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ON A LOGARITHMIC COEFFICIENTS INEQUALITY FOR THE CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In *Logarithmic coefficients problems in families related to starlike and convex functions*, J. Aust. Math. Soc., 109, pp. 230–249, 2020, Ponnusamy et al. stated the conjecture for the sharp bounds of the logarithmic coefficients γ_n for $f \in \mathscr{F}(3)$ as follows

$$|\gamma_n| \leq \frac{1}{n}\left(1-\frac{1}{2^{n+1}}\right), \quad n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6} + \frac{1}{4}\mathrm{Li}_2\left(\frac{1}{4}\right) - \mathrm{Li}_2\left(\frac{1}{2}\right),$$

where Li_2 is the Spence's (or dilogarithm) function. In this research we confirm that the conjecture for the above second inequality is true under some additional conditions.

***Key words**:* univalent functions, starlike, convex and close-to-convex functions, subordination, subordination function, logarithmic coefficients, dilogarithm function.
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1. INTRODUCTION

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane $\mathbb{C}$, and let $\mathscr{A}$ be the set of functions f analytic in $\mathbb{D}$ that has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Also, let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all univalent functions in $\mathbb{D}$. Then, *the logarithmic coefficients* $\gamma_n := \gamma_n(f)$ of a function $f \in \mathscr{S}$ are defined with the aid of the following series expansion

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{D}, \quad \log 1 := 0. \tag{2}$$

These coefficients are significant for various estimates in the theory of univalent functions, see for example [6, Chapter 2] and [5]. The logarithmic coefficient problems and their applications are also studied recently by several authors, for instance see [8, 10, 12]. Note that we use the notation γ_n instead of $\gamma_n(f)$ throughout the paper.

For $c \in (0,3]$, the class $\mathcal{F}(c)$ is defined (see [11]) by

$$\begin{aligned}\mathcal{F}(c) &:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 1 - \frac{c}{2},\, z \in \mathbb{D} \right\} \\ &= \left\{ f \in \mathcal{A} : \mathbb{D} \ni z \mapsto zf'(z) \in \mathcal{S}^*[c-1,-1] \right\},\end{aligned}$$

where

$$\mathcal{S}^*[A,B] := \left\{ \varphi \in \mathcal{A} : \frac{z\varphi'(z)}{\varphi(z)} \prec \frac{1+Az}{1+Bz},\, z \in \mathbb{D} \right\}, \quad A \in \mathbb{C},\ -1 \leq B \leq 0,\ A \neq B,$$

and the symbol "$\prec$" stands for the subordination. We recall that if f and F are two analytic functions in $\mathbb{D}$, the function f is called *subordinate* to F, written $f \prec F$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{C}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{D}$. The function ω that satisfies this property is called a *subordination function* (see [3, p. 125]). It is well-known that if F is univalent in $\mathbb{D}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$ (see [7, p. 15]).

If we take $\alpha := 1 - c/2 \in [0,1)$, then the family $\mathcal{F}(c)$ is the well-known class of *convex functions of order* α denoted by $\mathcal{C}(\alpha)$, and clearly $\mathcal{F}(2) = \mathcal{C}(0) =: \mathcal{C}$ is the class of *convex functions*. More specifically, for $c := 3$, we get the class $\mathcal{F}(3)$ which encouraged a lot of studies in recent years (see [9] and the references therein). It is also important to note that functions of $\mathcal{F}(3)$ are seen to be convex in one direction (and hence, univalent and close-to-convex) but are not necessarily starlike in $\mathbb{D}$ (see [15]).

In 2020 Ponnusamy et al. [11] investigated the bounds of the logarithmic coefficients for selected subfamilies of univalent functions and found the sharp upper bound for γ_n when $n = 1,2,3$, if f belongs to the classes $\mathcal{F}(c)$ for $c \in (0,3]$, (see also [1, 2]). Additionally, the authors of this study presented a conjecture for the logarithmic coefficients γ_n for $f \in \mathcal{F}(3)$ as follows:

CONJECTURE 1. *The logarithmic coefficients γ_n of $f \in \mathcal{F}(3)$ satisfy the inequalities*

$$|\gamma_n| \leq \frac{1}{n}\left(1 - \frac{1}{2^{n+1}}\right), \quad n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(2 - \frac{1}{2^n}\right)^2 = \frac{\pi^2}{6} + \frac{1}{4}\operatorname{Li}_2\left(\frac{1}{4}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right),$$

where

$$\operatorname{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad x \in (-1,1),$$

denotes the Spence's (or dilogarithm) function. Equalities in these inequalities are attained for the function $f_0 \in \mathcal{F}(3)$ of the form

$$f_0(z) := \frac{z - z^2/2}{(1-z)^2}, \quad z \in \mathbb{D}.$$

In the current study we confirm that this conjecture holds for the above second inequality under some additional conditions.

2. MAIN RESULTS

We will get our first main result by using the subsequent lemmas. The first one was shown by Rogosinski [13]; cf. [4, Theorem 6.2, p. 192].

LEMMA 1. *Let*

$$f(z)=\sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z)=\sum_{n=1}^{\infty} b_n z^n, \quad z\in\mathbb{D},$$

be analytic in $\mathbb{D}$*, and suppose that* $f\prec g$*. Then for every* $n\in\mathbb{N}$*,*

$$\sum_{k=1}^{n}|a_k|^2 \le \sum_{k=1}^{n}|b_k|^2.$$

Repeating argumentation from the proof of Theorem 9 from [3, p. 135] we can observe that Theorem 9 is also true for $\alpha := 3/2$ and then has the following form.

LEMMA 2. *Let* $h,q:\mathbb{D}\to\mathbb{C}$ *be given by*

$$h(z):=\frac{1+2z}{1-z} \quad \textit{and} \quad q(z):=\frac{2}{(1-z)(2-z)}, \quad z\in\mathbb{D}. \tag{3}$$

If p *is an analytic function in* $\mathbb{D}$ *with* $p(0)=1$ *and* ω *is a subordination function such that*

$$p(z)+\frac{zp'(z)}{p(z)}=h(\omega(z)), \quad z\in\mathbb{D}, \tag{4}$$

then the differential equation

$$\varphi'=\frac{\varphi\left[1-\omega+3(\omega-\varphi)-(2\omega+1)(1-\varphi)^3\right]}{z(1-\omega)\left[1-(2\varphi+1)(1-\varphi)^2\right]}, \quad z\in\mathbb{D}, \tag{5}$$

with $\varphi(0)=0$*, has a solution* φ *analytic in* $\mathbb{D}$ *such that* $p(z)=q(\varphi(z))$ *for* $z\in\mathbb{D}$*. Furthermore, if* φ *is also a subordination function, then* $p\prec q$ *and* q *is the best dominant.*

Using the notations of Theorem 3.1d of [7] (see also [14]), this theorem can be formulated for the special case $a=0$ and $n=1$, with $F(z):=zp'(z)$ for $z\in\mathbb{D}$, as follows:

LEMMA 3. *Let* h *be starlike in* $\mathbb{D}$*, with* $h(0)=0$*. If* F *is analytic in* $\mathbb{D}$ *with* $F(0)=0$*, and* $F\prec h$*, then*

$$\int_0^z \frac{F(t)}{t}\,\mathrm{d}t \prec \int_0^z \frac{h(t)}{t}\,\mathrm{d}t =: q(z), \quad z\in\mathbb{D}.$$

Moreover, q *is a convex function and the best dominant.*

In the next theorem we will prove that the second inequality of the Conjecture A holds under some additional conditions, and another inequality involving the logarithmic coefficient will be also obtained.

THEOREM 1. *Let* $f\in\mathscr{F}(3)$ *and* ω *be the subordination function such that*

$$1+\frac{zf''(z)}{f'(z)}=\frac{1+2\omega(z)}{1-\omega(z)}, \quad z\in\mathbb{D}, \tag{6}$$

and let φ *the analytic solution in* $\mathbb{D}$ *of the differential equation* (5) *with* $\varphi(0)=0$*. If* φ *is a subordination function, then the logarithmic coefficients of* f *fulfill the inequalities*

$$\sum_{n=1}^{\infty}|\gamma_n|^2 \le \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2-\frac{1}{2^n}\right)^2=\frac{\pi^2}{6}+\frac{1}{4}\mathrm{Li}_2\left(\frac{1}{4}\right)-\mathrm{Li}_2\left(\frac{1}{2}\right) \tag{7}$$

and

$$\sum_{n=1}^{\infty} n^2|\gamma_n|^2 \le \frac{1}{4}\sum_{n=1}^{\infty}\left(2-\frac{1}{2^n}\right)^2. \tag{8}$$

The equalities in these inequalities are attained for the function $f_0 \in \mathscr{F}(3)$ of the form

$$f_0(z) := \frac{z-z^2/2}{(1-z)^2}, \quad z \in \mathbb{D}.$$

Proof. Let $f \in \mathscr{F}(3)$ be of the form (1). By definition,

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+2z}{1-z}, \quad z \in \mathbb{D}.$$

Thus there exists a subordination function ω such that (6) hold. If we set

$$p(z) := \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}, \tag{9}$$

then (6) is equivalent to

$$p(z)+\frac{zp'(z)}{p(z)} = h(\omega(z)), \quad z \in \mathbb{D},$$

where the function h is defined by (3), i.e., (4) holds.

On the other hand, the function q defined by (3), i.e.,

$$q(z) = \frac{2}{(1-z)(2-z)}, \quad z \in \mathbb{D}, \tag{10}$$

is an analytic solution in $\mathbb{D}$ of the differential equation

$$q(z)+\frac{zq'(z)}{q(z)} = \frac{1+2z}{1-z} = h(z), \quad z \in \mathbb{D}.$$

Since φ is a subordination function, from Lemma 2 it follows that $p \prec q$ and q is the best dominant. Thus by (9) and (10) we obtained the sharp subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{2}{(1-z)(2-z)} = 1+\sum_{n=1}^{\infty} 2\left(1-\frac{1}{2^{n+1}}\right)z^n, \quad z \in \mathbb{D}. \tag{11}$$

Define the function $H : \mathbb{D} \to \mathbb{C}$ as

$$H(z) := \frac{f(z)}{z}, \quad z \in \mathbb{D}\setminus\{0\}, \quad H(0) := 1. \tag{12}$$

Clearly, H is an analytic function in $\mathbb{D}$. Since f is a univalent function in $\mathbb{D}$, it follows that $f(z) \neq 0$ for $z \in \mathbb{D}\setminus\{0\}$ and 0 is a simple zero for f. Thus the function $F : \mathbb{D} \to \mathbb{C}$ defined as

$$F(z) := \frac{zH'(z)}{H(z)}, \quad z \in \mathbb{D}\setminus\{0\}, \quad F(0) := 1, \tag{13}$$

is analytic in $\mathbb{D}$. Hence using (11) the following subordination holds

$$\frac{zH'(z)}{H(z)} = \frac{zf'(z)}{f(z)} - 1 \prec q(z)-1 =: \nu(z), \quad z \in \mathbb{D}.$$

We have $\nu(0)=0$, $\nu'(0)=q'(0)=3/4\neq 0$, and

$$\begin{aligned}\frac{z\nu'(z)}{\nu(z)}&=\frac{zq'(z)}{q(z)-1}\\&=\frac{2(2z-3)}{(z-3)(z-1)(z-2)}=\frac{3}{z-3}-\frac{2}{z-2}-\frac{1}{z-1},\quad z\in\mathbb{D}.\end{aligned}\tag{14}$$

Note that for $z:=\mathrm{e}^{\mathrm{i}t}$, $t\in(0,2\pi)$, we have

$$\operatorname{Re}\left(\frac{3}{z-3}-\frac{2}{z-2}-\frac{1}{z-1}\right)=G(\cos t),\tag{15}$$

where $G:[-1,1)\to\mathbb{R}$ is a function defined as

$$G(s):=\frac{3(s-3)}{10-6s}-\frac{2(s-2)}{5-4s}+\frac{1}{2},\quad s\in[-1,1).$$

Since

$$G'(s)=\frac{-42s^2+60s}{(3s-5)^2(4s-5)^2},\quad s\in(-1,1),$$

we get

$$\min\{G(s):s\in[-1,1)\}=G(0)=\frac{2}{5}>0.$$

Hence, from (14), (15) and minimum principle for harmonic functions it follows that

$$\operatorname{Re}\frac{z\nu'(z)}{\nu(z)}>\frac{2}{5}>0,\quad z\in\mathbb{D}.$$

Thus ν is a starlike univalent function in $\mathbb{D}$. Now, using Lemma 3 with F defined by (13) and $h:=\nu$, we conclude that

$$\int_0^z\frac{H'(t)}{H(t)}\mathrm{d}t\prec\int_0^z\frac{\nu(t)}{t}\mathrm{d}t,\quad z\in\mathbb{D},$$

i.e., by (12) that

$$\log\frac{f(z)}{z}\prec\int_0^z\frac{\nu(t)}{t}\mathrm{d}t,\quad z\in\mathbb{D}.$$

Moreover, the function

$$\mathbb{D}\ni z\mapsto\int_0^z\frac{\nu(t)}{t}\mathrm{d}t$$

is a convex function and is the best dominant. Now by (2) and (3) the previous subordination could be written as

$$\sum_{n=1}^{\infty}2\gamma_n z^n\prec\sum_{n=1}^{\infty}\frac{2}{n}\left(1-\frac{1}{2^{n+1}}\right)z^n,\quad z\in\mathbb{D}.$$

Hence by using Lemma 1 we get

$$\sum_{n=1}^{k}|\gamma_n|^2\le\sum_{n=1}^{k}\frac{1}{n^2}\left(1-\frac{1}{2^{n+1}}\right)^2\le\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{2^{n+1}}\right)^2,\quad k\in\mathbb{N},$$

and taking $k\to\infty$ we conclude that

$$\sum_{n=1}^{\infty}|\gamma_n|^2\le\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{2^{n+1}}\right)^2=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2-\frac{1}{2^n}\right)^2,$$

which shows (7).

Further, from (2) and (11) we deduce that

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n = z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec q(z) - 1 = \nu(z), \quad z \in \mathbb{D}.$$

Using now Lemma 1 we get

$$\sum_{n=1}^{k} n^2 |\gamma_n|^2 \le \sum_{n=1}^{k}\left(1 - \frac{1}{2^{n+1}}\right)^2 \le \frac{1}{4}\sum_{n=1}^{\infty}\left(2 - \frac{1}{2^n}\right)^2, \quad k \in \mathbb{N},$$

and letting $k \to +\infty$ shows the inequality (8).

Finally, it is sufficient to take into the account the equality

$$\frac{zf_0'(z)}{f_0(z)} = \frac{2}{(1-z)(2-z)}, \quad z \in \mathbb{D},$$

to prove the sharpness of inequalities (7) and (8). □

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