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NONDEGENERACY OF THE ENTIRE SOLUTION FOR THE *n*-LAPLACE HÉNON EQUATION OF LIOUVILLE TYPE

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Abstract. Motivated by the work of Takahashi [10], we establish nondegeneracy of the explicit family of solutions of the n-Laplace Hénon equation of Liouville type on the whole space.

***Key words**:* singular Liouville equation, n-Laplacian operator, linearization, nondegeneracy.
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1. INTRODUCTION AND STATEMENT OF RESULTS

For $n \geq 2$ and $\beta > -1$, consider the following quasilinear singular Liouville equation

$$\begin{cases} -\Delta_n u = |x|^{n\beta} e^u & \text{in} \quad \mathbb{R}^n \\ \displaystyle\int_{\mathbb{R}^n} |x|^{n\beta} e^u \mathrm{d}x < \infty, \end{cases} \tag{1}$$

where $\Delta_n u = \operatorname{div}(|\nabla u|^{n-2}\nabla u)$, denotes the n-Laplacian operator. Problem (1) has the explicit solution

$$u_\beta(x) = \log\left(\frac{n(\frac{n^2}{n-1})^{n-1}(1+\beta)^n}{\left(1+|x|^{\frac{n}{n-1}(1+\beta)}\right)^n}\right), \quad x \in \mathbb{R}^n. \tag{2}$$

Notice that equation (1) is invariant under dilation in the following sense: If u is a solution of (1) and if $\tau > 0$, then $u_\beta(\tau\cdot) + n(1+\beta)\log\tau$, is also a solution of (1). With this observation in mind, we define for all $\tau > 0$

$$u_{\beta,\tau}(x) = \log\left(\frac{n(\frac{n^2}{n-1})^{n-1}(1+\beta)^n \tau^{n(1+\beta)}}{\left(1+|\tau x|^{\frac{n}{n-1}(1+\beta)}\right)^n}\right), \quad x \in \mathbb{R}^n. \tag{3}$$

This note aims to generalize the result of Takahashi [10], who studied the case $\beta = 0$. Specifically, he considered the following quasilinear Liouville equation

$$-\Delta_n u = e^u \quad \text{in} \quad \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^u \mathrm{d}x < \infty,$$

he prove the linear nondegeneracy of the explicit entire solution

$$u(x) = \log \frac{C_n}{(1+|x|^{\frac{n}{n-1}})^n}, \quad x \in \mathbb{R}^n,$$

where $C_n = n(\frac{n^2}{n-1})^{n-1}$.

More precisely, we are concerned with the linear nondegeneracy of the explicit solution $u_{\beta,\tau}$ defined by (3). Thus, we define the linearized operator of (1) around $u_{1,\beta} := u_{\beta,\tau=1}$ as follows

$$Lh := -\operatorname{div}(|\nabla u_{1,\beta}|^{n-2}\nabla h) - (n-2)\operatorname{div}(|\nabla u_{1,\beta}|^{n-4}(\nabla u_{1,\beta}\cdot\nabla h)\nabla u_{1,\beta}) - |x|^{n\beta}e^{u_{1,\beta}}h, \tag{4}$$

here "." denotes the standard inner product in $\mathbb{R}^n$. We are interested in the classification of all bounded solutions of $Lh = 0$ in $\mathbb{R}^n$. It is easy to get that

$$\phi_0(r) = \frac{\partial u_{\beta,\tau}}{\partial \tau}|_{\tau=1} = \frac{n(1+\beta)}{n-1}\frac{(n-1) - r^{\frac{n}{n-1}(1+\beta)}}{1+r^{\frac{n}{n-1}(1+\beta)}}, \tag{5}$$

a bounded solution to the linearized equation $Lh = 0$, where $r = |x|$. This solution corresponds to the invariance of the equation under dilation. We say that $u_{\beta,\tau}(x)$ is non-degenerate if the kernel of the associated linearized operator (4) is spanned only by the function ϕ_0 defined by (5). Our main result states as follows.

THEOREM 1. *Suppose that $\beta > -1$ and $\beta \neq 0$. Let h be a solution in $L^\infty \cap C^2(\mathbb{R}^n)$ to the linearized equation $Lh = 0$ which defined by* (4). *Then h can be written as a linear combination of ϕ_0 defined by* (5).

When $n = 2$, the above Theorem was known already, see [3]. All solutions for the singular Liouville equation have been classified by Prajapat-Tarantello in [9], when $n = 2$. For $n \geq 3$, Esposito [5] proves the same classification result for (1), when $\beta = 0$. His method exploits a weighted Sobolev estimates at infinity for any solution to (1). Furthermore, he studied the behavior of solutions near an isolated singularity, as well as a quantization result for entire solutions of problem (1), see [4].

The rest of this note is devoted to proof our main result. Our proof is similar to that of [10]. See also [1,7,8].

2. PROOF OF THEOREM

This section is devoted to proof Theorem . To begin, let L be defined by (4), we rewrite the linear equation $Lh = 0$ as follows

$$\begin{aligned} &r^2\Delta h + n(n-2)(1+\beta)\frac{(x.\nabla h)}{1+r^{\frac{n}{n-1}(1+\beta)}} + (n-2)\sum_{i,j=1}^{n}\frac{\partial^2 h}{\partial x_i\partial x_j}x_i x_j \\ &+ \frac{n^3}{n-1}(1+\beta)^2\frac{r^{\frac{n}{n-1}(1+\beta)}}{(1+r^{\frac{n}{n-1}(1+\beta)})^2}h = 0, \end{aligned} \tag{6}$$

where $r = |x|$. Indeed, a straightforward computation shows that

$$\begin{aligned} Lh &= -\operatorname{div}(|\nabla u_{1,\beta}|^{n-2}\nabla h) - (n-2)\operatorname{div}(|\nabla u_{1,\beta}|^{n-4}(\nabla u_{1,\beta}\cdot\nabla h)\nabla u_{1,\beta}) - |x|^{n\beta}e^{u_{1,\beta}}h \\ &= -|\nabla u_{1,\beta}|^{n-2}\Delta h - \nabla(|\nabla u_{1,\beta}|^{n-2}).\nabla h - (n-2)|\nabla u_{1,\beta}|^{n-4}(\nabla u_{1,\beta}\cdot\nabla h)\Delta u_{1,\beta} \\ &\quad -(n-2)(\nabla u_{1,\beta}\cdot\nabla h)\nabla(|\nabla u_{1,\beta}|^{n-4}).\nabla u_{1,\beta} - (n-2)|\nabla u_{1,\beta}|^{n-4}\nabla(\frac{1}{2}|\nabla u_{1,\beta}|^2).\nabla h \\ &\quad -(n-2)|\nabla u_{1,\beta}|^{n-4}(D^2 h)(\nabla u_{1,\beta},\nabla u_{1,\beta}) - |x|^{n\beta}e^{u_{1,\beta}}h, \end{aligned}$$

with $(D^2h)(\nabla u_{1,\beta},\nabla u_{1,\beta}) = \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j}\frac{\partial u_{1,\beta}}{\partial x_i}\frac{\partial u_{1,\beta}}{\partial x_j}$. Now, we calculate that

$$\begin{aligned}
&\nabla u_{1,\beta} = \frac{-n^2}{n-1}(1+\beta)\frac{r^{\frac{n}{n-1}(1+\beta)-1}}{1+r^{\frac{n}{n-1}(1+\beta)}}\frac{x}{r},\\
&|\nabla u_{1,\beta}|^k = (\frac{n^2}{n-1})^k(1+\beta)^k\frac{r^{\frac{kn}{n-1}(1+\beta)-k}}{(1+r^{\frac{n}{n-1}(1+\beta)})^k},\\
&\nabla(|\nabla u_{1,\beta}|^k) = (\frac{n^2}{n-1})^k(1+\beta)^k\frac{k(1+n\beta)}{n-1}\frac{r^{\frac{nk}{n-1}(1+\beta)-k-1}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{k+1}}\big(1+\frac{1-n}{1+n\beta}r^{\frac{n}{n-1}(1+\beta)}\big)\frac{x}{r},
\end{aligned}$$

where $k\in\mathbb{Z}$ and $r=|x|$. Therefore, we get

$$\begin{aligned}
&\nabla u_{1,\beta}.\nabla h = \frac{-n^2}{n-1}(1+\beta)\frac{r^{\frac{n}{n-1}(1+\beta)-2}}{1+r^{\frac{n}{n-1}(1+\beta)}}(x.\nabla h),\\
&\nabla(|\nabla u_{1,\beta}|^{n-4}).\nabla u_{1,\beta} = -(\frac{n^2}{n-1})^{n-3}(1+\beta)^{n-3}\frac{(n-4)(1+n\beta)}{n-1}\frac{r^{\frac{n(n-3)}{n-1}(1+\beta)-(n-2)}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{n-2}}\\
&\qquad\qquad\times\big(1+\frac{1-n}{1+n\beta}r^{\frac{n}{n-1}(1+\beta)}\big),\\
&(D^2h)(\nabla u_{1,\beta},\nabla u_{1,\beta}) = (\frac{n^2}{n-1})^2(1+\beta)^2\frac{r^{\frac{2n}{n-1}(1+\beta)-4}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^2}\sum_{i,j=1}^{n}\frac{\partial^2 h}{\partial x_i\partial x_j}x_ix_j.
\end{aligned}$$

Furthermore, we have

$$\Delta u_{1,\beta} = \frac{-n^2}{n-1}(1+\beta)\frac{r^{\frac{n}{n-1}(1+\beta)-2}}{(1+r^{\frac{n}{n-1}(1+\beta)})^2}\big(\frac{1+n\beta}{n-1}+(n-1)+(n-2)r^{\frac{n}{n-1}(1+\beta)}\big).$$

From these, we obtain

$$\begin{aligned}
&|\nabla u_{1,\beta}|^{n-2}\Delta h = (\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2}\Big(\frac{r^{\frac{n}{n-1}(1+\beta)-1}}{1+r^{\frac{n}{n-1}(1+\beta)}}\Big)^{n-2}\Delta h,\\
&\nabla(|\nabla u_{1,\beta}|^{n-2}).\nabla h = (\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2}\frac{(n-2)(1+n\beta)}{n-1}\frac{r^{\frac{n(n-2)}{n-1}(1+\beta)-n}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{n-1}}\\
&\qquad\qquad\times\big(1+\frac{1-n}{1+n\beta}r^{\frac{n}{n-1}(1+\beta)}\big)(x.\nabla h),\\
&(n-2)|\nabla u_{1,\beta}|^{n-4}(\nabla u_{1,\beta}\cdot\nabla h)\Delta u_{1,\beta} = (n-2)(\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2}\frac{r^{\frac{n(n-2)}{n-1}(1+\beta)-n}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{n-1}}\\
&\qquad\qquad\times\big(\frac{1+n\beta}{n-1}+(n-1)+(n-2)r^{\frac{n}{n-1}(1+\beta)}\big)(x.\nabla h),\\
&(n-2)(\nabla u_{1,\beta}\cdot\nabla h)\nabla(|\nabla u_{1,\beta}|^{n-4}).\nabla u_{1,\beta} = (n-2)(\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2}\frac{(n-4)(1+n\beta)}{n-1}\\
&\qquad\qquad\times\frac{r^{\frac{n(n-2)}{n-1}(1+\beta)-n}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{n-1}}\big(1+\frac{1-n}{1+n\beta}r^{\frac{n}{n-1}(1+\beta)}\big)(x.\nabla h),\\
&(n-2)|\nabla u_{1,\beta}|^{n-4}\nabla(\frac{1}{2}|\nabla u_{1,\beta}|^2).\nabla h = (n-2)(\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2}\frac{1+n\beta}{n-1}\\
&\qquad\qquad\times\frac{r^{\frac{n(n-2)}{n-1}(1+\beta)-n}}{\big(1+r^{\frac{n}{n-1}(1+\beta)}\big)^{n-1}}\big(1+\frac{1-n}{1+n\beta}r^{\frac{n}{n-1}(1+\beta)}\big)(x.\nabla h),
\end{aligned}$$

$$
\begin{aligned}
(n-2)|\nabla u_{1,\beta}|^{n-4}(D^2h)(\nabla u_{1,\beta},\nabla u_{1,\beta}) &= (n-2)(\frac{n^2}{n-1})^{n-2}(1+\beta)^{n-2} \\
&\times \frac{r^{\frac{n(n-2)}{n-1}(1+\beta)-n}}{\left(1+r^{\frac{n}{n-1}(1+\beta)}\right)^{n-2}}\sum_{i,j=1}^{n}\frac{\partial^2 h}{\partial x_i\partial x_j}x_ix_j, \\
\lambda_1|x|^{n-2}e^{u_{1,\beta}}h &= n(\frac{n^2}{n-1})^{n-1}(1+\beta)^n\frac{r^{n\beta}}{(1-r^{\frac{n}{n-1}(1+\beta)})^n}h.
\end{aligned}
$$

Thus, with these expressions and after some manipulations, we get that $Lh=0$ is equivalent to h verifies (6).

Now, we decompose a solution h to (6) by using spherical harmonics. So we write h as follows

$$
h(x)=h(r,\theta)=\sum_{k=1}^{\infty}g_k(r)l_k(\theta), \quad g_k(r)=\int_{S^{n-1}}h(r,\theta)l_k(\theta)\mathrm{d}\theta, \tag{7}
$$

where $r=|x|$, $\theta=\frac{x}{r}\in S^{n-1}$ for a point $x\in\mathbb{R}^n$ and $l_k(\theta)$ denote the k-th spherical harmonics verifying

$$
-\Delta_{S^{n-1}}l_k=\lambda_k l_k, \quad \text{on } S^{n-1},
$$

with $\Delta_{S^{n-1}}$ denotes the Laplace-Beltrami operator on S^{n-1} and

$$
\lambda_k=k(k+n-2), \quad k=0,1,2,...,
$$

denotes the k-th eigenvalue. It is known that the multiplicity of λ_k is finite. In particular, $\lambda_0=0$ has multiplicity 1 and $\lambda_1=n-1$ has multiplicity n.

Let us now write the equations satisfied by the radial functions $g_k(r)$ for $k=0,1,2,....$ Let ∇_θ denote the spherical gradient operator on S^{n-1}. Since the decomposition of the gradient operator

$$
\nabla=\theta\frac{\partial}{\partial r}+\frac{1}{r}\nabla_\theta, \quad \theta.\nabla_\theta=0
$$

holds, for a function h of the form $h(x)=g(r)l(\theta)$, we have

$$
x.\nabla h=x.\nabla(g(r)l(\theta))=rg'(r)l(\theta),
$$

$$
\sum_{i,j=1}^{n}\frac{\partial^2 h}{\partial x_i\partial x_j}x_ix_j=\sum_{i,j=1}^{n}\frac{\partial^2(g(r)l(\theta))}{\partial x_i\partial x_j}x_ix_j=r^2g''(r)l(\theta).
$$

Furthermore recall the formula

$$
\Delta=\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{S^{n-1}}.
$$

Therefore we have, for h of the form $h(x)=g(r)l(\theta)$, the equation (6) becomes

$$
\begin{aligned}
&r^2(g''(r)+\frac{n-1}{r}g'(r))l(\theta)+g(r)\Delta_{S^{n-1}}l(\theta)+n(n-2)(1+\beta)\frac{rg'(r)l(\theta)}{1+r^{\frac{n}{n-1}(1+\beta)}} \\
&+(n-2)r^2g''(r)l(\theta)+\frac{n^3}{n-1}(1+\beta)^2\frac{r^{\frac{n}{n-1}(1+\beta)}}{(1+r^{\frac{n}{n-1}(1+\beta)})^2}g(r)l(\theta)=0.
\end{aligned}
$$

Inserting equation (7) into equation (6), we deduce that each g_k must be a solution to

$$
\begin{aligned}
L_k(g):=\quad & g''(r)+\frac{g'(r)}{r}\Big(1+\frac{n(n-2)}{n-1}(1+\beta)\frac{1}{1+r^{\frac{n}{n-1}(1+\beta)}}\Big)-\frac{\lambda_k}{n-1}\frac{g(r)}{r^2} \\
&+\frac{n^3}{(n-1)^2}(1+\beta)^2\frac{r^{\frac{n}{n-1}(1+\beta)}}{(1+r^{\frac{n}{n-1}(1+\beta)})^2}\frac{g(r)}{r^2}=0.
\end{aligned} \tag{8}
$$

For $h(x) = g(r)l(\theta)$ is equivalent to that g satisfies

$$\Big(r^{n-1}g'(r)|u'_{1,\beta}|^{n-2}\Big)' - \lambda_k r^{n-3}\frac{1}{n-1}|u'_{1,\beta}|^{n-2}g(r) + \frac{r^{n-1}}{n-1}r^{n\beta}e^{u_{1,\beta}(r)}g(r) = 0. \tag{9}$$

In the following, we treat the equation $L_k(g) = 0$ in (8) for $k = 0$ and $k \geq 1$ separately.

The case $k = 0$. By the invariance under the dilation, we know that $\phi_0(x)$ defined by (5) satisfies (6). Since

$$\phi_0(r) = \frac{n(1+\beta)}{n-1}\frac{(n-1) - r^{\frac{n}{n-1}(1+\beta)}}{1 + r^{\frac{n}{n-1}(1+\beta)}}. \tag{10}$$

It is clear to see that

$$g_0(r) = \frac{(n-1) - r^{\frac{n}{n-1}(1+\beta)}}{1 + r^{\frac{n}{n-1}(1+\beta)}},$$

is a solution of $L_0(g) = 0$, which is bounded on $[0, \infty)$.

We assert that any other bounded solution of $L_0(g) = 0$ must be a constant multiple of g_0. To prove this, let us assume the contrary, that there exists a second linearly independent bounded solution g satisfying $L_0(g) = 0$. Without loss of generality, we can assume that g is of the form

$$g(r) = c(r)g_0(r),$$

for some $c = c(r)$. Substituting this into the equation (8), and recognizing that $\lambda_0 = 0$, we derive the following result

$$\begin{aligned}
&c''(r)g_0(r) + c'(r)\Big(2g_0'(r) + \frac{g_0(r)}{r}\Big(1 + \frac{n(n-2)}{n-1}(1+\beta)\frac{1}{1 + r^{\frac{n}{n-1}(1+\beta)}}\Big)\Big) \\
&+c\Big(g_0''(r) + \frac{g_0'(r)}{r}\Big(1 + \frac{n(n-2)}{n-1}(1+\beta)\frac{1}{1 + r^{\frac{n}{n-1}(1+\beta)}}\Big) \\
&+\frac{n^3}{(n-1)^2}(1+\beta)^2\frac{r^{\frac{n}{n-1}(1+\beta)}}{(1 + r^{\frac{n}{n-1}(1+\beta)})^2}\frac{g_0(r)}{r^2}\Big) = 0,
\end{aligned}$$

which leads to

$$\frac{c''(r)}{c'(r)} = -2\frac{g_0'(r)}{g_0(r)} - \frac{1}{r}\Big(1 + \frac{n(n-2)}{n-1}(1+\beta)\frac{1}{1 + r^{\frac{n}{n-1}(1+\beta)}}\Big).$$

This can be written as

$$(\log|c'(r)|)' = -2(\log|g_0(r)|)' - (1 + \frac{n(n-2)}{n-1}(1+\beta)(\log r)' + (n-2)\big(\log(1 + r^{\frac{n}{n-1}(1+\beta)})\big)'.$$

So, we have that

$$c'(r) = K\frac{(1 + r^{\frac{n}{n-1}(1+\beta)})^{n-2}}{g_0(r)^2 r^{1+\frac{n(n-2)}{n-1}(1+\beta)}},$$

for some $K \neq 0$. Since $g_0(r) \sim -1$ near $r = \infty$, we have

$$c'(r) \sim K\frac{r^{\frac{n(n-2)}{n-1}(1+\beta)}}{r^{1+\frac{n(n-2)}{n-1}(1+\beta)}} = \frac{K}{r}, \quad \text{as } r \to \infty,$$

which implies $c(r) \sim K\log r + B$ as $r \to \infty$ for some $K \neq 0$ and $B \in \mathbb{R}$. However, in this case, $|g(r)| \sim |(K\log r + B)g_0(r)| \to \infty$ as $r \to \infty$, which contradicts the assumption that g is bounded. As a result, we can conclude or obtain the claim.

The case $k \geq 1$. In this case, we claim that all bounded solutions of $L_k(g) = 0$ are identically zero. To prove this, let us assume the contrary, that there exists $g \not\equiv 0$ satisfying $L_k(g) = 0$. We may assume that there exists

$R_k > 0$ such that $g(r) > 0$ on $(0, R_k)$ and $g'(R_k) \leq 0$. Now, g_k satisfies

$$\left(r^{n-1} g_k'(r)|u_{1,\beta}'|^{n-2}\right)' - \lambda_k r^{n-3} \frac{1}{n-1}|u_{1,\beta}'|^{n-2} g_k(r) + \frac{r^{n-1}}{n-1} r^{n\beta} e^{u_{1,\beta}(r)} g_k(r) = 0. \tag{11}$$

Furthermore g_0 is a solution of (9) for $k = 0$:

$$\left(r^{n-1} g_0'(r)|u_{1,\beta}'|^{n-2}\right)' + \frac{r^{n-1}}{n-1} r^{n\beta} e^{u_{1,\beta}(r)} g_0(r) = 0. \tag{12}$$

Multiplying (11) by g_0 and multiplying (12) by g_k and subtracting, we find

$$\left(r^{n-1} g_k'(r)|u_{1,\beta}'|^{n-2}\right)' g_0(r) - \left(r^{n-1} g_0'(r)|u_{1,\beta}'|^{n-2}\right)' g_k(r) = \lambda_k r^{n-3} \frac{1}{n-1}|u_{1,\beta}'|^{n-2} g_k(r) g_0(r). \tag{13}$$

Integrating both sides of the above from $r = 0$ to $r = R_k$ and using $g_k(R_k) = 0$, we obtain

$$R_k{}^{n-1}|u_{1,\beta}'|^{n-2} g_k'(R_k) g_0(R_k) = \lambda_k \int_0^{R_k} r^{n-3} \frac{1}{n-1}|u_{1,\beta}'|^{n-2} g_k(r) g_0(r) \mathrm{d}r. \tag{14}$$

Since $\lambda_k > 0$ for $k \geq 1$, $g_k(r) > 0$ on $(0, R_k)$, and $g_0(r) > 0$, the right-hand side of (14) is positive. On the other hand, the left-hand side of (14) is non positive since $g_k'(R_k) \leq 0$. This contradiction implies the claim. By combining all the facts and evidence presented throughout our proof, we can confidently conclude that Theorem has been successfully proven.

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