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BOUNDS OF THE THIRD AND THE FOURTH LOGARITHMIC COEFFICIENTS FOR CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper we study coefficient problems in some subclasses of close-to-convex functions. More precisely, we determine the upper bounds of the third and the fourth logarithmic coefficients, γ_3 and γ_4, for functions in some subclasses of analytic and univalent functions in the unit disc $\mathbb{D}$. In our research we use not only classical results, but also recent results obtained by Efraimidis.

***Key words**:* coefficient problems, logarithmic coefficients, univalent functions, close-to-convex functions, Schwarz functions.

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{A}$ denotes the class of all functions f analytic in $\mathbb{D}$, satisfying the condition $f(0) = f'(0) - 1 = 0$. It means that function $f \in \mathscr{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, . \tag{1}$$

Let $\mathscr{S}$ be the class of all functions in $\mathscr{A}$ which are univalent in $\mathbb{D}$. The logarithmic coefficients of $f \in \mathscr{S}$, denoted by $\gamma_n = \gamma_n(f)$, are defined by

$$\frac{1}{2} \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n \, . \tag{2}$$

The logarithmic coefficients γ_n are significant in the theory of univalent functions and play the important role in the proof of the well-known Bieberbach conjecture. The utility of the logarithmic coefficients in the context of Bieberbach conjecture was discovered by Milin [9], who conjectured that for $f \in \mathscr{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0 \, .$$

In 1985, De Branges [4] proving Milin's conjecture confirmed the Bieberbach conjecture.

If f is of the form (1), then the logarithmic coefficients are given by

$$
\begin{aligned}
\gamma_1 &= \frac{1}{2}a_2 \\
\gamma_2 &= \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \\
\gamma_3 &= \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) \\
\gamma_4 &= \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right) \\
\gamma_5 &= \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right) .
\end{aligned} \tag{3}
$$

Hence, if $f \in \mathscr{S}$, it is easy to see that $|\gamma_1| \le 1$ with equality for the Koebe function $f(z) = \frac{z}{(1-z)^2}$. For this function, it is known that $\gamma_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Since the Koebe function occurs as an extremal function for most of the extremal problems in the class $\mathscr{S}$, it is expected that $|\gamma_n| \le \frac{1}{n}$ for $f \in \mathscr{S}$ and $n \in \mathbb{N}$. Nevertheless, it is not true in general. Namely, it does not hold even for γ_2. The Fekete-Szegö inequality leads to $|\gamma_2| \le \frac{1}{2}\left(1+2e^{-2}\right) = 0.635\ldots$. Also, in 2020 Obradović and Tuneski proved that $|\gamma_3| \le \frac{\sqrt{133}}{15}$ for all $f \in \mathscr{S}$, see [10]. The problem of finding the sharp upper bounds for $|\gamma_n|$ for $f \in \mathscr{S}$ is still open for $n \ge 3$. However, if $f \in \mathscr{S}^*$, the class of starlike functions, the inequality $|\gamma_n| \le \frac{1}{n}$ holds for $n \in \mathbb{N}$, see [12].

Taking into account selected subclasses of the class $\mathscr{S}$, some partial results concerning logarithmic coefficients are known. Particularly, the upper bounds of γ_n for the class of strongly starlike functions of order β $(0 < \beta \le 1)$ were obtained by Thomas in [13], that is $|\gamma_n| \le \frac{\beta}{n}$, $n \in \mathbb{N}$. Whereas, the results for γ-starlike functions were given by Darus and Thomas in [3]. Moreover, non-sharp estimates for the class of Bazilevič, close-to-convex and different subclasses of close-to-convex functions were examined in [6], [1] and [14], respectively.

The sharp upper estimates of $|\gamma_3|$ in subclasses $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ of the class $\mathscr{S}$ of functions f satisfying respectively the following conditions

$$\operatorname{Re}\{(1-z)f'(z)\} > 0, \quad z \in \mathbb{D} \tag{4}$$

$$\operatorname{Re}\{(1-z^2)f'(z)\} > 0, \quad z \in \mathbb{D} \tag{5}$$

$$\operatorname{Re}\{(1-z+z^2)f'(z)\} > 0, \quad z \in \mathbb{D} \tag{6}$$

$$\operatorname{Re}\{(1-z)^2f'(z)\} > 0, \quad z \in \mathbb{D} \tag{7}$$

were investigated in [1], [8] and [2] with the additional assumptions about the coefficient a_2. Namely, the estimates related to $\mathscr{F}_k$ are as follows

$$|\gamma_3| \le \frac{1}{288}(11+15\sqrt{30}) = 0.3234\ldots \text{ for } f \in \mathscr{F}_1 \text{ and } \frac{1}{2} \le a_2 \le \frac{3}{2}, \text{ [8]}$$

$$|\gamma_3| \le \frac{1}{972}(95+23\sqrt{46}) = 0.2582\ldots \text{ for } f \in \mathscr{F}_2 \text{ and } 0 \le a_2 \le 1, \text{ [8]}$$

$$|\gamma_3| \le \frac{1}{7776}(743+131\sqrt{262}) = 0.3682\ldots \text{ for } f \in \mathscr{F}_3 \text{ and } \frac{1}{2} \le a_2 \le \frac{3}{2}, \text{ [8]}$$

$$|\gamma_3| \le \frac{1}{243}(28+19\sqrt{19}) = 0.4560\ldots \text{ for } f \in \mathscr{F}_4 \text{ and } 1 \le a_2 \le 2, \text{ [1]}$$

These results were generalized for all real a_2 by Cho et al. in [2].

In this paper we consider more general case, when a_2 is an arbitrarily complex number. Despite the rejection of the assumption $a_2 \in \mathbb{R}$, the derived results are only slightly worse than those obtained in [2].

It is worth noting that in Theorems 1, 2 and 3 we obtained similar results as in [2], the derived results are slightly worse than the sharp bounds obtained in [2], differing at the level of hundredths in $\mathscr{F}_4$, thousandths in $\mathscr{F}_1$ and $\mathscr{F}_3$, and ten-thousandths in $\mathscr{F}_2$ but without any additional assumptions for the coefficient a_2. Furthermore, we derive the bounds of $|\gamma_4|$ in the same subclasses of $\mathscr{S}$.

Note that, each class $\mathscr{F}_i$, $i = 1, \dots 4$ is the subclass of the well-known class of close-to-convex functions. Since all of these classes have a representation by using the Carathéodory class $\mathscr{P}$, i.e., the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}, \tag{8}$$

having a positive real part in $\mathbb{D}$, so both γ_3 and γ_4 have a suitable representations expressed by the coefficients of functions in $\mathscr{P}$.

Denote by $\mathscr{B}_0$ the class of Schwarz functions, i.e., the class of all analytic functions $\omega : \mathbb{D} \to \mathbb{D}$, $\omega(0) = 0$. A function $\omega \in \mathscr{B}_0$ can be written as a power series

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n . \tag{9}$$

There exists a close relationship between the class $\mathscr{P}$ and the class $\mathscr{B}_0$. Namely, $p = \frac{1+\omega}{1-\omega}$ is in $\mathscr{P}$ if and only if $\omega \in \mathscr{B}_0$. From this relation we conclude that

$$\begin{gathered} p_1 = 2c_1, \ p_2 = 2(c_2 + c_1^2), \ p_3 = 2(c_3 + 2c_1c_2 + c_1^3), \\ p_4 = 2(c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4) . \end{gathered} \tag{10}$$

To prove our results we need the following lemmas for Schwarz functions obtained by Prokhorov and Szynal [11] and Carlson [5].

LEMMA 1. *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_2| \le 1 - |c_1|^2, \quad |c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \quad |c_4| \le 1 - |c_1|^2 - |c_2|^2 .$$

We also need the results obtained by Efraimidis (see, [7]).

LEMMA 2. *Let $\omega = c_1 z + c_2 z^2 + \dots$ be a Schwarz function and $\lambda \in \mathbb{C}$. Then*

$$|c_3 + (1+\lambda)c_1c_2 + \lambda c_1^3| \le \max\{1, |\lambda|\} \tag{11}$$

$$|c_3 + 2\lambda c_1c_2 + \lambda^2 c_1^3| \le \max\{1, |\lambda|^2\} \tag{12}$$

and

$$|c_4 + (1+\lambda)c_1c_3 + c_2^2 + (1+2\lambda)c_1^2c_2 + \lambda c_1^4| \le \max\{1, |\lambda|\} \tag{13}$$

$$|c_4 + 2c_1c_3 + \lambda c_2^2 + (1+2\lambda)c_1^2c_2 + \lambda c_1^4| \le \max\{1, |\lambda|\} . \tag{14}$$

2. ESTIMATES FOR THE γ_3

To obtain our results we apply a different approach than those used in [1, 3, 9]. In order to estimate γ_3 we express it by coefficients of Schwarz functions.

THEOREM 1. *If $f \in \mathscr{F}_1$, then*

$$|\gamma_3| \le \frac{21}{64} .$$

Proof. Let $f \in \mathscr{F}_1$ be of the form (1). Then there exists $p \in \mathscr{P}$ of the form (8) such that

$$(1-z)f'(z) = p(z), \quad z \in \mathbb{D} . \tag{15}$$

Substituting the series (1) and (8) into (15) and equating the coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}(1+p_1) \\ a_3 &= \frac{1}{3}(1+p_1+p_2) \\ a_4 &= \frac{1}{4}(1+p_1+p_2+p_3) \\ a_5 &= \frac{1}{5}(1+p_1+p_2+p_3+p_4)\ . \end{aligned} \tag{16}$$

By (3) and (16) we get

$$\gamma_3 = \frac{1}{48}\left(p_1 - p_1^2 + p_1^3 - 4p_1p_2 + 2p_2 + 6p_3 + 3\right)\ .$$

Using now (10) we have

$$\gamma_3 = \frac{1}{48}\left(4c_1^3 + 8c_1c_2 + 2c_1 + 4c_2 + 12c_3 + 3\right)\ .$$

Then obviously

$$|\gamma_3| \le \frac{1}{48}\left(4|c_1|^3 + 8|c_1||c_2| + 2|c_1| + 4|c_2| + 12|c_3| + 3\right)$$

and applying Lemma 1 we get

$$|\gamma_3| \le \frac{1}{48}\left(4|c_1|^3 + 8|c_1||c_2| + 2|c_1| + 4|c_2| + 12\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 3\right)\ .$$

Let $h_1(x,y)$ denotes the right hand side of the above inequality with $x = |c_1|$ and $y = |c_2|$. Then

$$h_1(x,y) = \frac{1}{48}\left(4x^3 + 8xy + 2x + 4y + 12\left(1 - x^2 - \frac{y^2}{1+x}\right) + 3\right)\ .$$

The shape of the region of variability of (x,y) is a simple consequence of the Schwarz-Pick Lemma. It coincides with

$$\Omega = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1 - x^2\}\ . \tag{17}$$

The critical points of h_1 are the solutions of the system

$$\begin{cases} 12x^2 + \frac{12y^2}{(x+1)^2} - 24x + 8y + 2 = 0 \\ -\frac{24y}{x+1} + 8x + 4 = 0\ . \end{cases}$$

Hence, the only critical point of h_1 in Ω is $\left(\frac{1}{4}, \frac{5}{16}\right)$ and $h_1\left(\frac{1}{4}, \frac{5}{16}\right) = \frac{21}{64}$. Now, it is enough to derive the greatest value of h_1 on the boundary of Ω. We have the following

$$\begin{aligned} h_1(x,0) &= \frac{1}{48}\left(4x^3 - 12x^2 + 2x + 15\right) \le \frac{3}{16} + \frac{5\sqrt{\frac{5}{6}}}{36} = 0.3142\ldots \\ h_1(0,y) &= \frac{1}{48}\left(4y + 15 - 12y^2\right) \le \frac{23}{72} \\ h_1(x,1-x^2) &= \frac{1}{48}\left(-16x^3 - 4x^2 + 22x + 7\right) \le \frac{278 + 67\sqrt{67}}{2592} = 0.3188\ldots\ . \end{aligned}$$

Combinig all these inequalities we get

$$h_1(x,y) \le \frac{21}{64} \quad \text{for all} \quad (x,y) \in \Omega\ ,$$

which results in the desired bound. □

THEOREM 2. *If* $f \in \mathscr{F}_2$, *then*

$$|\gamma_3| \le 0.2587\dots\ .$$

Proof. Suppose that $f \in \mathscr{F}_2$ is given by (1). Then there exists $p \in \mathscr{P}$ of the form (8) such that

$$(1-z^2)f'(z) = p(z), \quad z \in \mathbb{D}\ . \tag{18}$$

Applying in (18) the expansions of f and p given by (1) and (8), we obtain

$$\begin{aligned} a_2 &= \frac{1}{2}p_1 \\ a_3 &= \frac{1}{3}(1+p_2) \\ a_4 &= \frac{1}{4}(p_1+p_3) \\ a_5 &= \frac{1}{5}(1+p_2+p_4)\ . \end{aligned} \tag{19}$$

Using (3) and (19) we have

$$\gamma_3 = \frac{1}{48}\left(p_1^3+2p_1-4p_1p_2+6p_3\right)\ .$$

From (10) we get

$$\gamma_3 = \frac{1}{12}\left(c_1^3+2c_1c_2+c_1+3c_3\right)\ .$$

Hence from the triangle inequality we have

$$|\gamma_3| \le \frac{1}{12}\left(|c_1|^2+2|c_1||c_2|+|c_1|+3|c_3|\right)\ .$$

Now, using Lemma 1 we get

$$|\gamma_3| \le \frac{1}{12}\left(|c_1|^3+2|c_1||c_2|+|c_1|+3\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)\right)\ .$$

Let us denote by $h_2(x,y)$ the right hand side of the above inequality, where $x=|c_1|$ and $y=|c_2|$. Therefore,

$$h_2(x,y) = \frac{1}{12}\left(x^3+2xy+x+3-3x^2-\frac{3y^2}{1+x}\right)\ .$$

From

$$\begin{cases} 3x^2+\frac{3y^2}{(x+1)^2}-6x+2y+1=0 \\ x-\frac{3y}{x+1}=0 \end{cases}$$

it follows that $(0.2257\dots,0.0922\dots)$ is the only critical point of h_2 in Ω. In this case $h_2(0.2257\dots,0.0922\dots) = 0.2587\dots$.

To complete the proof we need to verify the behavior of the function h_2 on the boundary of Ω. We have

$$\begin{aligned} h_2(x,0) &= \frac{1}{12}\left(3+x-3x^2+x^3\right) \le \frac{1}{54}(9+2\sqrt{6}) = 0.2573\dots \\ h_2(0,y) &= \frac{1}{4}\left(1-y^2\right) \le \frac{1}{4} \\ h_2(x,1-x^2) &= \frac{1}{6}\left(3x-2x^3\right) \le \frac{1}{3\sqrt{2}} = 0.2357\dots\ . \end{aligned}$$

Taking everything into account we obtain

$$h_2(x,y) \le 0.2587\ldots \ .$$

This ends the proof of the theorem. □

THEOREM 3. *If $f \in \mathscr{F}_3$, then*

$$|\gamma_3| \le \frac{71}{192}\ .$$

Proof. Assume that $f \in \mathscr{F}_3$ is of the form (1). Then there exists $p \in \mathscr{P}$ of the form (8) such that

$$(1-z+z^2)f'(z) = p(z), \quad z \in \mathbb{D}\ . \tag{20}$$

Putting the series (1) and (8) into (20) by equating the coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}(1+p_1) \\ a_3 &= \frac{1}{3}(p_1+p_2) \\ a_4 &= \frac{1}{4}(p_2+p_3-1) \\ a_5 &= \frac{1}{5}(p_3+p_4-p_1-1)\ . \end{aligned} \tag{21}$$

By (3) and (21) we obtain

$$\gamma_3 = \frac{1}{48}\left(p_1^3 - p_1^2 - p_1 + 2p_2 - 4p_1p_2 + 6p_3 - 5\right)\ .$$

Again, applying (10) we have

$$\gamma_3 = \frac{1}{48}\left(4c_1^3 + 8c_1c_2 - 2c_1 + 4c_2 + 12c_3 - 5\right)\ .$$

The traingle inequality and Lemma 1 result in

$$\begin{aligned} |\gamma_3| &\le \frac{1}{48}\left(4|c_1|^3 + 8|c_1||c_2| + 2|c_1| + 4|c_2| + 12\left(1-|c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 5\right) \\ &= h_1\left(|c_1|,|c_2|\right) + \frac{1}{24}\ , \end{aligned}$$

where h_1 is defined in the proof of Theorem 1.

Therefore, from Theorem 1, the declared result follows. □

THEOREM 4. *If $f \in \mathscr{F}_4$, then*

$$|\gamma_3| \le \frac{185}{384}\ .$$

Proof. Let $f \in \mathscr{F}_4$ be of the form (1). Then there exists $p \in \mathscr{P}$ of the form (8) such that

$$(1-z)^2 f'(z) = p(z), \quad z \in \mathbb{D}\ . \tag{22}$$

Substituting the series (1) and (8) into (22) and equating the coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}(p_1+2) \\ a_3 &= \frac{1}{3}(2p_1+p_2+3) \\ a_4 &= \frac{1}{4}(3p_1+2p_2+p_3+4) \\ a_5 &= \frac{1}{5}(4p_1+3p_2+2p_3+p_4+5)\ . \end{aligned} \tag{23}$$

From (3) and (23) we get

$$\gamma_3 = \frac{1}{48}\left(p_1^3-2p_1^2+2p_1-4p_1p_2+4p_2+6p_3+8\right)$$

and, using (10),

$$\gamma_3 = \frac{1}{12}\left(c_1^3+2c_1c_2+c_1+2c_2+3c_3+2\right)\ .$$

From (11), the traingle inequality and Lemma 1 we get

$$\begin{aligned} |\gamma_3| &\le \frac{1}{12}\left|(c_3+2c_1c_2+c_1^3)+2c_3+2c_2+c_1+2\right| \\ &\le \frac{1}{12}\left(1+2\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)+2|c_2|+|c_1|+2\right)\ . \end{aligned}$$

Denote by $h_4(x,y)$ the right hand side of the above inequality with $x=|c_1|$ and $y=|c_2|$. Then

$$h_4(x,y) = \frac{1}{12}\left(5+x-2x^2+2y-\frac{2y^2}{1+x}\right)\ .$$

Now we consider the system

$$\begin{cases} 1-4x+\frac{2y^2}{(1+x)^2}=0 \\ 2-\frac{4y}{1+x}=0\ . \end{cases}$$

Therefore, the only critical point of h_4 in Ω is $\left(\frac{3}{8},\frac{11}{16}\right)$ and $h_4\left(\frac{3}{8},\frac{11}{16}\right)=\frac{185}{384}$.

On the boundary of the set Ω, we have

$$\begin{aligned} h_4(x,0) &= \frac{1}{12}\left(5+x-2x^2\right) \le \frac{41}{96} \\ h_4(0,y) &= \frac{1}{12}\left(5+2y-2y^2\right) \le \frac{11}{24} \\ h_4(x,1-x^2) &= \frac{1}{12}\left(5+3x-2x^2-2x^3\right) \le \frac{1}{324}\left(104+11\sqrt{22}\right) = 0.4802\ldots\ . \end{aligned}$$

Finally, we obtain

$$h_4(x,y) \le \frac{185}{384}\ ,$$

which ends the proof.

□

3. ESTIMATES FOR THE γ_4

Now we will prove the results concerning the bounds of γ_4 for $\mathscr{F}_k$. For this purpose we will use the results obtained by Efraimidis.

THEOREM 5. *If* $f \in \mathscr{F}_1$, *then*

$$|\gamma_4| \leq 0.3245\ldots\ .$$

Proof. Let $f \in \mathscr{F}_1$ be of the form (1). Using (3) and (16) we get

$$\gamma_4 = \frac{1}{5760}\left(-45p_1^4+60p_1^3+240p_1^2p_2-70p_1^2-200p_1p_2-360p_1p_3+76p_1 \right.$$
$$\left. -160p_2^2+136p_2+216p_3+576p_4+251\right)\ .$$

Applying now (10) we obtain

$$|\gamma_4| = \frac{1}{5760}\left|864c_1c_3+432c_3+512c_2^2+1216c_1^2c_2+64c_1c_2+272c_2+272c_1^4 \right.$$
$$\left. +112c_1^3-8c_1^2+152c_1+1152c_4+251\right|\ .$$

Using the triangle inequality we have

$$|\gamma_4| \leq \frac{1}{5760}\left(432S_1+80S_2+32S_3+S_4\right),$$

where

$$S_1 = \left|c_4+2c_1c_3+c_2^2+3c_1^2c_2+c_1^4\right| \leq 1 \text{ by (13) with } \lambda = 1\ ,$$
$$S_2 = \left|c_4+c_2^2-c_1^2c_2-c_1^4\right| \leq 1 \text{ by (13) with } \lambda = -1\ ,$$
$$S_3 = \left|c_3+2c_1c_2+c_1^3\right| \leq 1 \text{ by (12) with } \lambda = 1\ ,$$
$$S_4 = \left|400c_3+80c_1^3+640c_4-80c_1^4+272c_2-8c_1^2+152c_1+251\right|\ .$$

Applying the triangle inequality once more we have

$$|\gamma_4| \leq \frac{1}{5760}\left[795+400|c_3|+80|c_1|^3+640|c_4|+80|c_1|^4+272|c_2|+8|c_1|^2+152|c_1|\right].$$

Now, by Lemma 1, we receive

$$|\gamma_4| \leq \frac{1}{5760}\left(640(1-|c_1|^2-|c_2|^2)+400\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)\right.$$
$$\left. +272|c_2|+80|c_1|^4+80|c_1|^3+8|c_1|^2+152|c_1|+795\right) \leq g_1\left(|c_1|,|c_2|\right)\ ,$$

where

$$g_1(x,y) = \frac{1}{5760}\left(80x^4+80x^3-1032x^2+152x-640y^2+272y+1835\right)\ .$$

The critical points of g_1 are the solutions of the system

$$\begin{cases} 40x^3+30x^2-258x+19=0 \\ 272-1280y=0\ . \end{cases}$$

Taking into account the second equation we get rational solution of the form $y_0 = \frac{17}{80}$, whereas $x_0 = 0.0743\ldots$ is the solution of the polynomial of the third degree. So, the value of g_1 at this point is $g_1\left(0.0743\ldots,\frac{17}{80}\right) = 0.3245\ldots$.

Now, we need to find the gratest value of g_1 on the boundary of Ω. We have

$$g_1(x,0)=\frac{80x^4+80x^3-1032x^2+152x+1835}{5760}\leq 0.3195\ldots$$
$$g_1(0,y)=\frac{-640y^2+272y+1835}{5760}\leq\frac{2071}{6400}$$
$$g_1(x,1-x^2)=\frac{-560x^4+80x^3-24x^2+152x+1467}{5760}\leq 0.2630\ldots .$$

Finally, we obtain

$$g_1(x,y)\leq 0.3245\ldots ,$$

which gives the desired result. □

THEOREM 6. *If $f\in\mathscr{F}_2$, then*

$$|\gamma_4|\leq\frac{29}{100}.$$

Proof. Assume that $f\in\mathscr{F}_2$ be of the form (1). Then from (3) and (19) we obtain

$$\gamma_4=\frac{1}{5760}\left(-45p_1^4+240p_1^2p_2-120p_1^2-360p_1p_3-160p_2^2+256p_2+576p_4+416\right)$$

and from (10)

$$|\gamma_4|=\frac{1}{360}\left|54c_1c_3+32c_2^2+76c_1^2c_2+32c_2+17c_1^4+2c_1^2+72c_4+26\right| .$$

By the triangle inequality we obtain

$$|\gamma_4|\leq\frac{1}{360}\left(27S_1+5S_2+S_3\right),$$

where

$$\begin{aligned}
S_1&=\left|c_4+2c_1c_3+c_2^2+3c_1^2c_2+c_1^4\right|\leq 1 \text{ by (13) with } \lambda=1 ,\\
S_2&=\left|c_4+c_2^2-c_1^2c_2-c_1^4\right|\leq 1 \text{ by (13) with } \lambda=-1 ,\\
S_3&=\left|40c_4-5c_1^4+32c_2+2c_1^2+26\right| .
\end{aligned}$$

Consequently,

$$|\gamma_4|\leq\frac{1}{360}\left(58+40|c_4|+5|c_1|^4+32|c_2|+2|c_1|^2\right) .$$

By Lemma 1,

$$|\gamma_4|\leq\frac{1}{360}\left(58+40(1-|c_1|^2-|c_2|^2)+5|c_1|^4+32|c_2|+2|c_1|^2\right)=g_2\left(|c_1|,|c_2|\right) ,$$

where

$$g_2(x,y)=\frac{1}{360}\left(98-38x^2-40y^2+5x^4+32y\right) .$$

It is easy to check that there are no critical points inside the set Ω. Now, it is enough to examine the behavior of g_2 on the boundary of Ω. Hence,

$$\begin{aligned} g_2(x,0) &= \frac{1}{360}(98-38x^2+5x^4) \le \frac{49}{180} \\ g_2(0,y) &= \frac{1}{360}(98-40y^2+32y) \le \frac{29}{100} \\ g_2(x,1-x^2) &= \frac{1}{72}(18+2x^2-7x^4) \le \frac{127}{504} . \end{aligned}$$

Taking into account all these inequalities we have

$$g_2(x,y) \le \frac{29}{100} ,$$

which ends the proof of the theorem. □

THEOREM 7. *If $f \in \mathscr{F}_3$, then*

$$|\gamma_4| \le 0.3287\ldots .$$

Proof. Let $f \in \mathscr{F}_3$ be of the form (1). Using (3) and (21) we have

$$\begin{aligned} \gamma_4 = \frac{1}{5760}\big(&576p_4+216p_3-360p_1p_3-160p_2^2+240p_1^2p_2-200p_1p_2-120p_2 \\ &-45p_1^4+60p_1^3+50p_1^2-156p_1-261\big) . \end{aligned}$$

By (10),

$$\begin{aligned} |\gamma_4| = \frac{1}{5760}\big|&864c_1c_3+432c_3+512c_2^2+1216c_1^2c_2+64c_1c_2-240c_2 \\ &+272c_1^4+112c_1^3-40c_1^2-312c_1+1152c_4-261\big| . \end{aligned}$$

Hence,

$$|\gamma_4| \le \frac{1}{5760}(432S_1+80S_2+S_3)$$

where

$$\begin{aligned} S_1 &= \left|c_4+2c_1c_3+c_2^2+3c_1^2c_2+c_1^4\right| \le 1 \text{ by (13) with } \lambda=1 , \\ S_2 &= \left|c_4+c_2^2-c_1^2c_2-c_1^4\right| \le 1 \text{ by (13) with } \lambda=-1 , \\ S_3 &= \left|640c_4-80c_1^4+432c_3+64c_1c_2-240c_2+112c_1^3-40c_1^2-312c_1+261\right| . \end{aligned}$$

The triangle inequality and Lemma 1 results in

$$\begin{aligned} |\gamma_4| \le \frac{1}{5760}\big(&773+640(1-|c_1|^2-|c_2|^2)+80|c_1|^4+432(1-|c_1|^2) \\ &+64|c_1||c_2|+240|c_2|+112|c_1|^3+40|c_1|^2+312|c_1|\big) = g_3(|c_1|,|c_2|) , \end{aligned}$$

where

$$g_3(x,y) = \frac{1}{5760}\left(80x^4+112x^3-1032x^2+64xy+312x-640y^2+240y+1845\right) .$$

The critical points of g_3 inside the set Ω coincide with the solution of the system of equations

$$\begin{cases} 39-258x+42x^2+40x^3+8y=0 \\ 15+4x-80y=0 . \end{cases}$$

There is only one critical point $(0.1621\ldots,0.1956\ldots)$ and $g_3(0.1621\ldots,0.1956\ldots)=0.3287\ldots$. On the boundary of Ω,

$$\begin{aligned}
g_3(x,0) &= \frac{1}{5760}(1845+312x-1032x^2+112x^3+80x^4) \le 0.3244\ldots\\
g_3(0,y) &= \frac{1}{5760}(1845+240y-640y^2) \le \frac{83}{256}\\
g_3(x,1-x^2) &= \frac{1}{5760}(1445+376x+8x^2+48x^3-560x^4) \le 0.2798\ldots\ .
\end{aligned}$$

Finally, considering all the inequalities, we obtain

$$g_3(x,y) \le 0.3287\ldots\ ,$$

which is the desired result. □

THEOREM 8. *If $f \in \mathscr{F}_4$, then*

$$|\gamma_4| \le 0.5027\ldots\ .$$

Proof. Let $f \in \mathscr{F}_4$ be of the form (1). Therefore, using (3) and (23), we have

$$\begin{aligned}
|\gamma_4| = \frac{1}{5760}\big(&576p_4+432p_3-360p_1p_3-160p_2^2+240p_1^2p_2-400p_1p_2+288p_2\\
&-45p_1^4+120p_1^3-160p_1^2+144p_1+720\big)\ .
\end{aligned}$$

It is easy to conclude from (10) that

$$\begin{aligned}
|\gamma_4| = \frac{1}{360}\big|&54c_1c_3+54c_3+32c_2^2+76c_1^2c_2+8c_1c_2+36c_2+17c_1^4+14c_1^3-4c_1^2\\
&+18c_1+72c_4+45\big|\ .
\end{aligned}$$

Using the triangle inequality, we obtain

$$|\gamma_4| \le \frac{1}{360}(27S_1+5S_2+S_3)$$

where

$$\begin{aligned}
S_1 &= \left|c_4+2c_1c_3+c_2^2+3c_1^2c_2+c_1^4\right| \le 1 \text{ by (13) with } \lambda=1\ ,\\
S_2 &= \left|c_4+c_2^2-c_1^2c_2-c_1^4\right| \le 1 \text{ by (13) with } \lambda=-1\ ,\\
S_3 &= \left|40c_4-5c_1^4+54c_3+8c_1c_2+36c_2+14c_1^3-4c_1^2+18c_1+45\right|\ .
\end{aligned}$$

By the triangle inequality and Lemma 1,

$$\begin{aligned}
|\gamma_4| &\le \frac{1}{360}\Big(77+40(1-|c_1|^2-|c_2|^2)+5|c_1|^4+54\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)\\
&\quad +8|c_1||c_2|+36|c_2|+14|c_1|^3+4|c_1|^2+18|c_1|\Big)\\
&\le \frac{1}{360}\Big(171+18|c_1|-90|c_1|^2+14|c_1|^3+5|c_1|^4+36|c_2|-40|c_2|^2\\
&\quad +8|c_1|\left(1-|c_1|^2\right)=g_4\left(|c_1|,|c_2|\right)\ ,
\end{aligned}$$

where

$$g_4(x,y) = \frac{1}{360}\left(171+26x-90x^2+6x^3+5x^4+36y-40y^2\right)\ .$$

We can observe that $\left(0.1469\ldots, \frac{9}{20}\right)$ is the only critical point of g_4 inside the set Ω. Therefore, $g_4\left(0.1469\ldots, \frac{9}{20}\right) = 0.5027\ldots$.

Moreover,

$$g_4(x,0) = \frac{1}{360}(171 + 26x - 90x^2 + 6x^3 + 5x^4) \le 0.4802\ldots$$
$$g_4(0,y) = \frac{1}{360}(171 + 36y - 40y^2) \le \frac{199}{400}$$
$$g_4(x,1-x^2) = \frac{1}{360}(167 + 26x - 46x^2 + 6x^3 - 35x^4) \le 0.4738\ldots\ .$$

This means that

$$g_4(x,y) \le 0.5027\ldots\ .$$

In this way we have proved the theorem. □

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