $b_3 \in \{3,4,6,7\} \setminus \{b_1, b_2\}$ to color v.

• $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 5, respectively. Then use color $b_1 \in \{2,3,5,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{4,6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{3,6,7\} \setminus \{b_1, b_2\}$ to color v. Next, assume that $\phi(v_2u_2) = 4$. It firstly suffices to color v_2, v_1 with 2, 5, respectively. Then use color $b_1 \in \{3,5,6,7\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{4,6,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{3,4,6,7\} \setminus \{b_1, b_2\}$ to color v.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

(3) $\phi(vv_3) \in \{5,6,7\}$. Say $\phi(vv_3) = 5$ by symmetry. There are two possibilities to be handled:

• $\phi(u_2) \neq 1$. By $\phi(v_2u_2) \neq 1$, it firstly suffices to color v_2, v_1 with 1, 6, respectively. Then use color $b_1 \in \{2,3,4,5,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}, u_2^{''}$ and v_3, $b_2 \in \{4,7\} \setminus \{b_1\}$ to color vv_1 such that v_1 is compatible with v_2, and $b_3 \in \{2,3,4,7\} \setminus \{b_1, b_2\}$ to color v. It is easy to notice that vv_2 can be colored with at least two colors, then vv_1 can be colored with at least one color. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$.

• $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) \neq 4$. It firstly suffices to color v_2, v_1 with 4, 6, respectively. Then use color $b_1 \in \{2,3,5,6,7\} \setminus \{\phi(v_2u_2)\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{4,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{2,3,7\} \setminus \{b_1, b_2\}$ to color v. It is easy to notice that v_1, v_2 and v_3 do not conflict with each other. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3$. Next, assume that $\phi(v_2u_2) = 4$. We firstly color v_2, v_1 with 2, 6, respectively. Then use color $b_1 \in \{3,5,6,7\}$ to color vv_2 such that v_2 is compatible with $u_2^{'}$ and $u_2^{''}$, $b_2 \in \{4,7\} \setminus \{b_1\}$ to color vv_1, and $b_3 \in \{3,4,7\} \setminus \{b_1, b_2\}$ to color v. It is easy to notice that v_1, v_2 and v_3 do not conflict with each other. If vv_2 can only be colored by 3 and 7, and $C(u_3) = \{1,3,4,7\}$. Then $C(u_2) \neq \{1,3,4,7\}$ and G can not be legally colored. Thus, it suffices to recolor v_1, vv_1, v with 7, 6, 4, respectively. Otherwise, we are done. □

CLAIM 6. *G does not contain 3_2-vertex.*

Proof. Assume to the contrary, G contains a 3_2-vertex v. Let $N_G(v) = \{v_1, v_2, v_3\}$, $d_G(v_1) = d_G(v_2) = 2$ and let u_i be the neighbor of v_i other than v for $i = 1,2$. By Claim 3 and Claim 5, $d_G(u_1) = d_G(u_2) = d_G(v_3) = 3$. Let $N_G(u_i) = \{v_i, u_i^{'}, u_i^{''}\}$ for $i = 1,2$, and $N_G(v_3) = \{v, u_3, u_4\}$. Assume without loss of generality that $2 \leq d_G(u_1^{'}) \leq d_G(u_2^{'}) \leq 3$ and $d_G(u_1^{''}) = d_G(u_2^{''}) = 3$ by Claim 5. By Claim 4, $u_1 \neq u_2$. Then consider the subgraph $G^{'} = G - \{v_1\} - \{vv_2\} + \{u_1v_2\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(v_2u_1) = 1, \phi(u_1u_1^{'}) = 2, \phi(u_1u_1^{''}) = 3$ and $\phi(u_1) = 4$. It easy to notice that $\phi(v_2) \notin \{1,4\}$ and $\phi(v_2u_2) \neq 1$. Based on ϕ, in G, we color u_1v_1 with 1. Remove the color of v. There are two possibilities to be handled:

Case 1. $\phi(vv_3) \neq 1$.

We color vv_2 with 1. It is easy to notice that the color set of v_2 doesn't change in G. We have to handle three situations:

(1) $\phi(vv_3) \in \{2,3\}$. Say $\phi(vv_3) = 2$ by symmetry. It suffice to use 3 to color v_1, $b_1 \in \{4,5,6,7\}$ to color vv_1 such that v_1 is compatible with v_2, and $b_3 \in \{4,5,6,7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$ to color v. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3,4$.

(2) $\phi(vv_3) = 4$. The following four possibilities are discussed:

• Assume that $\phi(v_2) = 2$ and $(C(u_2), C(u_3), C(u_4)) = (\{1,4,5,6\}, \{1,4,5,7\}, \{1,4,6,7\})$. Then it respectively suffices to color v_1, vv_1 with 5, 3, and color v with $\{5,6,7\} \setminus \{\phi(v_3)\}$.

• Assume that $\phi(v_2) \in \{5,6,7\}$ and $\phi(v_3) = 2$. Say $\phi(v_2) = 5$ by symmetry. If $\phi(v_2u_2) = 3$ and $(C(u_2), C(u_3), C(u_4)) = (\{1,4,3,5\}, \{1,4,3,7\}, \{1,4,6,7\})$. We recolor vv_2 with 2 or 6 such that v_2 is compatible with $u_2^{'}$, and color v_1, vv_1, v with 6, 7, 3, respectively. If $\phi(v_2u_2) = 3, C(u_2) \neq \{1,3,4,i\}$ for $i = 5,7$, and $\{1,4,6,7\} \in (C(u_3), C(u_4))$. We respectively color v_1, v with 6, 3, and color vv_1 with 5 or 7. If $\phi(v_2u_2) = 3$ and $\{1,4,6,7\} \notin (C(u_3), C(u_4))$. We color v_1, vv_1, v with 5, 6, 7. If $\phi(v_2u_2) \neq 3$ and $(C(u_2), C(u_3), C(u_4)) = (\{1,4,5,6\}, \{1,4,5,7\}, \{1,4,6,7\})$. We color v_1, vv_1, v with 5, 3, 6. Otherwise, it suffices to color v_1 with 3, vv_1 with $b_1 \in \{5,6,7\}$ and v with color in $\{6,7\} \setminus \{b_1\}$.

• Assume that $\phi(v_2), \phi(v_3) \in \{5,6,7\}$. Say $\phi(v_2) = 5, \phi(v_3) = 6$ by symmetry. It suffices to color v_1 with 3, vv_1 with $b_1 \in \{5,6,7\}$ such that v_1 is compatible with v_2, and color v with color in $\{2,7\} \setminus \{b_1\}$.

• Otherwise, we color v_1 with 3, vv_1 with $b_1 \in \{5,6,7\}$ such that v_1 is compatible with v_2, and color v with color in $\{2,5,6,7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3,4$.

(3) $\phi(vv_3) \in \{5,6,7\}$. Say $\phi(vv_3) = 5$ by symmetry. It suffices to color v_1 with 5, vv_1 with $b_1 \in \{3,4,,6,7\}$ such that v_1 is compatible with v_2, and color v with color in $\{2,3,4,6,7\} \setminus \{\phi(v_2), \phi(v_3), b_1\}$. Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3,4$.

Case 2 $\phi(vv_3) = 1$.

Remove the color of v_2. There are three situations to be handled depending on the value of $\phi(v_3)$:

(1) $\phi(v_3) \in \{2,3,4\}$. Say $\phi(v_3) = 2$ by symmetry. Firstly, we color v_1 with 3. Next, there are two possibilities to be discussed:

• $\phi(u_2) \neq 1$. It suffices to color v_2 with 1, vv_2 with $b_1 \in \{2,3,4,5,6,7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{4,5,6,7\} \setminus \{b_1\}$ such that v_1 is compatible with v_2, and v with color in $\{4,5,6,7\} \setminus \{b_1, b_2\}$.

• $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) = 4$. It suffices to color v_2 with 2, vv_2 with $b_1 \in \{3,5,6,7\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{4,5,6,7\} \setminus \{b_1\}$, and v with color in $\{4,5,6,7\} \setminus \{b_1,b_2\}$. Next, assume that $\phi(v_2u_2) \neq 4$. It suffices to color v_2 with 4, vv_2 with $b_1 \in \{2,3,5,6,7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{4,5,6,7\} \setminus \{b_1\}$, and v with color in $\{5,6,7\} \setminus \{b_1,b_2\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3,4$.

(2) $\phi(v_3) \in \{5,6,7\}$. Say $\phi(v_3) = 5$ by symmetry. Firstly, we color v_1 with 5. Next, we have two possibilities to be discussed:

• $\phi(u_2) \neq 1$. It suffices to color v_2 with 1, vv_2 with $b_1 \in \{2,3,4,5,6,7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{3,4,6,7\} \setminus \{b_1\}$ such that v_1 is compatible with v_2, and v with color in $\{2,3,4,6,7\} \setminus \{b_1,b_2\}$.

• $\phi(u_2) = 1$. Firstly, assume that $\phi(v_2u_2) = 4$. It suffices to color v_2 with 5, vv_2 with $b_1 \in \{2,3,6,7\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{3,4,6,7\} \setminus \{b_1\}$, and v with color in $\{2,3,4,6,7\} \setminus \{b_1,b_2\}$. Next, assume that $\phi(v_2u_2) \neq 4$. It suffices to color v_2 with 4, vv_2 with $b_1 \in \{2,3,5,6,7\} \setminus \{\phi(v_2u_2)\}$ such that v_2 is compatible with $u_2^{'}$, vv_1 with $b_2 \in \{3,4,6,7\} \setminus \{b_1\}$, and v with color in $\{2,3,6,7\} \setminus \{b_1,b_2\}$.

Obviously, v has at least one color set that can be distinguished from u_i for $i = 1,2,3,4$. □

CLAIM 7. *G does not contain* 3_1*-vertex.*

Proof. Assume to the contrary, G contains a 3_1-vertex v. Let $N_G(v) = \{v_1,v_2,v_3\}$, $d_G(v_1) = 2$, and let u_1 be the neighbor of v_1 other than v. By Claim 3 and Claim 6, $d_G(u_1) = d_G(v_2) = d_G(v_3) = 3$. Let $N_G(u_1) = \{v_1,u_1^{'},u_1^{''}\}$, and $N_G(v_i) = \{v,v_i^{'},v_i^{''}\}$ for $i = 2,3$. By Claim 6, $d_G(u_1^{'}) = d_G(u_1^{''}) = 3$. By Claim 2, $u_1 \neq v_2$. Then consider the subgraph $G^{'} = G - \{v_1\} + \{vu_1\}$. Then $G^{'}$ has a 2DVDT-7-coloring ϕ using the color set C by the minimality of G. Assume without loss of generality that $\phi(vu_1) = 1, \phi(vv_2) = 2, \phi(vv_3) = 3$ and $\phi(v) = 4$. Based on ϕ, in G, we color u_1v_1 with 1. It is easy notice that the color set of u_1 doesn't change in G. If $\{2,3,4,i\} \in (C(v_2^{'}),C(v_2^{''}),C(v_3^{'}),C(v_3^{''}))$ for $i = 5,6,7$. We color v_1 with color $\{2,3\} \setminus \{\phi(u_1)\}$, recolor v with $b \in \{5,6,7\} \setminus \{\phi(v_2),v_3\}$, and color vv_1 with color in $\{5,6,7\} \setminus \{b\}$. Otherwise, it suffices to color v_1 with color $\{2,3\} \setminus \{\phi(u_1)\}$ and vv_1 with color in $\{5,6,7\}$. Obviously, v has at least one color set that can be distinguished from $u_1,v_i^{'},v_i^{''}$ for $i = 2,3$. □

CLAIM 8. *G does not contain a cut edge.*

Proof. Assume to the contrary that G contains a cut edge uv. It follows that $G^{'} = G - \{uv\}$ consists of two components $G_1^{'}$ and $G_2^{'}$ for $u \in V(G_1^{'})$ and $v \in V(G_2^{'})$. Let $G_1 = G[V(G_1^{'}) \cup \{v\}]$ and $G_2 = G[V(G_2^{'}) \cup \{u\}]$. It follows that G_1 and G_2 are proper subgraphs of G. By the minimalization of G, G_1 and G_2 have 2DVDT-7-coloring ϕ_1 such that $C_{\phi_1}(u) = \{1,2,3,4\}$ with $\phi_1(uv) = 1$ and ϕ_2 such that $C_{\phi_2}(v) = \{1,5,6,7\}$ with $\phi_2(uv) = 1$ use the color set C, respectively.(This can be accomplished by exchanging reasonably the colors of G_2 under ϕ_2.) Noticed that combining ϕ_1 and ϕ_2 to produce a 2DVDT-7-coloring of G. This is a contradiction. □

So far it follows that G is a 3-regular simple graph without cut edges. Since K_4 can be 2DVDT-7-colorable with color set C by [12], we assume that $G \neq K_4$. By Lemma 2, G is 3-colorable. Thus, first we use 1, 2, 3 to color all vertices of G such that adjacent vertices receive distinct colors.

Next, we color all edges of G with colors in $\{3,4,5,6,7\}$. By Lemma 3, G can be edge-partitioned into a perfect matching M and a class $\mathscr{A}$ of cycles. Color all edges of M with the same color 7. For each cycle $A = v_1v_2v_3 \cdots v_kv_1$ in $\mathscr{A}$, we use colors of $\{3,4,5,6\}$ to color its edges. For convenience, we define each edge v_iv_{i+1} as e_i for $v_{k+1} = v_1$. There are two situations to be handled as follows:

Case 1. Only two colors appear on the vertices of A.

Assume without loss of generality that $\phi(v_1) = 1$ and $\phi(v_2) = 2$.

If $k \equiv 0 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-5}, e_{k-2}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-4}, e_{k-1}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-3}, e_k\} \to \{6\}$.

If $k \equiv 1 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-6}, e_{k-3}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-5}, e_{k-2}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-4}, e_{k-1}\} \to \{6\}$, $\{e_k\} \to \{5\}$.

If $k \equiv 2 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-7}, e_{k-4}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-6}, e_{k-3}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-5}, e_{k-2}\} \to \{6\}$, $\{e_{k-1}, e_k\} \to \{4, 3\}$.

Case 2. There are three consecutive vertices of A that receive distinct colors.

Assume without loss of generality that $\phi(v_1) = 1$, $\phi(v_2) = 2$ and $\phi(v_3) = 3$.

If $k \equiv 0 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-5}, e_{k-2}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-4}, e_{k-1}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-3}, e_k\} \to \{6\}$.

If $k \equiv 1 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-6}, e_{k-3}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-5}, e_{k-2}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-4}, e_{k-1}\} \to \{6\}$, $\{e_k\} \to \{5\}$.

If $k \equiv 2 \pmod 3$, then $\{e_1, e_4, \cdots, e_{k-7}, e_{k-4}\} \to \{4\}$, $\{e_2, e_5, \cdots, e_{k-6}, e_{k-3}\} \to \{5\}$, $\{e_3, e_6, \cdots, e_{k-5}, e_{k-2}\} \to \{6\}$, $\{e_{k-1}\} \to \{4\}$. If $\phi(v_k) = 2$, then color e_k with 3. If $\phi(v_k) = 3$, then color e_k with 5.

This makes G 2DVDT-7-colorable. Thus, it is a contradiction. The whole proof of Theorem 1 is completed. □

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